Contributions to the phase separation problem and Ornstein–Zernike theory

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Summary

This doctoral dissertation is concerned with the study of the behaviour of models of Statistical mechanics far away or near criticality. We essentially focus on two instances of such models, namely the *FK percolation* and the *Potts model*. We investigate two theories developed to study these models in such non-critical regimes: the *phase separation theory* and the *Ornstein–Zernike theory*. While the former is concerned with the study of the coexistence of distinct phases in supercritical spin models, the latter is a tool allowing to understand the fine behaviour of correlation functions for models in which exponential decay of correlation hold; in the case of FK percolation, this is the whole subcritical regime.

The thesis contains three separate works providing original research contributions to these two problems.

The first contribution — based on a research article in collaboration with Romain Panis — is concerned with the study of the phase separation problem for random walks. Indeed, this work is devoted to the study of the behaviour of a (1+1)-dimensional model of random walk conditioned to enclose an area of order N^2. Such a conditioning enforces a globally concave trajectory. We study the local deviations of the walk from its convex hull. To this end, we introduce two quantities — the mean local roughness MeanLR and the mean facet length MeanFL — measuring the typical transversal and longitudinal fluctuations around the boundary of the convex hull of the random walk. Our main result is that MeanFL is of order $N^{2/3}$ and MeanLR is of order $N^{1/3}$. This model is intended to be a toy model for the interface of a two-dimensional statistical mechanics model (such as the Ising model) in the phase separation regime — we discuss this issue at the end of this work

Our second contribution regards the study of a finite system of long clusters of subcritical 2-dimensional FK-percolation with $q \geq 1$, conditioned on mutual avoidance. We show that the diffusive scaling limit of such a system is given by a system of Brownian bridges conditioned not to intersect: the so-called *Brownian watermelon*. Moreover, we give an estimate of the probability that two sets of r points at distance n of each other are connected by distinct clusters. As a byproduct, we obtain the asymptotics of the probability of the occurrence of a large finite cluster in a supercritical random-cluster model. The proofs heavily relies on Ornstein–Zernike theory for *non-independent* systems of interfaces.

Finally the last contribution of this doctoral dissertation — based on a research article to be written in collaboration with Ioan Manolescu — regards an extension of the Ornstein–Zernike theory to the *near-critical* regime of FK percolation. In particular, the formula shown to drive the behaviour of the two-point correlation function of the model is uniform in $p < p_c$. The analysis is carried out by studying the renewal properties of a subcritical percolation cluster, *at the scale of the correlation length*. In particular, we show that the uniform probability of crossing boxes at the scale of the correlation length is sufficient to obtain the required mixing properties of the underlying renewal process. We also derive the Brownian asymptotics for long subcritical clusters, uniformly in p.

Cette thèse de doctorat se propose d'étudier le comportement de certain modèles issus de la mécanique statistique, dans leurs régimes non critiques ou presque critiques. En particulier, nous nous concentrons sur l'étude de deux modèles classiques, à savoir la *FK percolation* et le *modèle de Potts*. Nous étudions deux théories développées dans le but d'étudier ces modèles hors de leur régime critique : la *théorie de la séparation des phases* et la *théorie d'Ornstein–Zernike*. La première concerne l'étude de la coexistence de phases distinctes dans les modèles de spins surcritiques, tandis que la seconde permet de comprendre le comportement précis des fonctions de corrélations des modèles dans lesquels ces dernières décroissent exponentiellement.

Cette thèse contient trois contributions distinctes et originales à ces deux problèmes.

La première — basée sur un article de recherche co-écrit avec Romain Panis — concerne l'étude du problème de la séparation des phases pour les marches aléatoires. En effet, ce travail est consacré à l'étude du comportement d'un modèle de marche aléatoire (1+1)-dimensionnelle conditionnée à capturer une aire d'ordre N^2. Ce conditionnement impose une trajectoire globalement concave. Nous étudions les déviations locales de la marche par rapport à son enveloppe convexe. Dans ce but, nous introduisons deux quantités — la rugosité locale moyenne MeanLR et la longueur moyenne des facettes MeanFL — mesurant les fluctuations transversales et longitudinales typiques autour de la frontière de l'enveloppe convexe de la marche aléatoire. Notre résultat principal est que MeanFL est d'ordre $N^{2/3}$ et MeanLR est d'ordre $N^{1/3}$. Ce modèle est conçu pour servir de modèle simplifié pour l'interface obtenue dans le modèle d'Ising dans le régime de séparation des phases.

Notre seconde contribution concerne l'étude d'un système fini de longs amas de percolation FK sous-critique bidimensionnelle avec $q \geq 1$, conditionnées à s'éviter mutuellement. Nous montrons que la limite d'échelle diffusive d'un tel système est donnée par un système de ponts browniens conditionnés à ne pas s'intersecter : *la pastèque brownienne*. De plus, nous fournissons une estimation de la probabilité que deux ensembles de r points distants de n soient connectés par des amas distincts. En passant, nous obtenons également le comportement asymptotique de la probabilité d'occurrence d'un grand amas fini pour la FK percolation surcritique. Les preuves reposent fortement sur la théorie d'Ornstein–Zernike pour les systèmes d'interfaces *non-indépendants*.

Enfin, la dernière contribution de cette thèse de doctorat — basée sur un futur article de recherche en collaboration avec Ioan Manolescu — concerne une extension de la théorie d'Ornstein–Zernike au régime *presque-critique* de la percolation FK. En particulier, la formule régissant le comportement de la fonction de corrélation à deux points du modèle obtenue dans ce travail est uniforme en $p < p_c$. L'analyse est réalisée en étudiant les propriétés de renouvellement d'un amas de percolation sous-critique, *à l'échelle de la longueur de corrélation*. En particulier, nous montrons que la probabilité uniforme de traverser des boîtes à l'échelle de la longueur de corrélation est suffisante pour obtenir les propriétés de mélange requises du processus de renouvellement sous-jacent. Nous dérivons également les asymptotiques browniennes pour les longs amas sous-critiques, uniformément en p.

À mon père.

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Contents

Chapter 1

Introduction: the phase separation problem and the Ornstein–Zernike theory

What happens when a small quantity of oil is mixed in some water? Reading that question in the very beginning of a PhD dissertation in Mathematics might put a slight smile on some faces (at least we hope it does!). Depending on who you ask it to, this question may have several different answers. If you are discussing with a child, they would probably tell you after a moment of hesitation that the oil creates a round droplet inside the water. If you talk to a physicist, however, you would probably get a more sophisticated answer such as: “the oil creates a droplet which adopts a shape minimising its surface energy”. Finally if you ask a mathematician, well, you might end up embarking on a 4-years journey, ultimately leading to a doctoral dissertation such as the present one.

More precisely, consider the following thought experiment, introduced in [32]. Pour a very small quantity of oil inside a jar of water. As one knows, oil and water tend to repel each other. However, at very small densities, the oil will be soluble in the water up to some critical density ρ_c above which it is not soluble anymore. Moreover, it is known that the critical density increases with the temperature T: the larger T is, the larger ρ_c will be. The experiment proposed by Cerf and Pisztora is thus the following: pour a small quantity of oil in the water, such that its density ρ is just slightly smaller than ρ_c. The oil is then totally dissolved in the water. Once the mixture is stabilised, decrease T, in such a way that ρ_c drops below ρ. One thus expects to see a *condensation* phenomenon: all of a sudden, a visible droplet will appear in the mixture.

On the other hand, it is now widely acknowledged that the behaviour of many physical systems is intrinsically *random*, at least at very small (atomic or *microscopic*) scales. One might think for instance of the theory of Quantum mechanics, but most importantly of the development of so-called *Statistical mechanics*. Very roughly speaking, the purpose of this approach is trying to understand how the large scale properties of a physical system can be understood from its microscopic properties. For instance, assume that one has n molecules interacting in a finite volume. If $n = 2$, then one might expect to rigorously compute the way in which those particles evolve in the space, as the interaction between them is known. If n is slightly larger, then the problem already starts to be very complicated to solve rigorously[1]. Of course, if n is of realistic physical order, it should be taken extremely large if one wants to describe a physical phenomenon at the visible scale (coming back to our previous discussion, there are roughly $n \approx 10^{26}$ molecules that interact together), making any dream of an exact computation totally hopeless. However, it has progressively been understood during the last century that considering that the molecules adopt a *random* configuration in which the probability of a given configuration depends on the *total energy* of the configuration provides an accurate description of the physical system, in the sense that large-scale properties of the system can be derived by probabilistic techniques. To mention just a few examples, this approach has revealed itself very fruitful in the understanding of ferromagnetism, supraconductivity, turbulence, phase transitions, and many other physical phenomena. Our goal is not to provide a comprehensive history of Statistical mechanics; however it would be almost criminal not to mention the names of James C. Maxwell, Ludwig Boltzmann and Josiah W. Gibbs as the three founding fathers of this theory.

Coming back to our original problem, the question driving this dissertation is then the following: how does the previously described condensation phenomenon fit into the conceptual framework provided by the rigorous Statistical mechanics theory? This turns out to be a pretty deep and mesmerizing question, and its investigation has given rise to very fruitful methods and results that shall be described in more detail later on in this introduction. For the mathematical side of the story, it would also probably be *lèse-majesté* not to mention the names of Roland D. Dobrushin, Roman Kotecký and Senya Shlosman as the first ones to provide a rigorous and comprehensive treatment of that question in two dimensions, in the context of the Ising model.

It is now time to leave the physical realm, as this dissertation will mostly be concerned with the study of this problem *in two dimensions*. As will be described later, the shape of the oil droplet in that case is well-understood. It adopts a deterministic (*i.e.*, non-random) profile, that can be described as the solution of a variational problem in two dimensions. However, imagine that one now turns into a tiny ant[2] walking on the boundary of the droplet, so small that it does not feel the curvature of the droplet at all. At this scale, it seems reasonable to expect that the intrinsic randomness of the model starts to play a role, and that the ant might actually walk on a *random* curve of the plane. If it were indeed the case, what can be said about the distribution of this random curve? Can some interesting scaling limits be

[1] Think about the infamous *three body problem*, in which three massive point masses orbit each other in space. The resulting system of equations does not have a closed-form solution and is known to be chaotic in most cases.

[2] Of course, this is not really a good point of view, for at least two different reasons: first, an ant is still formally a macroscopic entity whose size is comparable to the size of the droplet. Second, an ant could not survive for a very long time when immersed in a mixture of oil and water. Nevertheless, the idea of being a tiny ant walking on a random curve is so surprisingly seducing that we decide to keep it anyway.

identified? This dissertation is exactly concerned with this type of questions. It gathers three different pieces of work related to this topic in the framework of the so-called *random-cluster* and *directed random walk* models.

We will start with an historical review of the problem in the introduction, rigorously prove a number of well-known properties of the random-cluster model in the second chapter, and the remaining three chapters are devoted to the research output of this PhD thesis.

1.1 Models to be considered

As this dissertation is concerned with probability measures on graphs, we introduce the notation and some basic graph-theoretical notions.

1.1.1 Graph theoretical notions and notation

An *unoriented graph* is a pair of sets $G = (V(G), E(G))$, where $V(G)$ is referred to as the set of *vertices*, and $E(G)$ is some collection of pairs $\{x, y\}$ where $x, y \in V(G)$. The set $E(G)$ is referred to as the *edge set* of the graph G, and a given edge $\{x, y\} \in E(G)$ will be denoted by $\{xy\}$. In this dissertation we will mostly be concerned with a special family of graphs, namely subgraphs of the hypercubic lattice. Fix $d \in \mathbb{N}^*$ a positive integer. The hypercubic lattice is the graph $(\mathbb{Z}^d, E(\mathbb{Z}^d))$, where $E(\mathbb{Z}^d) = \{\{xy\}, x, y \in \mathbb{Z}^d, \|x - y\| = 1\}$ ($\| \cdot \|$ denotes the standard Euclidean distance on $\mathbb{R}^d$). In what follows, we will often make the slight notational abuse of writing $\mathbb{Z}^d$ instead of $(\mathbb{Z}^d, E(\mathbb{Z}^d))$ to refer to the lattice itself, but it will be clear depending on the context.

In our context, percolation theory studies the connectivity properties of a model of random subgraphs of the hypercubic lattice. A subgraph $G' = (V(G'), E(G'))$ of G is defined as a pair consisting of a subset of $V(G)$ and a set of edges between vertices of $V(G')$. In what follows, the vertex-set will almost always taken to be equal to $\mathbb{Z}^d$ itself: we will mostly consider subgraphs of $\mathbb{Z}^d$ obtained by randomly removing some edge of the hypercubic lattice.

The following notion of connectivity will be thoroughly used in what follows: let $x, y \in V(G)$. We say that x and y are *connected* in G if there exists an integer $n \geq 0$, a sequence of vertices $x_0, \dots, x_{n+1}$ satisfying $x_0 = x, x_{n+1} = y$ and $\{x_i x_{i+1}\} \in V(G)$ for any $i \in \{0, \dots, n\}$. In that case, we will write $\{x \longleftrightarrow y\}$, and the sequence of edges $(\{x_0 x_1\}, \{x_1 x_2\}, \dots, \{x_n x_{n+1}\})$ will be referred to as an *edge-path* (or simply *path*) from x to y. If x and y are *not* connected in the graph G, we will write $\{x \not\longleftrightarrow y\}$. Moreover, we will say that $x \in V(G)$ *percolates* if there exists an infinite self-avoiding edge-path starting at x or equivalently if there exists an infinite sequence of vertices $(x_0, x_1, \dots)$ such that $x_0 = x$, for any $i \geq 0$, $\{x_i x_{i+1}\} \in E(G)$ and for any $0 \leq i \neq j$, $x_i \neq x_j$). In that case we will write $\{x \longleftrightarrow \infty\}$.

If $G' = (V', E')$ is a subgraph of G, we define its *inner boundary*, denoted by $\partial G'$ as follows

$$\partial G' = \{x \in V', \exists y \in V(G), \{xy\} \in E(G) \setminus E'\},$$

and its *outer boundary*, denoted by $\partial^{\text{ext}} G'$ as

$$\partial^{\text{ext}} G' = \{x \in V(G) \setminus V', \exists y \in V', \{xy\} \in E(G)\}.$$

We introduce the following family of subsets of $\mathbb{Z}^d$ that we call *boxes*. For $n \geq 1$, define $\Lambda_n := \{-n, \dots, n\}^d$ (we make no reference to the dimension d as it will always be fixed and no confusion will be possible in what follows). Observe that $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$. Finally if $x \in \mathbb{Z}^d$, we write $\Lambda_n(x) := x + \Lambda_n$.

1.1.2 The q-states Potts model

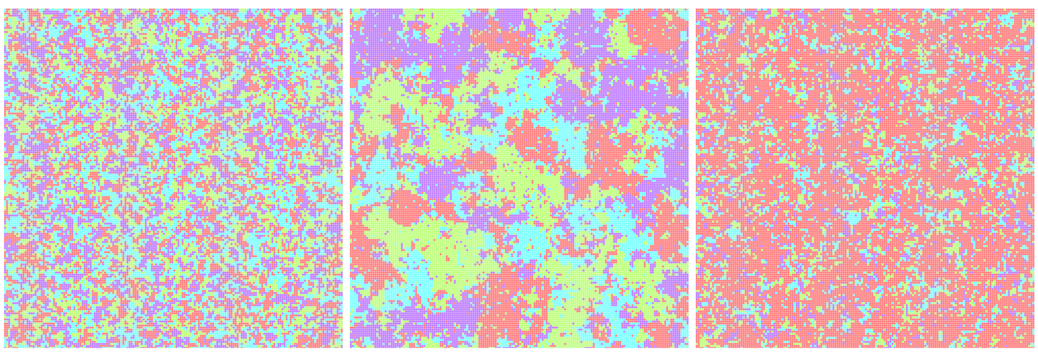

Figure 1.1: Three simulations of the Potts model with $q = 4$, sampled via Glauber dynamics. From left to right, $\beta < \beta_c, \beta \approx \beta_c, \beta > \beta_c$.

Definition of the measure

Fix an integer $q \geq 2$. The q-states Potts model, named after the Australian mathematician Renfrey Potts, was introduced [105] in 1952 as a generalisation of the Ising model. It is an important protagonist of the modern Statistical mechanics theory, as it displays a rich behaviour despite its quite simple definition.

We chose to restrict our setting to the Potts model on the hypercubic lattice, even though the model has been extensively studied in different types of graphs (*e.g.* trees). Fix a dimension $d \geq 1$ and some finite subgraph $G \subset \mathbb{Z}^d$. The q-states Potts measure is a probability measure on the set of functions $\sigma : V(G) \to \{1, \dots, q\}$, where $V(G)$ denotes the set of vertices of the graph G, and the numbers $1, \dots, q$ are interpreted as different *colours*. We will refer to such σ as *spin configurations* on G, and — sticking to usual conventions — will write $\sigma_x := \sigma(x)$. As often in Statistical mechanics, we start by defining the so-called *Hamiltonian* of the model. We fix η a spin configuration on $\partial^{\mathsf{ext}}G$, which we will refer to as the *boundary condition*. For σ a spin configuration on G, we define

$$\mathcal{H}^{\eta}(\sigma) := -\sum_{\substack{x,y\in V(G)\\ x\sim y}} \mathbb{1}_{\sigma_x=\sigma_y} - \sum_{\substack{x\in V(G),y\notin V(G)\\ x\sim y}} \mathbb{1}_{\sigma_x=\eta_y}.$$

In the previous expression, we say that $x \sim y$ whenever $\|x-y\| = 1$.[3]

We fix a parameter $\beta > 0$,[4] which shall be called the *inverse temperature* in what follows. The q-states Potts measure on G is defined by setting for any function F from the set of spin configurations to $\mathbb{R}$,

$$\mu^{\eta}_{G,\beta,q}[F] = \frac{1}{Z^{\eta}_{G,\beta,q}} \sum_{\sigma:V(G)\to\{1,\dots,q\}} F(\sigma)\mathrm{e}^{-\beta\mathcal{H}^{\eta}(\sigma)},$$

where $Z^{\eta}_{G,\beta,q} > 0$ is the unique normalization constant ensuring that $\mu^{\eta}_{G,\beta,q}[1] = 1$. It is usually referred to as the *partition function* of the model, and it is an object of crucial importance in mathematical physics.

Some choices of boundary conditions are classical. Fix $j \in \{1,\dots,q\}$. We shall refer to the choice $\eta \equiv j$ (resp. $\eta \equiv 0$) as j-*monochromatic* boundary conditions (resp. *free* boundary conditions) and will write the measures as $\mu^{j}_{G,\beta,q}$ (resp. $\mu^{\mathsf{f}}_{G,\beta,q}$). It is classical, and will be a consequence of what follows, that with those particular choices of boundary conditions, one can extend those measures to the whole $\mathbb{Z}^d$, by taking limits along sequences of growing boxes.

Lemma 1.1.1. *For any $q \in \mathbb{N}^*$ and $\beta > 0$, there exist $q+1$ measures (not necessarily distinct) $\mu^1_{\beta,q},\dots,\mu^q_{\beta,q},\mu^{\mathsf{f}}_{\beta,q}$ on $\{1,\dots,q\}^{V(\mathbb{Z}^d)}$ such that for any event $\mathcal{A}$ depending on a finite number of spins, for any $j \in \{1,\dots,q\}$,*

$$\lim_{n\to\infty} \mu^{j}_{\Lambda_n,\beta,q}[\mathcal{A}] = \mu^{j}_{\beta,q}[\mathcal{A}] \qquad \textit{and} \qquad \lim_{n\to\infty} \mu^{\mathsf{f}}_{\Lambda_n,\beta,q}[\mathcal{A}] = \mu^{\mathsf{f}}_{\beta,q}[\mathcal{A}].$$

Those measures are called *infinite-volume measures*, and the question of determining whether they are distinct or equal is one of the starting points of the modern Statistical mechanics theory.

[3] In this dissertation, we are interested only in the nearest-neighbour Potts model. Of course, one can generalise the definition of the model by allowing more complex interactions by means of adding a family of coupling constants $\{J_{xy}\}$ in the sum defining the Hamiltonian and summing over all the $(x,y) \in V(G)^2$. Our setting then corresponds to choosing $J_{xy} = \mathbb{1}_{x\sim y}$.

[4] The choice $\beta > 0$ corresponds to the so-called *ferromagnetic* Potts model: the measure favours neighbouring spins to have the same colour. The model could however be defined in a similar way for $\beta < 0$ which corresponds to the antiferromagnetic Potts model. This model is expected to display a very different behaviour than in the ferromagnetic case; for instance it does not enjoy the positive association property. In this dissertation, we will be focusing on the ferromagnetic case.

The phase transition of the Potts model

We now describe one of the salient features of the q-states Potts model: the existence of the phase transition.

Theorem. Fix $q \in \mathbb{N}^*$. There exists $\beta_c \in (0, +\infty)$ such that

1. For any $\beta < \beta_c$, any $j \in \{1, \dots, q\}$, $\mu^j_{q,\beta} = \mu^{\mathsf{f}}_{q,\beta}$. Moreover,
$$\lim_{\|x-y\| \to \infty} \mu^j_{q,\beta}[\sigma_x = \sigma_y] = \tfrac{1}{q}.$$

2. For any $\beta > \beta_c$, the $q+1$ measures $\mu^1_{\beta,q}, \dots, \mu^q_{\beta,q}, \mu^{\mathsf{f}}_{\beta,q}$ are distinct. Moreover,
$$\lim_{\|x-y\| \to \infty} \mu^j_{q,\beta}[\sigma_x = \sigma_y] - \tfrac{1}{q} > 0.$$

The behaviour of the model will be very different when $\beta < \beta_c, \beta = \beta_c$ and $\beta > \beta_c$. Those regimes will be called *subcritical, critical* and *supercritical* respectively. Even though this PhD thesis will be mostly concerned with properties of the model off criticality, we do not resist gathering a few properties of the *critical* model in two dimensions, as it displays a fascinating behaviour.

The phase transition of the planar Potts model

Among the many questions regarding the behaviour of the phase transition in Potts model, one of the most crucial is the question of *continuity* of the phase transition. It can be phrased in different ways; we choose to formulate it in terms of the number of extremal infinite-volume limit measures at $\beta = \beta_c$. For the square lattice, the question has been solved using percolation methods in a remarkable series of works [53, 50], see also Chapter 2 for a discussion on the random-cluster model. The statement can be summed up as follows.

Theorem. Let $d = 2$.

1. When $q \in [1, 4]$, there is a unique infinite-volume limit measure at $\beta = \beta_c$. For any $j \in \{1, \dots, q\}$, $\mu^j_{\beta_c,q} = \mu^{\mathsf{f}}_{\beta_c,q}$. Moreover,
$$\lim_{\|x-y\| \to \infty} \mu^j_{\beta_c,q}[\sigma_x = \sigma_y] = \tfrac{1}{q}.$$

2. When $q > 4$, the measures $\mu^1_{\beta_c,q}, \dots, \mu^q_{\beta_c,q}, \mu^{\mathsf{f}}_{\beta_c,q}$ are distinct. Moreover, for any $j \in \{1, \dots, q\}$,
$$\lim_{\|x-y\| \to \infty} \mu^j_{\beta_c,q}[\sigma_x = \sigma_y] - \tfrac{1}{q} > 0.$$

We further mention a recent result [65] fully identifying the set of infinite-volume limi measures of the q-states critical Potts model with $q > 4$.

Theorem. Let $d = 2$ and $q > 4$. Let μ be a Gibbs measure for the critical q-states Potts model on $\mathbb{Z}^2$. Then there exist $\alpha_0, \alpha_1, \dots, \alpha_q \geq 0$, with $\alpha_0 + \cdots + \alpha_q = 1$ such that

$$\mu = \alpha_0 \mu^{\mathsf{f}}_{\beta_c,q} + \sum_{j=1}^{q} \alpha_j \mu^{j}_{\beta_c,q}.$$

1.1.3 Graphical representations of spin systems: the example of the Edwards-Sokal coupling

One of the natural important questions in the field of Statistical mechanics is the study of the behaviour of the so-called *correlations functions* of spin models. For instance, for any $x \in \mathbb{Z}^d$, define $G_\beta(x) := \mu^1_{\beta,q}[\sigma_0 = \sigma_x] - \frac{1}{q}$. Understanding the fine behaviour of the function G_β is of particular interest, as this function encodes the correlation structure of the model. One of the main contributions of this PhD thesis is to provide a precise understanding of the function G_β in the joint limit $\|x\| \to \infty$ and $\beta \nearrow \beta_c$.

This problem is obviously very difficult to decipher in general. However a crucial tool has been investigated for more than 50 years now: the so-called *graphical representations* of spin models (see for instance the monograph [48]). Instead of studying spin models on the vertex set of $\mathbb{Z}^d$, we now turn to models of random subgraphs of $\mathbb{Z}^d$: so-called *percolation models*. The idea of graphical representations is to identify correlation functions of the original spin model with the probabilities of some geometrical events in an adequate percolation model. The latter being sometimes easier to study, thanks to modern percolation techniques, this motivates the interest of the study of such models. In what follows, we introduce the main protagonist of this dissertation: the *random-cluster measure*. It is one of the most useful graphical representations of the Potts model and an object of great interest in itself.

The random-cluster model

As done previously for the definition of the Potts measure, we start by defining the model in finite volume. Fix a dimension $d \geq 1$ and a finite subgraph G of $\mathbb{Z}^d$. The random-cluster measure is a probability measure on subgraphs of G. Its state space is given by $\Omega^G := \{0,1\}^{E(G)}$. For an edge $e \in E(G)$ and any percolation configuration $\omega \in \Omega^G$, we say that e is *open* if $\omega(e) = 1$ and *closed* else. The connectivity properties of the subgraph induced by ω play a crucial role in the definition of the measure. Let $x, y \in V(G)$. We say that x and y are connected in ω if there exists $k \geq 0$ and a sequence of vertices $x_0 = x, x_1, \dots, x_k, x_{k+1} = y$ such that for any $0 \leq j \leq k$, $\|x_{j+1} - x_j\| = 1$ and the edge $\{x_j, x_{j+1}\}$ is open in ω. The maximal connected components for this notion of connectivity shall be called *open clusters* of ω.

Like in the Potts model, the boundary conditions will play a crucial role in the definition of the measure. A *boundary condition* η is formally defined as a partition of ∂G. To any boundary condition η and any percolation configuration $\omega \in \{0,1\}^{E(G)}$, one can associate the percolation configuration ω^η which is obtained by identifying together the vertices of ∂G that lie in the same set of the partition η. Moreover, we shall almost always consider a special type of boundary conditions, namely those that are induced by a percolation configuration outside of G. Indeed, let η be a percolation configuration on $\mathbb{Z}^d \setminus G$. Observe that it induces a boundary condition on ∂G by saying that two vertices of ∂G belong to the same set of the partition if and only if they are connected in η. In that case, we shall make a slight notational abuse by identifying the percolation configuration η with the boundary condition it induces on ∂G, and keeping the notation ω^η when η is a percolation configuration on $\mathbb{Z}^d \setminus G$.

Let ω be a percolation configuration on G. We define $o(\omega)$ to be the total number of open edges in ω, and call $k^\eta(\omega)$ the number of open clusters of the configuration ω^η.

Fix two parameters $p \in [0,1]$ and $q > 0$, and a boundary condition η. The random-cluster measure on G with parameters p and q and boundary conditions η is defined as follows: for any percolation configuration ω on G, we set

$$\phi^\eta_{G,p,q}[\omega] = \frac{1}{Z^\eta_{G,p,q}} \left(\frac{p}{1-p}\right)^{o(\omega)} q^{k^\eta(\omega)}.$$

As previously, $Z^\eta_{G,p,q}$ is the *partition function* of the model, that is, the only constant guaranteeing that $\phi^\eta_{G,p,q}$ is indeed a probability measure.

Some choices of q are of particular interest: observe that when $q = 1$, the measure is a product measure. It can be realized a follows: for each edge open it independently of the state of the other edges with probability p and close it else. This model is called *Bernoulli percolation* and is historically the first model of percolation. In that case, the model is insensitive to the choice of boundary conditions, and $Z_{G,p,1} = 1$. The choice $q = 2$ is also relevant, as it can be coupled to the so-called *Ising model* thanks to the Edwards–Sokal coupling (see the next subsection). In that case, the model in $\mathbb{Z}^2$ is integrable, meaning that almost all the relevant quantities such as the correlation length, the critical exponents, etc ., can be explicitly computed.

Two choices of boundary conditions will be particularly relevant: the one corresponding to $\eta \equiv 0$ (resp. $\eta \equiv 1$) on $\mathbb{Z}^d \setminus G$ will be referred to as the *free* (resp. *wired*) boundary condition and the corresponding random-cluster measure will be denoted by $\phi^0_{G,p,q}$ (resp. $\phi^1_{G,p,q}$). For these two choices of boundary conditions, it is classical — and it will be proved in the next chapter — that the corresponding measures can be extended to the whole $\mathbb{Z}^d$ when $q \geq 1$.

Lemma 1.1.2. *Let $d \geq 1$, $q \geq 1$ and $p \in [0,1]$. There exist two measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ on $\{0,1\}^{E(\mathbb{Z}^d)}$ such that for any event $\mathcal{A}$ depending on a finite number of edges,*

$$\lim_{n\to\infty} \phi^0_{\Lambda_n,p,q}[\mathcal{A}] = \phi^0_{p,q}[\mathcal{A}] \qquad \textit{and} \qquad \lim_{n\to\infty} \phi^1_{\Lambda_n,p,q}[\mathcal{A}] = \phi^1_{p,q}[\mathcal{A}].$$

Edwards–Sokal coupling

This section is devoted to the description of the so-called *Edwards–Sokal coupling*, named after Robert Edwards and Alan Sokal [55]. This coupling is very useful as it allows to relate correlations of the Potts model and probabilities of connectivity events in the random cluster model. The coupling can be described in different ways, starting either with a random-cluster configuration or a spin configuration. It also needs to be adapted to the boundary conditions in consideration. Let us describe all the versions of the coupling.

DESCRIPTION OF THE EDWARDS–SOKAL COUPLING

FROM THE RANDOM-CLUSTER TO THE POTTS MODEL WITH FREE BOUNDARY CONDITIONS. Let $q \geq 2$ be an integer, G be a fixed finite graph and let $\omega \sim \phi^0_{G,p,q}$. Let $U_{q,\mathcal{C}}$ be a family of independent uniform random variables on $\{1,\ldots,q\}$ indexed by the family $\mathcal{C}$ of open clusters of ω. We set σ to be the spin configuration constructed as follows. To each cluster $\mathcal{C}$ of ω, we assign the colour $U_{q,\mathcal{C}}$. Then $\sigma \sim \mu^{\mathrm{f}}_{G,\beta,q}$, with $\beta := -\log(1-p)$.

FROM THE RANDOM-CLUSTER TO THE POTTS MODEL WITH MONOCHROMATIC BOUNDARY CONDITIONS. Let $q \geq 2$ be an integer, $j \in \{1,\ldots,q\}$ a fixed colour, G a fixed finite graph and let $\omega \sim \phi^1_{G,p,q}$. We construct a spin configuration σ as described previously, except that every cluster intersecting the boundary of G is automatically assigned the colour j. Then, $\sigma \sim \mu^j_{G,\beta,q}$, with $\beta := -\log(1-p)$.

FROM THE POTTS MODEL TO THE RANDOM-CLUSTER WITH FREE BOUNDARY CONDITIONS. Let $q \geq 2$ be an integer , $\beta > 0$, and G be a fixed finite graph. Sample a spin configuration $\sigma \sim \mu^{\mathrm{f}}_{G,\beta,q}$. Set $p := 1 - \mathrm{e}^{-\beta}$. We construct a percolation configuration ω as follows: for any edge $e = \{xy\} \in E(G)$, set $\omega(e) = 0$ if $\sigma_x \neq \sigma_y$, and if $\sigma_x = \sigma_y$, open e with probability p and close e else. Then, $\omega \sim \phi^0_{G,p,q}$.

FROM THE POTTS MODEL TO THE RANDOM-CLUSTER WITH MONOCHROMATIC BOUNDARY CONDITIONS. Let $q \geq 2$ be an integer , $\beta > 0$, $j \in \{1,\ldots,q\}$ and G be a fixed finite graph. Sample a spin configuration $\sigma \sim \mu^j_{G,\beta,q}$ and construct ω as previously described. Then, $\omega \sim \phi^1_{G,p,q}$, with $p = 1 - \mathrm{e}^{-\beta}$.

Proof. We only prove the first and the third statement, as the other ones follow similarly. Say that a spin configuration σ and a percolation configuration ω are *compatible* if for any $\{xy\} \in E(G)$, $\omega_{\{xy\}} = 1 \Rightarrow \sigma_x = \sigma_y$. We denote by $\mathbf{P}$ the joint law of (ω,σ) in the previously described coupling. Our task is to show that its second marginal is the free q-Potts measure on G. Observe that $\mathbf{P}(\omega,\sigma) = 0$ if ω and σ are not compatible. If ω and σ are compatible then it is clear by definition of the coupling that

$$\mathbf{P}(\omega,\sigma) = \phi^0_{G,p,q}[\omega] q^{-k(\omega)} \propto p^{o(\omega)}(1-p)^{|E(G)|-o(\omega)}.$$

Fix a spin configuration σ, and introduce its set of disagreement edges $D(\sigma) = \{\{xy\} \in E(G), \sigma_x \neq \sigma_y\}$. Summing over the possible ω, we get

$$\mathbf{P}(\sigma) \propto \sum_{\omega} p^{o(\omega)}(1-p)^{|E(G)|-o(\omega)} \mathbb{1}_{\omega \text{ and } \sigma \text{ are compatible}}.$$

Now observe that the compatibility condition is restrictive only on the set $D(\sigma)$: if $e \notin D(\sigma)$, then the state of the edge e in ω does not affect the compatibility of ω and σ. Then,

$$\mathbf{P}(\sigma) \propto (1-p)^{|D(\sigma)|-|E|} \sum_{\tilde{\omega} \in \{0,1\}^{E(G) \setminus D(\sigma)}} p^{o(\tilde{\omega})}(1-p)^{|D(G)|-o(\tilde{\omega})}.$$

Observe that the sum appearing in the latter expression is equal to 1 (it is the sum of all the possible configurations under the Bernoulli percolation measure on $E(G) \setminus D(\sigma)$). Using the fact that $(1-p) = \mathrm{e}^{-\beta}$, we obtain that

$$\mathbf{P}(\sigma) \propto (1-p)^{|D(\sigma)|} = \exp(-\beta \sum_{x \sim y} \mathbb{1}_{\sigma_x \neq \sigma_y}) = \exp(|E(G)|) \exp(\beta \sum_{x \sim y} \mathbb{1}_{\sigma_x = \sigma_y}).$$

As $\exp(|E(G)|)$ does not depend on σ, we conclude that $\mathbf{P}(\sigma) \propto \exp(-\beta \mathcal{H}^0(\sigma))$. Hence, the marginal of $\mathbf{P}$ on σ is the q-states Potts measure.

Let us turn to the third statement. Fix a percolation configuration ω on G. Observe that

$$\begin{aligned}
\mathbf{P}(\omega) \propto & \sum_{\sigma \text{ compatible with } \omega} \exp(-\beta \mathcal{H}^0(\sigma)) p^{o(\omega)}(1-p)^{|E(G)|-|D(\sigma)|-o(\omega)} \\
&= \left(\frac{p}{1-p}\right)^{o(\omega)} \sum_{\sigma \text{ compatible with } \omega} \exp(\beta(|E| - |D(\sigma)|))(1-p)^{|E(G)|-|D(\sigma)|} \\
&= \left(\frac{p}{1-p}\right)^{o(\omega)} |\{\sigma, \sigma \text{ is compatible with } \omega\}|.
\end{aligned}$$

In the last line we used that $1-p = \mathrm{e}^{-\beta}$. Observe that a spin configuration σ is compatible with ω if and only if σ is constant on the open clusters of ω. Thus, $|\{\sigma, \sigma \text{ is compatible with } \omega\}| = q^{k(\omega)}$. Inserting this in the latter equation yields that $\mathbf{P}(\omega) \propto \left(\frac{p}{1-p}\right)^{o(\omega)} q^{k(\omega)}$, and thus that the law of ω is the random-cluster measure with parameters p and q. □

One of the following consequences of the Edwards–Sokal coupling is the following identity.

Lemma 1.1.3. *Let $d \geq 1$ be the dimension, $q \geq 2$ be an integer and $p \in (0,1), \beta \geq 0$ satisfying the relation $p = 1 - \mathrm{e}^{-\beta}$. Let G be a finite subgraph of $\mathbb{Z}^d$, and $x, y \in V(G)$. Then, for any fixed colour $j \in \{1, \ldots, q\}$,*

$$\mu^{\mathsf{f}}_{G,\beta,q}[\sigma_x = \sigma_y] - \tfrac{1}{q} = \tfrac{q-1}{q} \phi^0_{G,p,q}[x \longleftrightarrow y]$$

and

$$\mu^{j}_{G,\beta,q}[\sigma_x = j] - \tfrac{1}{q} = \tfrac{q-1}{q}\phi^{1}_{G,p,q}[x \longleftrightarrow \partial G].$$

Proof. The proof is almost immediate by definition of the coupling. We start by proving the first item by considering the coupling in which the Potts measure is obtained starting with a random-cluster configuration. Recall that we denote this coupling by $\mathbf{P}$. Observe that by definition of the coupling,

$$\begin{aligned}\mu^{\mathsf{f}}_{G,\beta,q}[\sigma_x = \sigma_y] &= \mathbf{P}[\sigma_x = \sigma_y, x \longleftrightarrow y] + \mathbf{P}[\sigma_x = \sigma_y, x \not\longleftrightarrow y] \\ &= \phi^0[x \longleftrightarrow y] + \tfrac{1}{q}\phi^0[x \not\longleftrightarrow y] \\ &= \tfrac{1}{q} + (1 - \tfrac{1}{q})\phi^0[x \longleftrightarrow y].\end{aligned}$$

We used the fact that if $\{x \longleftrightarrow y\}$ occurs in ω, then $\sigma_x = \sigma_y$, while if x and y are not connected in ω, then the spins σ_x and σ_y are independent and thus have a probability $\frac{1}{q}$ to be equal. The second item is proved similarly. □

Observe that this demonstrates that $\mu^{\mathsf{f}}_{G,\beta,q}[\sigma_x = \sigma_y] - \frac{1}{q} \geq 0$ and $\mu^{j}_{G,\beta,q}[\sigma_x = j] \geq \frac{1}{q}$. This follows very easily from the Edwards–Sokal coupling, but already is a non-trivial result for the Potts model, as it already demonstrates a weak form of positive association (see Chapter 2). Finally, observe that inserting $G = \Lambda_n$ and letting n tend to infinity in Lemma 1.1.3 yields the following result.

Lemma 1.1.4. *For any $q \geq 2$ and any choice of $p \in (0,1), \beta \geq 0$ such that $p = 1 - \mathrm{e}^{-\beta}$, any $x, y \in \mathbb{Z}^d$ and any colour $j \in \{1, \dots, q\}$,*

$$\mu^{\mathsf{f}}_{\beta,q}[\sigma_x = \sigma_y] - \tfrac{1}{q} = \tfrac{q-1}{q}\phi^{0}_{p,q}[x \longleftrightarrow y]$$

and

$$\mu^{j}_{\beta,q}[\sigma_x = j] - \tfrac{1}{q} = \tfrac{q-1}{q}\phi^{1}_{p,q}[x \longleftrightarrow \infty].$$

1.2 The phase separation problem

1.2.1 Historical overview

The phase separation problem is concerned with the following vague question: what happens to a physical system in which two or more phases are forced to coexist? Giving a precise mathematical meaning to that question is already a bit of a challenge, and we shall see that this question appears in several different contexts. Before diving into the history of the phase separation problem, we want to mention that only Chapter 3 is directly concerned with the phase separation problem. Chapters 4 and 5 are related to *Ornstein–Zernike theory*, which is a crucial tool in the study of the phase separation problem in the planar case. This is the reason we chose to unite these three separate works under the setting of the phase separation problem. In what follows, we choose to restrict ourselves to the planar case $d = 2$ (even though the Wulff theory has been proved to remain valid in dimensions 3 and higher).

1.2.2 The droplet at equilibrium: the Wulff construction

For the sake of the exposition, consider the Ising model[5] (the Potts model with two colours) on $\mathbb{Z}^2$ in the supercritical regime $\beta > \beta_c$. In what follows, we omit the subscript β as this parameter will be fixed in the sequel. In that case, it is standard to name the colours $+$ and $-$ instead of 1 and 2. We adopt that convention in the sequel and shall consequently name the monochromatic measures of the Ising model μ^+ and μ^-. In the supercritical regime, it is known that $\mu^+ \neq \mu^-$, and that $m^+ := \mu^+[\sigma_0] > 0$. Moreover, it is easy to prove that in the box Λ_n, the average number of "plus" spins is at first order $\frac{1+m^+}{2}n^2$. Fix $\varepsilon \in (0, \frac{1+m^+}{2})$. The phase separation problem in this setting can be rephrased in the following way.

Phase separation problem, version 1

Question: What are the geometrical properties of the measure

$$\mu^+_{n,\varepsilon} := \mu^+_{\Lambda_n}\Big[\ \cdot\ \Big| \#\{x \in \Lambda_n, \sigma_x = -\} > \Big(\tfrac{1-m^+}{2} + \varepsilon\Big) n^2\Big].$$

This problem received a lot of attention for more than a century. It is of high physical relevance, as its answer allows to understand the behaviour of two phases in equilibrium (one may think of a crystal at the equilibrium point with its vapour for instance, but also of the condensation problem described at the very beginning of the dissertation). In the measure previously described, we condition on very atypical event — a large deviation event for the average magnetization. The question is to understand the geometry of a typical configuration sampled under $\mu^+_{n,\varepsilon}$: how do the excessive "minus" spins occupy the space within the box? *A priori*, the reader might think about 3 possible scenarios (even more if they are creative enough!):

- **The sparse scenario.** The $-$ spins are distributed approximately uniformly within the box Λ_n.
- **The "small islands" scenario.** The $-$ spins aggregate in a large number of droplets, which are approximately of the same size, tending to infinity with a rate $o(n^2)$.
- **The "giant island" scenario.** The $-$ spins aggregate in one "giant" droplet, the volume of which is of order αn^2 with $\alpha > 0$.

Surprisingly, it turns out that for any $\varepsilon \in (0, \frac{1+m^+}{2})$, the right answer is the third scenario. The $-$ spins aggregate, forming a droplet of macroscopic size. Let us define the collection of *contours* of a spin configuration σ as the set of edges of the box $(1/2, 1/2) + \Lambda_n$ such that the two vertices of $\mathbb{Z}^2$ bordering those edges carry a spin of opposite sign: it is easy to see that when σ is sampled according to a monochromatic measure, the collection of contours

[5]The following discussion should not be specific to the choice $q = 2$. We chose to only mention the case of the Ising model as it is by far the most studied setting. Moreover, the results are slightly more natural to formulate in this setting

forms a family of non-intersecting closed loops in the plane. In total rigour, for the collection of contours to be non-intersecting, one needs to fix a so-called *splitting rule* such as the one graphically described in Figure 1.2. However, we do not pay too much attention to the precise definition of the contour model, as we will not work with it in the dissertation.

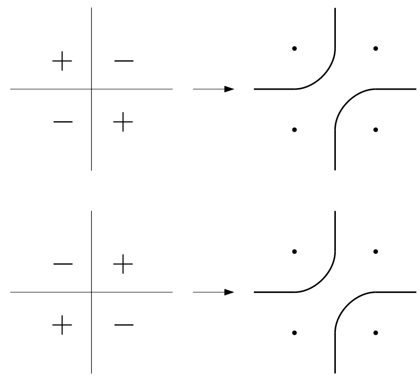

Figure 1.2: An example of a splitting rule.

For any $c_1, c_2 > 0$ and any closed curve γ in $[0,1]\times[0,1]$, let us call $\mathcal{E}(\gamma, c_1, c_2)$ the following event (d_{H} denotes the Haussdorf distance between compact subsets of $\mathbb{R}^2$):

- There exists a contour γ_0 such that

$$\min_{x\in\Lambda_n} \mathrm{d}_{\mathsf{H}}(n^{-1}\gamma_0, x+\gamma) \leq c_1 n^{-1/4}(\log n)^{1/2}.$$

- All the other contours have diameter smaller than $c_2 \log n$.

In words, the event $\mathcal{E}(\gamma, c_1, c_2)$ corresponds to the fact that there exists a unique "large" contour γ_0 which is very close to the curve γ (the distance between γ_0 and γ being quantified by the constants c_1 and c_2). Then, we have the following result, coming from [47]

Theorem 1.2.1. *For any $\varepsilon \in (0, \frac{1+m^+}{2})$ and $\beta > \beta_c$, there exists a (deterministic) closed curve Γ in $[0,1]\times[0,1]$ and two constants $c_1, c_2 > 0$ such that:*

$$\lim_{n\to+\infty} \mu^+_{n,\varepsilon}[\mathcal{E}(\Gamma, c_1, c_2)] = 1.$$

This theorem has a long and rich history. It was first proved by Dobrushin, Kotecký and Shlosman in the very low temperature regime (*i.e.*, $\beta \gg \beta_c$) in the influential work [47]. Later on, the result was extended by Ioffe and Schonmann to *any* $\beta > \beta_c$ [85]. We also refer to the very complete survey of the problem of [15]. It paves the way to a vast number of very delicate and interesting questions, among which are the following:

1. What can be said about the curve Γ? What are its regularity properties, and how can one compute it as a function of the microscopic model?

2. What are the properties of the random contour γ_0 beyond its global shape given by Γ? In particular, how does this random interface fluctuates around Γ, can interesting limit theorems can be obtained at a finer level than the macroscopic one?

While the first question has been totally solved, the second one remains mostly open. One of the goals of this dissertation is to provide some results and techniques in that direction. We first explain how the first question can be answered; this section is mostly review and discussion about the history of the problem. The interested reader may consult [15] for further explanation and proofs.

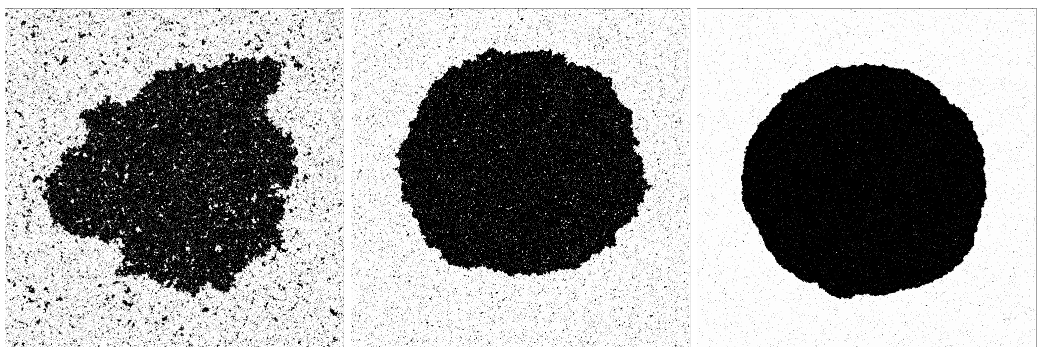

Figure 1.3: Three simulations of the Wulff droplet in the case of the Ising model, at different values of $\beta > \beta_c$, increasing from left to right. Pictures used by courtesy of R. Cerf.

The purpose of what follows is a very broad discussion of the construction of the curve Γ. Of course, it is hopeless to summarize a whole theory in a few pages; we only introduce the main protagonists and do not give any proofs at all, as this dissertation is concerned with the *fluctuations* of the phase separation line, and does not formally require the existence of Γ.

The surface tension

A crucial quantity for the investigation of the properties of the curve Γ is the so-called *surface tension*. Informally, it represents the exponential cost for the interface to move in a given direction. As the model — at the microscopic level — is defined on a lattice, it is not isotropic. Thus, the “cost” of a given microscopic portion of interface is very dependent on the direction of its displacement. We start by defining the surface tension.

Let $\mathbf{n} \in \mathbb{R}^2$, with $\|\mathbf{n}\| = 1$. Define the $\mathbf{n}$-tilted Dobrushin boundary conditions as follows

$$\eta_x^{\mathbf{n}} = \begin{cases} +1 & \text{if } \langle x, \mathbf{n} \rangle \geq 0 \\ -1 & \text{if } \langle x, \mathbf{n} \rangle < 0 \end{cases}.$$

As a side note, when $\mathbf{n} = (0,1)$ we will refer to $\eta^{\mathbf{n}}$ as the *standard Dobrushin boundary* *condition* and denote it by $\pm$.

The surface tension is classically defined as the following limit:

$$\tau(\mathbf{n}) = \lim_{n\to+\infty} -\frac{1}{n}\log\frac{Z^{\eta^{\mathbf{n}}}_{\Lambda_n,\beta}}{Z^{+}_{\Lambda_n,\beta}}. \tag{1.1}$$

The proposition gathering the properties of the surface tension is the following:

Proposition 1.2.2. *The limit in* (1.1) *exists and is positive as long as* $\beta > \beta_c$*. Moreover its homogeneous extension of order 1 defines a norm in* $\mathbb{R}^d$ *(in particular it is a convex function).*

A proof at a greater level of generality can be found in [96]. As the surface tension $\tau(\mathbf{n})$ represents the local cost for the interface to move in a direction orthogonal to $\mathbf{n}$, it is natural — at least on a broad heuristic level — to think that the curve Γ is the minimizer of the integral of the function τ along the curve. This motivates us to introduce the following *Wulff functional*: for a closed, locally $\mathcal{C}^1$ curve γ, define

$$\mathcal{W}(\gamma) = \int_\gamma \tau(\mathbf{n}_x)\mathrm{d}x,$$

where $\mathbf{n}_x$ is the unique unit vector normal to the tangent line to γ at x. The Wulff variational problem corresponds to finding the minimiser of the functional $\mathcal{W}$, subject to a volume constraint. It is often noted:

$$\mathcal{W}(\gamma) \to \min, \qquad \text{with the constraint} \qquad \mathrm{Vol}(\gamma) = \tfrac{\varepsilon}{m^+}. \tag{Wulff}$$

In other words, one seeks for a curve minimising the functional $\mathcal{W}$ under a volume constraint. As we aim for a presentation at a very heuristic level, we do not specify to which set of closed curves we restrict this minimisation problem; the reader should keep in mind that they need to be regular enough for the tangent normal vector to be defined at each point. A whole theory has been developed to study problems such as (Wulff). Let us state its principal output.

Theorem. Let $\mathsf{W} := \{x \in \mathbb{R}^2, \langle x, \mathbf{n}\rangle \le \tau(\mathbf{n}), \forall \mathbf{n} \in \mathbb{S}^1\}$. Then, there exists a unique $\lambda^*_\varepsilon > 0$ such that the curve $\lambda^*_\varepsilon\partial\mathsf{W}$ solves (Wulff). Moreover, it is the unique solution of (Wulff).

The heuristic reasoning presented previously is then an argument in favour of the identification of the "macroscopic profile of the giant contour" Γ with $\lambda^*_\varepsilon\partial\mathsf{W}$. This is indeed the case and is the content of the celebrated Dobrushin–Kotecký–Shlosman (DKS) theory, later on extended all the way down to the critical temperature by Ioffe and Schonmann. Recall the notation of Theorem 1.2.1.

Theorem ([47, 85])**.** For any $\beta > \beta_c$, any $\varepsilon \in (0, \frac{1+m^+}{2})$,

$$\Gamma = \lambda^*_\varepsilon\partial\mathsf{W}$$

as curves in the plane. In particular for any $\varepsilon' > 0$,

$$\mu^{+}_{n,\varepsilon}[\min_{x\in\Lambda_n} \mathrm{d}_{\mathsf{H}}(n^{-1}\gamma_0, x + \lambda^*_\varepsilon \partial \mathsf{W}) > \varepsilon'] \xrightarrow[n\to\infty]{} 0.$$

In other words, there is a phenomenon of *condensation*: as predicted by the classical reasoning in physics, the droplet tends to adopt the shape that minimises its surface energy. The mathematical derivation of that well-known fact is a spectacular *tour de force*, and essentially solves the phase separation problem in two dimensions[6] for the macroscopic profile of the droplet.

Regarding the *fluctuations* of the random contour γ_0, very low temperature results were derived in [46, 81], but their extension to the whole subcritical regime remained out of reach, until the apparition of a new tool that we shall discuss extensively in this dissertation: the so-called *Ornstein–Zernike theory*.

1.3 Ornstein–Zernike theory

Consider the random-cluster model with parameters $q \geq 1$ and$0 < p < p_c(q)$. In this regime, we already know that there is a unique infinite volume random-cluster measure, that we call ϕ (for convenience we drop p and q from the notation when there is no ambiguity). In this setting, the cluster of 0 is almost surely finite. Moreover, sharpness results actually tell us that its volume decays exponentially fast (see Chapter 2). We are interested in the following two questions:

- **Question 1**: Beyond its exponential decay, what is the asymptotic behaviour of $\phi[x \longleftrightarrow y]$ when $\|x - y\| \to \infty$?
- **Question 2**: What are the geometric properties of the measure $\phi[\,\cdot\,|x \longleftrightarrow y]$, when $\|x - y\| \to \infty$?

Even though these questions — at least at first sight — seem unrelated to the previous section, there is actually a very explicit connection between them and the phase separation problem *in the planar case*.

1.3.1 Ornstein–Zernike theory and the phase separation problem

Consider the following problem, which is a milder version of the phase separation problem[7].

[6]In three dimensions, the rigorous Wulff construction could also be implemented by R. Cerf and A. Pizstora in a very influential series of papers [31, 32, 33].

[7]As previously mentioned, the discussion here is not restricted to the Ising case. The same exact reasoning could be done with the planar Potts model with q colours, replacing the Dobrushin boundary conditions $\pm$ by j_1/j_2, where $j_1 \neq j_2$ are two arbitrary colours of $\{1, \ldots, q\}$.

PHASE SEPARATION PROBLEM, VERSION 2

Consider the Ising model at a supercritical temperature $\beta > \beta_c$ in the planar box $\Lambda_n \subset \mathbb{Z}^2$. Equip the box with standard Dobrushin boundary conditions $\pm$. What are the properties of the interface between the $+$ phase and the $-$ phase?

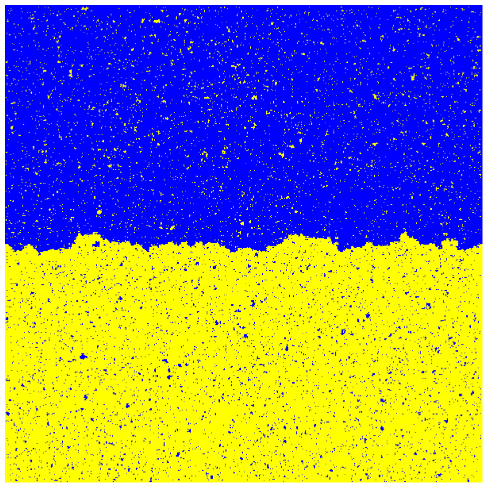

Figure 1.4: A simulation of an Ising configuration in a finite volume with Dobrushin boundary conditions at $\beta > \beta_c$. Picture used by courtesy of S. Ott.

For describing the connection between the two questions, we first need to give a precise definition of what we call the *interface*. Several choices can be made; in the planar case, it turns out that every "reasonable" choice of definition yields the same type of result[8]. For convenience, we choose the one that is the most adapted to our setting.

Definition 1.3.1. Call ω^+ the $+$ cluster of the upper boundary of the box Λ_n. Formally, let Λ_n^+ (resp Λ_n^-) be the intersection of Λ_n with the hyperplane $\{x \in \mathbb{R}^2, \langle x, \vec{e}_1 \rangle \geq 0\}$ (resp the hyperplane $\{x \in \mathbb{R}^2, \langle x, \vec{e}_1 \rangle < 0\}$ and

$$\omega^+ := \{x \in \Lambda_n, \exists \text{ a nearest neigbour path}(\gamma_i)_{0\leq i\leq N} \text{ with } \gamma_0 = x, \dots, \gamma_N \in \partial\Lambda_n^+, \forall 1 \leq i \leq N, \sigma_{\gamma_i} = +\}.$$

Similarly, call ω^- the $-$ cluster of the lower boundary. We define the top interface γ^+ (resp. the bottom interface γ^-) to be the unique connected portion of the dual boundary of ω^+ (resp ω^-) that intersects $(\partial\Lambda_n)^*$.

[8]This fact is no longer true in dimension 3 and more, in which the definition of the interface is unclear. For example, our definition would not be the right one, as Aizenman and Lebowitz proved in [3] that in some regime of the half-line $\beta > \beta_c$, the $+$ spins might percolate in the $-$ phase in high dimension, and mentioned that the result should hold for any $d \geq 3$.

Proposition 1.3.2. *Set $p = \frac{q\mathrm{e}^{-\beta}}{1+(q-1)\mathrm{e}^{-\beta}}$. Then, the random variable (γ^-, γ^+) has the same asymptotic joint distribution (when n tends to infinity) as the bottom dual boundary and the top dual boundary of the cluster of $(-n, 0)$ under the law $\phi_{\Lambda_n,p,q}[\ \cdot \mid (-n,0) \longleftrightarrow (n,0)]$.*

Proof. The proof relies on two inputs: the Edwards–Sokal coupling and the duality of planar random-cluster models. As formal duality is introduced only in Chapter 2, we remain elliptical at this stage of the dissertation. Consider the Edwards–Sokal coupling and observe that the measure induced by the Dobrushin boundary conditions is a supercritical random measure with wired boundary conditions both on top and on the bottom of the box, conditioned on the following event: $\{\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset\}$, where $\mathcal{C}^+$ (resp $\mathcal{C}^-$) is the open cluster of Λ_n^+ (resp. Λ_n^-). Now use duality to map the dual measure on a subcritical measure and observe that this measure is supported on the set of configurations in which there exists an open cluster in Λ containing both $(-n/2, 0)$ and $(n/2, 0)$. The precise value of the parameter p is computed using the Edwards–Sokal formula together with the duality formula (2.4). □

This setting is slightly simpler than the Wulff setting introduced in the former section: indeed in that case, the convexity of the surface tension implies that the system of interfaces (γ^-, γ^+) converges to the line segment $[-1/2, 1/2]$ when rescaled by n, as the size of the box goes to infinity. Ornstein–Zernike theory will then be useful to understand the *fluctuations* of the phase separation interface around the limit shape.

Before diving into the history of the theory, let us state its two most fundamental outputs.

Theorem (Ornstein–Zernike theorem)**.** Let $q \geq 1$ and $0 < p < p_c(q)$ be a pair of subcritical parameters. Let $d \geq 2$ be the ambient dimension. Then there exist two analytic functions $\tau, \Psi : \mathbb{S}^{d-1} \to \mathbb{R}^+$, bounded away from 0, such that for any $x \in \mathbb{S}^{d-1}$, as $n \to \infty$,

$$\phi[0 \longleftrightarrow \lfloor nx \rfloor] = \frac{\Psi(x)}{n^{\frac{d-1}{2}}} \mathrm{e}^{-n\tau(x)}(1 + o(1)), \tag{1.2}$$

where $\lfloor nx \rfloor$ denotes the point of the lattice that is the closest to nx.

A second important output of the Ornstein–Zernike approach is the following result, describing the scaling limit of a long subcritical cluster. For that we need to introduce the object we are taking the scaling limit of. Several definitions are possible; once again, we stick to the simplest of them. Consider the measure $\phi[\ \cdot \mid 0 \longleftrightarrow (n,0)]$. In that case, for any $0 \leq k \leq n$, the cluster of 0 — called $\mathcal{C}_0$ hereafter — intersects the hyperplane $\{k\} \times \mathbb{Z}^{d-1}$ at least once. Call Γ_k^+ (resp. Γ_k^-) the maximal (resp. minimal) $x \in \{k\} \times \mathbb{Z}^{d-1} \cap \mathcal{C}_0$ for the lexicographical order. Moreover, we extend $(\Gamma_k^+)_k$ and $(\Gamma_k^-)_k$ as piecewise linear functions Γ^+ and Γ^- from $[0, n]$ to $\mathbb{Z}^d$ satisfying $\Gamma_k^+ = \Gamma^+(k)$ and $\Gamma_k^- = \Gamma^-(k)$ We are ready to formulate the second Ornstein–Zernike result.

Theorem. Fix a set of subcritical parameters $q \geq 1$ and $0 < p < p_c(q)$. There exists a $\sigma > 0$ such that under the family of measures $\phi[\ \cdot \mid 0 \longleftrightarrow (n,0)]$, the following convergence occurs in distribution when $n \to \infty$:

$$\left(\frac{1}{\sqrt{n}}\Gamma^+(nt)\right)_{0\le t\le 1} \Rightarrow (\mathsf{BB}^\sigma_t)_{0\le t\le 1},$$

where BB^σ denotes the Brownian bridge of variance σ. Moreover. with probability tending to 1,

$$\sup_{t\in[0,1]} |\Gamma^+(t) - \Gamma^-(t)| \le o(\log^2 n),$$

which implies that Γ^+ and Γ^- converge towards the *same* Brownian bridge after diffusive scaling.

Remark 1.3.3 (Absence of roughening transition in dimension 2)**.** In dimension 2, this result has an interesting consequence: it proves the absence of so-called *roughening transition*. We take this remark as an opportunity to state this fascinating conjecture. Consider the supercritical ($\beta > \beta_c$) Ising model in dimension $d \ge 2$, in the finite volume Λ_n with $\pm$ boundary conditions. An important question in statistical mechanics concerns the behaviour of the interface[9] separating the $+$ and $-$ phases. With this level of vagueness in the definitions, call $\phi(0)$ the height (in the transverse direction) of the interface at 0. We say that the interface is *localised* if $\mathrm{Var}[\phi(0)]$ is uniformly bounded as $n \to \infty$, and *delocalised* if it tends to infinity. To determine whether the interface is delocalised or not is a very important and difficult question, and the following behaviour is expected:

- In dimension $d = 2$, the interface should be delocalised as soon as $\beta > \beta_c$
- In dimension $d = 3$, it is expected that there exists a $\beta_r \in (\beta_c, \infty)$ such that the interface is delocalised when $\beta \in (\beta_c, \beta_r)$ and becomes localised when $\beta > \beta_r$. Moreover, the transition at β_r is expected to be of BKT type (the free energy of the model being smooth but non-analytic at β_r). This conjectural transition is called the *roughening transition*, and rigorously proving its existence remains one of the main challenges of modern Statistical mechanics.
- In dimension $d \ge 4$, the interface is expected to be localised as soon as $\beta > \beta_c$. This was proved for the so–called Discrete Gaussian model in [66].

For a more complete exposition of these questions, we refer to the very complete review [110]. Observe that Proposition 1.3.2 together with the previous Theorem asserting that the interface has Gaussian fluctuations for any $p < p_c$ rules out the possibility of a roughening transition in dimension 2: indeed, as long as $\beta > \beta_c$, the interface has transverse fluctuations of order $\sqrt{n}$. In that case, much more is known, as Ioffe and Greenberg proved that when suitably rescaled, for any $\beta > \beta_c$, the interface converges towards a Brownian bridge [68].

[9] Once again, we do not define rigorously the terminology *interface*. Unlike in dimension 2, the definition of this object requires some attention, as mentioned in the footnote preceding Proposition 1.3.2. Proposing an accurate definition of the object is one part — though probably the easiest — of the proof of the roughening conjecture.

1.3.2 The historical approach

The asymptotic given by (1.2) was first conjectured in two very influential works by Ornstein and Zernike in 1914 [101], and Zernike [114]. Trying to correct a formula describing the phenomenon of *opalescence* in a crystal, they provided a heuristic in favour of the formula (1.2). Of course, the formula was not stated in the setting of the random-cluster model, as the latter was introduced around 1970 by Cees Fortuin and Piet Kasteleyn [60], but rather in terms of deviations from the average density of molecules in a crystal. We now try to discuss their approach in the modern mathematical language. For proving that a certain function G displays the behaviour given by (1.2) (in the case of the random-cluster model for instance, $G(x) = \phi[0 \longleftrightarrow x]$), Ornstein and Zernike assumed the existence of another function $f : \mathbb{Z}^d \to \mathbb{R}^+$ (that they referred to as the *direct influence function*), such that the three following conditions hold

(i) The following limit exists in $\mathbb{R}^+ \setminus \{0\}$:

$$\lim_{|x| \to +\infty} -\frac{1}{x} \log G(x).$$

(ii) The following renewal equation holds:

$$G(x) = f(x) + \sum_{y \in \mathbb{Z}^d} f(y) G(x-y).$$

(iii) There exists $\varepsilon > 0$ such that for any $x \in \mathbb{Z}^d$,

$$f(x) \leq \mathrm{e}^{-\varepsilon \|x\|} G(x).$$

Property (iii) is known as the *mass gap* condition and is very important as it plays a role in all the approaches to Ornstein–Zernike theory (though under slightly different forms). The idea of [114] roughly follows the next lines. Introducing the Fourier transforms $\hat{G}(k) := \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{\langle k, x \rangle} G(x)$ and $\hat{f}(k) := \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{\langle k, x \rangle} f(x)$, it follows from the renewal equation that one has the following identity:

$$\hat{G}(k) = \frac{\hat{f}(k)}{1 - \hat{f}(k)}.$$

The next step is to carefully Taylor expand $\hat{f}$ around 0, and show that there exist three constants $A, B, C > 0$ (property (iii) is important for proving the positivity of B) such that:

$$\hat{G}(k) = \frac{A}{B + C|k|^2 + o(|k|^3)} (1 + o(1)).$$

The decay announced in (1.2) then follows by the application of an adequate Tauberian Theorem.

Later on, Abraham and Kunz [2] and Paes-Leme [104] were able to derive independently the first rigorous implementation of Ornstein and Zernike's reasoning, by constructing the function f explicitly in the case of classical lattice gases theory, by means of a graphical representation of the partition function of the model. The idea of using graphical representations to derive Ornstein–Zernike behaviour of the correlations of off-critical spin models has then become a central idea in the modern and rigorous developments of Ornstein–Zernike theory.

1.3.3 The modern approach

Later on, the Ornstein–Zernike result for the order of decay of the correlations has been shown to be true for a number of models *in a perturbative regime* (namely very far away from criticality; in the case of the Ising model, it would correspond to considering $\beta \gg \beta_c$), see [17, 97]. Then a rigorous derivation of the Ornstein–Zernike asymptotic result was done in the case of the self-avoiding walk along a direction given by the axis in [34] and later on in any direction in [82]. For percolation models, the case of Bernoulli percolation was first treated for an on-axis direction in [21] and later on in any direction in [22]. The case of subcritical Ising models was treated in [24] via the random-line graphical representation of the two-point correlation function. Finally, the analysis was carried out for all the subcritical random-cluster measures in [25]. In recent developments, the theory has been extended to Ising models with long-range interactions [10], and a new direction of research has been studied regarding the *failure* of Ornstein–Zernike behaviour in some long-range Ising models, when the coupling constants decay too slowly [7]. In what follows, we try to streamline the argument of [25] and insist on the difficulties of the proof. The interested reader might compare this approach and the content of Chapter 5, where we develop a slightly different approach of the theory that allows to treat the near-critical regime of the model.

We first recall the main statement of [25].

Theorem (Ornstein–Zernike theorem)**.** Let $q \geq 1$ and $0 < p < p_c(q)$ be a pair of subcritical parameters. Let $d \geq 2$ be the ambient dimension. Then there exist two analytic functions $\tau, \Psi : \mathbb{S}^{d-1} \to \mathbb{R}^+$ bounded away from 0 such that for any $x \in \mathbb{S}^1$, as $n \to \infty$,

$$\phi[0 \longleftrightarrow \lfloor nx \rfloor] = \frac{\Psi(x)}{n^{\frac{d-1}{2}}} \mathrm{e}^{-n\tau(x)}(1 + o(1)),$$

where $\lfloor nx \rfloor$ denotes the point of the lattice that is the closest to nx.

Unlike the first approach of the Ornstein–Zernike theorem consisting in the abstract analysis of the poles of the Fourier transform of the function $\phi[0 \longleftrightarrow \lfloor nx \rfloor]$, the "modern" proof of this statement consists in an explicit coupling between a system of carefully selected vertices of a long percolation cluster and the trajectory of a so-called "directed random walk" on $\mathbb{Z}^d$. The latter having the same asymptotic behaviour as a regular random walk on $\mathbb{Z}^{d-1}$, (1.2) follows classically from a local limit theorem. We now streamline the principal steps of the analysis.

Step 1: Coarse-graining of the cluster.

The first step consists in reducing the microscopical geometrical complexity of a cluster containing both 0 and $\lfloor nx \rfloor$ by coarse-graining it. Fix a scale $K \geq 0$, which will be chosen later on. For any $x \in \mathbb{Z}^d$, the following limit exists and is positive

$$\tau(x) := \lim_{n\to\infty} -\frac{1}{n} \log \phi[0 \longleftrightarrow \lfloor nx \rfloor]. \tag{1.3}$$

At this stage of the dissertation, we do not prove the statement as it will be a consequence of the results of Chapter 2. For any $y \in \mathbb{Z}^d$, define $B_K(y) = y + \{x \in \mathbb{Z}^d, \tau(x) \leq K\}$. The following algorithm, described in [111], allows to extract a directed tree-like structure from a cluster $\mathcal{C}$, at scale K. Fix an arbitrary order on the vertices of $\mathbb{Z}^d$, and consider the following algorithm, directly extracted from [111].

Algorithm 1: Extraction of the skeleton of a cluster

Data: A percolation cluster $\mathcal{C}$ containing 0 and x
Result: A sequence of vertices $x_0, \dots, x_N$
$x_0 \leftarrow 0$;
$A \leftarrow B_K(0)$;
$n \leftarrow 1$;
while *there exists some* $y \in \partial A$ *such that* $y \overset{\mathcal{C}\setminus A}{\longleftrightarrow} \partial B_K(y)$ **do**
  Let x_n be the minimal such vertex ;
  $A \leftarrow A \cup B_K(x_i n$;
  $n \leftarrow n + 1$
end

This sequence of vertices can be transformed in a tree by the following procedure: add a vertex between x_k and $\min\{x_j, j \in \{0, \dots, k-1\}, x_k \in \partial B_K(x_j)\}$. The tree $\mathfrak{T}$ shall be called the *skeleton* of the cluster $\mathcal{C}$. It is easy to check that its output is a tree, and that $\mathcal{C} \subset \bigcup_{0\leq k\leq N} B_{2K}(x_k)$. In that sense, the skeleton provides an approximation of the cluster at scale K. Of course, its geometry is much simpler than the one of the original cluster, as it contains no cycles and is relatively insensitive to the microscopic geometry of $\mathcal{C}$.

Step 2: Entropy/energy analysis of the geometry of the skeleton

The interest of the above-mentioned procedure lies in the following fact: when the scale K is chosen to be sufficiently large, the competition between entropy (the number of possible skeletons) and energy (the probabilistic cost of one given skeleton) is won by the energy term for any $p < p_c$. Indeed, fix some $N \in \mathbb{N}$ to be the number of vertices of the skeleton, and fix a skeleton $\mathfrak{T}$ with N vertices. On the one hand, it is easy, thanks to the FKG inequality (see

Chapter 2) and by definition of the balls B_K, to show that, as $K \to \infty$,

$$\sum_{C \sim \mathfrak{T}} \phi[\mathcal{C}_0 = C] \le \prod_{i=0}^{N} \phi^1_{B_K(x_i)}[x_i \longleftrightarrow \partial B_K(x_i)] \le \exp(-KN(1 + o_K(1))).$$

On the other hand, it is an easy combinatorial fact that the number of possible trees with branching number at most CK^{d-1} with N vertices is bounded above by $(CK^{d-1})^N$.

Hence, the energy bound is exponential in K while the entropy bound is exponential in $\log K$. As K grows, the geometry of a typical skeleton will be more and more regular due to that observation. Let us be more precise about the last sentence. Fix $\delta > 0$. For $j \in \{1, \ldots, N\}$, say that x_j is a *cone point* of the skeleton $\{x_0, \ldots, x_N\}$ if the set $\{x_0, \ldots x_{j-1}\}$ is contained into the "backward cone" $\mathcal{Y}^{\mathsf{b}}(x_j) := x_j + \{y \in \mathbb{Z}^d, \langle x, y\rangle \le -\delta\|y\|\}$, and if the set $\{x_{j+1}, \ldots, x_N\}$ is contained inside the "forward cone" $\mathcal{Y}^{\mathsf{f}}(x_j) := x_j + \{y \in \mathbb{Z}^d, \langle x, y\rangle \ge \delta\|y\|\}$. In particular, some more refined analysis (though in the fashion of the energy/entropy argument presented just above) allows the authors to derive the following result:

Proposition 1.3.4. *There exists* $\delta > 0$ *and* $c_1, c_2 > 0$ *such that*

$$\phi[\#\{k \ge 0, x_k \textit{ is a cone point of the skeleton of } \mathcal{C}\} < c_1\|x\| \mid 0 \longleftrightarrow x] \le \mathrm{e}^{-c_2\|x\|}.$$

The skeleton of a cluster is thus quite regular as it is contained in the following "diamond envelope":

$$\bigcup_{x_i \text{ cone-points of } \mathfrak{T}} \mathcal{Y}^{\mathsf{f}}(x_i) \cap \mathcal{Y}^{\mathsf{b}}(x_{i+1}).$$

Moreover, there is a density of such cone-points.

Step 3: From the geometry of the skeleton to the geometry of the cluster via surgery

We want to go back from the behaviour of the skeleton to the behaviour of the cluster under the conditional measure $\phi[\cdot|0 \longleftrightarrow x]$. At this stage, one knows two crucial facts about it. First, the skeleton of a typical cluster has a density of cone points. Second, the cluster is contained into $\bigcup_{i=1}^N B_{2K}(x_i)$. Now observe that, up to increasing δ, this proves that the cluster is contained in the slightly fattened diamond enveloppe:

$$\mathcal{C} \subset \bigcup_{x_i \text{ cone-points of } \mathfrak{T}} (\mathcal{Y}^{\mathsf{f}}(x_i) \cap \mathcal{Y}^{\mathsf{b}}(x_{i+1})) \cup B_K(x_i).$$

If x_i is a cone point of $\mathfrak{T}(\mathcal{C})$, the finite energy property implies that — conditionally on the event $\{0 \longleftrightarrow x\}$ — one can perform a local surgery and close all the edges of $B_K(x_i) \setminus (\mathcal{Y}^{\mathsf{b}}(x_i) \cup \mathcal{Y}^{\mathsf{f}}(x_i))$ to transform that cone point of the skeleton into a cone point *for the cluster* by paying a probabilistic cost that is at most of order $\exp(-K^d)$, uniformly in the boundary conditions induced by the cluster $\mathcal{C}$. As the scale K is now fixed, this demonstrates the following:

Proposition 1.3.5. *There exists $\delta > 0$ and $c_1, c_2 > 0$ such that*

$$\phi[\#\{k \geq 0, x_k \text{ is a cone point of } \mathcal{C}\} < c_1\|x\| \mid 0 \longleftrightarrow x] \leq \mathrm{e}^{-c_2\|x\|}.$$

Step 4: Coupling with the effective random walk

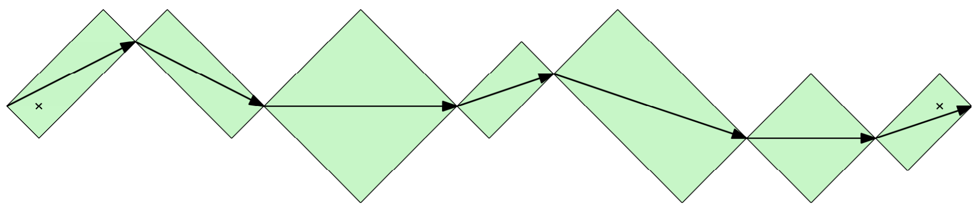

Figure 1.5: The effective random walk. The percolation cluster is contained in the shaded green "diamond enveloppe".

We now make the assumption that $x = n\vec{e_1}$ and shall write $\tau := \tau(\vec{e}_1)$. This is not too much of a simplification, as the reader might check that the analysis is essentially the same in any other given direction. The latter Proposition suggests a convenient decomposition of a cluster into irreducible pieces: due to the linear number of cone points, one might see a typical cluster as a succession of little "sausages" contained in diamonds. This motivates to resum over those sausages and to only take into account their displacement, as the sum of these displacements will already provide a very accurate description of the cluster shape. Thus, say that a connected subgraph $G \subset \mathbb{Z}^d$ is *irreducible* if

1. There exists $x_L, x_R \in V(G)$ such that

$$G \subset \mathcal{Y}^{\mathsf{f}}(x_L) \cap \mathcal{Y}^{\mathsf{b}}(x_R),$$

2. G does not have any cone points other than x_L and x_R.

For such an irreducible G, define its displacement by $X(G) = x_R - x_L$ (observe that $\langle X(G), \vec{e_1}\rangle > 0$).

Observe that a given cluster containing 0 and $n\vec{e_1}$ admits a unique decomposition into irreducible subpieces (in some sense, one might be tempted towards an analogy with the primes factors of a natural number), and that this decomposition has been previously proved to contain a linear number of pieces. We now introduce the following distribution on $\mathbb{N}^* \times \mathbb{Z}^{d-1}$:

$$\mathbf{P}[y] := \sum_{\substack{G \text{ irreducible} \\ X(G)=y}} \mathrm{e}^{\tau\langle y, \vec{e_1}\rangle} \phi[\mathcal{C}_0 = G].$$

Assume for a minute that we are in the case of Bernoulli percolation (*i.e.*, $q = 1$ in that case). By independence it would be true that for any cluster C containing 0 and $n\vec{e}_1$, one has the following identity[10]

$$\mathrm{e}^{\tau n}\phi[\mathcal{C}_0 = C] \approx \prod_{i=1}^{N} \mathbf{P}[C_i], \tag{1.4}$$

where $(C_1, \dots, C_N)$ is the unique irreducible decomposition of the cluster C. This identity in law provides us with a coupling between the set of the cone points of the cluster and the trajectory of a directed random walk.

The crux observations are the following:

1. $\mathbf{P}$ is a probability measure on $\mathbb{N}^* \times \mathbb{Z}^{d-1}$.
2. $\mathbf{P}$ has exponential tails.

The second item comes from all the reasoning of steps 1 – 3. For the first one, it comes from the renewal structure of the cluster. Indeed, observe that introducing the following generating series,

$$G(t) = \sum_{n\geq 1} t^n \mathrm{e}^{\tau n}\phi[n\vec{e}_1 \in \mathcal{C}_0] \qquad \text{and} \qquad K(t) = \sum_{n\geq 1} t^n \sum_{y\in\mathbb{Z}^d, \langle y, \vec{e}_1\rangle = n} \mathbf{P}(y).$$

Then, equation (1.4) yields

$$G(t) = \frac{K(t)}{1 - K(t)} + R(t),$$

where the remainder R takes into account the clusters that are irreducible all the way to the point $n\vec{e}_1$ (by the exponential tails of $\mathbf{P}$ however, its radius of convergence is larger than 1, actually it is trivial that $R(t) \leq K(t)$, which is sufficient to conclude). Now, the radius of convergence of G is 1 by definition of τ, and the radius of convergence of H is strictly larger than 1 by the exponential tails of the measure $\mathbf{P}$. This demonstrates that $K(1) = 1$, which means that $\mathbf{P}$ is indeed a probability measure.

Now, sample independent and identically distributed random variables $(X_n)_{n\geq 1}$ with law $\mathbf{P}$. Their sum $S_n := X_1 + \cdots + X_n$ will be called a *directed random walk* on $\mathbb{Z}^d$. It is actually more accurate to see it as the trajectory of a $(d-1)$-dimensional random walk with a random time reference given by the variables $\langle X_i, \vec{e_1}\rangle$. The idea is now to compare the law of this object conditioned to eventually reach the vertex $n\vec{e_1}$ with the law of the original cluster conditioned to contain the point $n\vec{e_1}$.

In particular, one can recover (1.2) by means of a local limit theorem for directed random walks on $\mathbb{Z}^d$: indeed,

$$\mathrm{e}^{\tau n}\phi[0 \longleftrightarrow n\vec{e}_1] \approx \mathbb{P}_0^{\mathsf{DRW}}[\exists k \geq 0, S_k = n\vec{e}_1] = Cn^{-\frac{d-1}{2}}(1 + o_n(1)),$$

[10] To make this true, one needs to take into account that both the very first and the very last step of the walk have a different distribution. This is a minor technicality and we do not take it into account to clarify the exposition.

where we used that a directed random walk on $\mathbb{Z}^d$ satisfies a $(d-1)$-dimensional local limit theorem, a fact that is not too difficult to establish and follows by classical theory of random walks. Moreover the convergence of the process of the cone points towards a $(d-1)$-dimensional random bridge follows directly from Donsker's invariance principle.

Step 5: Factorisation of measure

The previous reasoning shows how to conclude in the case of Bernoulli percolation. However, the crucial factorisation property given by (1.4) is not true anymore in the case of the random-cluster model, as the different irreducible pieces interact through the boundary conditions they induce on each other. This means that the effective random walk introduced in the previous step have steps that are no longer independent. To derive a local limit Theorem ultimately yielding the Ornstein–Zernike asymptotic formula, two options have been analyzed: the first one consists in the abstract analysis of the kernel of a Ruelle–Perron–Frobenius operator and using ideas coming from ergodic theory [1] to obtain the Gaussian behaviour of the directed random walk. The other option, more probabilistic in nature, consists in using ideas coming from perfect simulation to argue that — up to enlarging the probability space and adding additional randomness — one can randomly concatenate some steps of the directed random walk to obtain an embedded random walk with independent increments, and thus mimic the reasoning of Step 4. We do not develop on that aspect as Chapter 5 relies on a similar technique, which will be thoroughly explained. Let us simply mention that the important property of the subcritical random-cluster models that is used in both of these approaches is its exponential mixing property: the influence of the boundary conditions on a finite region of the space decays exponentially fast in the distance between these boundary conditions and the region, see Chapter 2 for a precise statement.

Remark 1.3.6. Coming back to the historical approach detailed in Subsection 1.3.2, this scheme of proof illuminates the construction of the function f, the existence of which was implicitly assumed by Ornstein and Zernike. Indeed, in that case, it would correspond to the "irreducible connectivity function", suitably tilted by the factor $\mathrm{e}^{\tau\langle y,\vec{e}_1\rangle}$. The renewal equation is trivial to prove in the case of Bernoulli percolation, and the mass gap property has been explained to hold in steps 1–3 of the previous reasoning. This approach demonstrates the power of graphical representations of spin models: indeed, in the case in which $G(0,x) = \langle\sigma_0\sigma_x\rangle$ is the 2-points correlation function of the Ising model, it is not clear at all how the "direct correlation function" f should be defined. However, going through the Edwards–Sokal coupling and looking at the previous reasoning, the function f acquires a geometrical significance and is much easier to understand. This gives a hope to extend the Ornstein–Zernike picture to all the subcritical models of statistical mechanics for which graphical representations of the correlation functions are available. In particular, it would be of interest to investigate the case of the so-called *XY model*, which is a lattice model in which the spins take their values in the non-discrete group $\mathbb{S}^1$ but possesses nonetheless a graphical representation of its correlation through the so-called *Brydges–Fröhlich–Spencer* (BFS in short) representation.

1.4 Questions and motivations of the dissertation

Let us briefly summarize the preceding discussion about the geometry of the phase separation interface. At the macroscopic scale (the scale n of the size of the box), the phase separation line is very well understood: it is shown to converge towards a deterministic curve given by the solution of a variational problem. On the other hand, at scale $\sqrt{n}$, Ornstein–Zernike theory allows to understand the *fluctuations* of the phase separation line around this deterministic shape (in a milder setting, even if partial results exist in the Wulff setting at low temperatures [46, 81]). The three main questions driving this dissertation are the following:

1. At scale n, the smoothing effect induced by the Wulff conditioning takes over the intrinsic randomness of the model. On the other hand, at scale $\sqrt{n}$, as the interface displays Brownian fluctuations, it is a hint that it does not "feel the conditioning", as Ornstein–Zernike theory asserts that it is the behaviour of a regular (*i.e.*, unconstrained) interface. Is there a scale at which those two effects (the smoothing induced by the conditioning and the roughening due to the randomness of the model) have the same order of magnitude? Can one identify this scale and study the phase separation line at this scale?

2. In the setting in which more than two phases have to coexist, what is the joint behaviour of the finite number of interfaces coexisting in the system? In particular, do the several interfaces coexisting in the system still display a Brownian behaviour, or is there a more subtle effect due to the interactions between them?

3. Finally, what is the effect of the temperature parameter on the behaviour of the Ornstein–Zernike formula (1.2)? In particular, what happens to the formula (1.2) when $p \nearrow p_c$ (equivalently when $\beta \nearrow \beta_c$ in the Potts model)?

1.5 Overview of the remainder of the dissertation

Let us describe now at a very broad level the content of the remaining four chapters of the dissertation. We try to introduce the main results of each section in a rather informal way; all the statements are made at a higher level of rigor in the corresponding chapters. We also try to motivate them at a "bird's-eye level", and focus on the broad links with the phase separation phenomenon and the rigorous Ornstein-Zernike theory.

1.5.1 Description of Chapter 2

Chapter 2 is devoted to a detailed introduction of the random-cluster model and of some of its basic properties. We shall extensively use these in the remainder of the dissertation.

1.5.2 Description of Chapter 3

Chapter 3 corresponds to the article [40], written in collaboration with Romain Panis. The goal of this work is to introduce a toy model for the two-dimensional Potts phase separation interface and to study its behaviour at a *mesoscopic scale* (the meaning of that expression will be made clearer later on). Let us briefly describe the model and summarize the results. We study a model of *random oriented paths constrained by area trapping*. Let us fix a parameter $\lambda \in (0, \frac{1}{2})$. We define Λ to be the set of finite oriented edge-paths in the first quadrant of $\mathbb{Z}^2$, starting at the y-axis and ending at the x-axis. The word "oriented" refers to the fact that we only consider paths that can only take rightward or downward steps. We equip Λ with the following probability measure: for any $\gamma \in \Lambda$, denoting by $|\gamma|$ its number of steps, we define

$$\mathbb{P}_\lambda[\gamma] = \frac{\lambda^{|\gamma|}}{Z_\lambda},$$

where $Z_\lambda = (1-2\lambda)^{-1}$ is the only constant ensuring that $\mathbb{P}_\lambda$ is indeed a probability measure. This measure can also be seen as a random walk measure with geometrically randomized length. Fix a parameter $N \geq 1$. If Γ denotes a random variable of law $\mathbb{P}_\lambda$, we introduce $\mathbb{P}_\Lambda^{N^2}$ to be the distribution of Γ conditioned on the event that $\{\mathcal{A}(\Gamma) \geq N^2\}$, where $\mathcal{A}(\Gamma)$ designates the area enclosed by the path Γ and the two coordinate axes. This is our basic model; our goal is to study its fine geometric properties.

Let us quickly explain why despite being quite simple, this model exhibits all the features of the phase separation interface for the two-dimensional Potts model (and we refer to Section 3.5 of Chapter 3 for a more detailed explanation). A typical sample of $\mathbb{P}_\lambda^{N^2}$ experiences a competition between the area requirement and the exponential tails of the background measure $\mathbb{P}_\lambda$. More precisely, the essential features of a sample Γ are its *local Brownian behaviour* (as the background measure is "random walk-like") and its *global curvature* induced by the conditioning.

The (rather imprecise) question that drives this piece of work is the following: what is the scale at which those two competing effects have the same order of magnitude? In order to answer it, we study deviations of Γ from its convex hull. Indeed, observe that the convex hull of Γ is made of a union of line segments that we shall call *facets*. We introduce the random variable MeanFL (for "mean facet length") to be the length of the unique facet intersecting the line $\{y = x\}$[11] . Let x_{mid} be the point of $\mathbb{N}^2 \cap \gamma$ that is the closest to the line $\{y = x\}$. We define MeanLR (for "mean local roughness") as the Euclidean distance between x_{mid} and the facet intercepted by the line $\{y = x\}$. Even though their definition might seem quite intricate, those random variables encapsulate the persistence of the randomness in the horizontal and vertical direction under the conditioning.

A very rough heuristic was proposed by Hammond [77] to identify the scaling of MeanFL.

[11] If the line $\{y = x\}$ intersects the common endpoint of two facets, we arbitrarily set MeanFL to be the length of the leftmost of them.

Indeed, assume that the conditioning forces the path to adopt a globally parabolic profile at scale N (this is a strong assumption that might actually not be satisfied by this model). Then, the scale ℓ at which this global curvature is of the same order as the Brownian fluctuations of the path should satisfy

$$\ell^{1/2} \approx \frac{\ell^2}{N},$$

which implies that ℓ is of order $N^{2/3}$. In this work, we validate this conjecture, proving the following:

Theorem. Let $0 < \lambda < \frac{1}{2}$. For any $\varepsilon > 0$, there exist $c, C > 0$ and $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$,

$$\mathbb{P}_\lambda^{N^2}\left[cN^{\frac{2}{3}} < \mathsf{MeanFL}(\Gamma) < CN^{\frac{2}{3}}\right] > 1 - \varepsilon,$$

and,

$$\mathbb{P}_\lambda^{N^2}\left[cN^{\frac{1}{3}} < \mathsf{MeanLR}(\Gamma) < CN^{\frac{1}{3}}\right] > 1 - \varepsilon.$$

We also prove an additional result regarding the behaviour of the *maximal* facet length and local roughness, namely:

Theorem. Let $0 < \lambda < \frac{1}{2}$. There exist $c, C > 0$ such that,

$$\mathbb{P}_\lambda^{N^2}\left[cN^{\frac{2}{3}}(\log N)^{\frac{1}{3}} < \mathsf{MaxFL}(\Gamma) < CN^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\right] \xrightarrow[N\to\infty]{} 1,$$

and

$$\mathbb{P}_\lambda^{N^2}[cN^{\frac{1}{3}}(\log N)^{\frac{2}{3}} < \mathsf{MaxLR}(\Gamma) < CN^{\frac{1}{3}}(\log N)^{\frac{2}{3}}] \xrightarrow[N\to\infty]{} 1.$$

As explained in more detail in Chapter 3, this work opens a new interesting perspective on the behaviour of the phase separation interface at the mesoscopic scale $N^{2/3}$, which encompasses precisely the competition between the Gaussian randomness of the interface and its curvature inherited from the global conditioning. In particular, a number of open problems regarding the *scaling limit* of the phase separation interface are discussed at the end of the chapter.

1.5.3 Description of Chapter 4

This chapter is based on the article [38]. In this work, we use the Ornstein–Zernike theory to analyse a system of non-intersecting long percolation clusters in the framework of the subcritical random-cluster model. This work implies a few concrete and novel consequences for the random-cluster model; however, it also demonstrates that the Ornstein–Zernike picture remains valid in a setting that is more complex than a single long cluster in infinite volume. As discussed in the introduction of Chapter 4, it thus adds a piece to an already subsequent series of papers implementing the Ornstein–Zernike in dependent or inhomogeneous environments (see for instance [84, 102] to mention just two of them; a more detailed bibliographical review is available in the introduction of Chapter 4).

Consider ϕ the (unique) random-cluster measure on $\mathbb{Z}^2$ with parameters $q \geq 1$ and $p < p_c(q)$. We also consider a finite set of "source points" $(0, x_1), \dots, (0, x_r)$ and a set of "target points" $(n, y_1), \dots, (n, y_r)$. We introduce the following events $\mathsf{Con} := \bigcap_{i \in \{1,\dots,r\}} \{(0, x_i) \longleftrightarrow (n, y_i)\}$, and $\mathsf{NI} := \bigcap_{1 \leq i \neq j \leq r} \{\mathcal{C}_{(0,x_i)} \cap \mathcal{C}_{(0,x_j)} = \emptyset\}$ ($\mathcal{C}_x$ is the cluster of the vertex $x \in \mathbb{Z}^2$). In words, under the event $\{\mathsf{Con}, \mathsf{NI}\}$ the points $(0, x_i)$ and (n, y_i) are pairwise connected by *distinct* percolation clusters. We are interested in two questions: what is the behaviour of the quantity $\phi[\mathsf{Con}, \mathsf{NI}]$ as n tends to infinity, and what is the typical behaviour of a percolation configuration sampled under the measure $\phi[\cdot|\mathsf{Con}, \mathsf{NI}]$? The answer to the first question is given by the following result[12](recall that the function τ was introduced in (1.3); we choose to write $\tau := \tau(\vec{e_1})$):

Theorem. Let $q \geq 1$, $0 < p < p_c(q)$ and $r \geq 1$ be a fixed integer. Then, when $n \to \infty$,

$$\phi\,[\mathsf{Con}, \mathsf{NI}] \asymp n^{-\frac{r^2}{2}} \mathrm{e}^{-\tau r n}.$$

We also describe the scaling limit of the shape of the clusters under the conditioning on $\{\mathsf{Con}, \mathsf{NI}\}$. For that, we need to introduce the *interfaces* of a cluster. They are described rigorously in the introduction of Chapter 4; for our purpose of exposition, let us simply say that they are the piecewise linear functions bordering the top and the bottom of a cluster (see Figure 4.1 for an illustration). They are called Γ^+ and Γ^- in this prequel. Our second result states that under a diffusive scaling (*i.e.*, a scaling of order n in the horizontal direction and $\sqrt{n}$ in the transverse direction), the system of interfaces converges towards a system of r Brownian bridges conditioned not to intersect: the so-called *Brownian watermelon*.

Theorem. Under the family of measures $\phi\,[\;\cdot\;|\mathsf{Con}, \mathsf{NI}]$ (we recall that $\mathsf{Con}, \mathsf{NI}$ depend on n), there exists $\sigma > 0$ such that:

$$\left(\frac{1}{\sqrt{n}}\left(\Gamma_1^+(nt), \dots, \Gamma_r^+(nt)\right)\right)_{0 \leq t \leq 1} \xrightarrow[n \to \infty]{(d)} (\sigma \mathsf{BW}_t^{(r)})_{0 \leq t \leq 1},$$

where $\mathsf{BW}^{(r)}$ is the Brownian watermelon with r bridges, and where the convergence holds in the space $\mathcal{C}\left([0,1], \mathbb{R}^r\right)$ endowed with the topology of uniform convergence. Moreover, almost surely, for all $1 \leq i \leq r$,

$$\frac{1}{\sqrt{n}} \left\| \Gamma_i^+ - \Gamma_i^- \right\|_\infty \xrightarrow[n \to \infty]{} 0.$$

[12]The result stated here is less general than the statement presented in Chapter 4. In particular, in full generality, we allow the source and target points to depend on n. We obtain a uniform estimate of $\phi[\mathsf{Con}, \mathsf{NI}]$ in the regime where their norm is of order $o(\sqrt{n})$.

We also notice *en passant* that the first result in the case $r = 2$ has an interesting consequence regarding the *supercritical* random-cluster model. Indeed, a simple duality argument yields the following formula computing the asymptotic behaviour of the probability of existence of a large but finite percolation cluster:

Theorem. For any $p > p_c(q)$, the following estimate holds when $n \to \infty$:

$$\phi[(n,0) \in \mathcal{C}_0, |\mathcal{C}_0| < \infty] \asymp \frac{1}{n^2} \mathrm{e}^{-2\tau n},$$

where $\tau := \lim_{n\to\infty} -\frac{1}{n} \log \phi_{p^*}[0 \longleftrightarrow (n,0)]$.

In the last statement, p^* correspond to the *dual* parameter of p. In particular, this theorem implies that the truncated correlation length of the planar random-cluster at $p > p_c$ equals twice the correlation length at $p^* < p_c$, a fact that was previously known only in the case $q = 1$ [35].

The analysis is carried out by proving that under the conditioning, the behaviour of the clusters is similar to that of a system of random bridges conditioned not to intersect; a conclusion that is reminiscent of the single-cluster Ornstein–Zernike theory. However, a number of difficulties are encountered in implementing this heuristic, due to the *a priori* long-range interactions of the random-cluster measure. Indeed, the interaction between the different clusters turns out to be *attractive*, allowing the *a priori* existence of a pinning transition — a regime in which this attraction is so strong that the clusters actually remain at a bounded distance from each other. We rule out the existence of such a transition, proving that the system obeys an *entropic repulsion phenomenon* at any subcritical temperature. In words, the entropy due to the large number of possible trajectories for the percolation clusters always beats the attractive term, causing the random walk-like behaviour of the clusters. This entropic repulsion phenomenon is established through a careful analysis of the random-cluster interactions.

1.5.4 Description of Chapter 5

This chapter is based on the preprint [39], written in collaboration with Ioan Manolescu. The goal of this work is to extend the Ornstein–Zernike formula to the near-critical regime of the planar random-cluster model when $q \in [1,4]$. Recall that the Ornstein–Zernike formula provides a precise understanding of the quantity $\phi[0 \longleftrightarrow \lfloor nx \rfloor]$ as a function of n, where ϕ is the unique subcritical infinite-volume limit of the random-cluster measure. In this work, we prove a formula for the connection probabilities that is uniform *both in* n *and in* $p_c - p$, in the planar case. This work has several interesting consequences. First, it allows to take limits in the Ornstein–Zernike formula when both $n \to \infty$ and $p \nearrow p_c$ simultaneously, thus allowing to understand the probability of long connections in the near-critical limit of the random-cluster model. Moreover, it also highlights an interesting — and perhaps slightly more subtle — feature of the Ornstein–Zernike construction. Indeed, in [25], the authors classically couple a long subcritical percolation cluster with a directed random walk having independent increments. Our walk can be summarized as the answer to the question "what is the size of the random walk steps as a function of p?". Looking carefully at the construction of [25], it appears that the events used by the authors to decouple the steps of the random walk and to factorize the measure are the so-called "4-arm events". In our work, we prove that the decoupling actually occurs at a scale which is *strictly smaller* (presumably polynomially smaller as $p \to p_c$) than the one at which the 4-arm events typically occur; namely the scale of the *correlation length*. This the main feature of this work: instead of relying on the 4-arm events to factorise the random walk measure, we prove that having uniformly positive probability of crossing square boxes is sufficient to get the mixing necessary to factorise the random walk measure.

Let us introduce formally our result. We need to introduce the so-called "one-arm critical probability", namely $\pi^1(p) := \phi_{p_c}[0 \longleftrightarrow \partial\Lambda_{(\tau_p)^{-1}}]$.

Theorem. Let $q \in [1,4]$. The following estimate holds uniformly in n and in $p < p_c$. Let $\phi_{p,q}$ be the unique random-cluster measure on $\mathbb{Z}^2$ with parameters p and q. Then for any $x \in \mathbb{S}^1$,

$$\phi_{p,q}[0 \longleftrightarrow \lfloor nx \rfloor] \asymp \pi^1(p)^2 \, (\tau_p n)^{-\frac{1}{2}} \, \mathrm{e}^{-\tau_p(x)n}$$

Observe that the constants implicitly appearing in the theorem depend neither on $p < p_c$ nor on n. This formula has the following feature: it interpolates between the critical behaviour of the system when p is very close to p_c (essentially the term $\pi^1(p)$ is a *critical term*) and the classical Ornstein–Zernike behaviour at subcriticality. In particular when $p < p_c$ is fixed and $n \to \infty$, we retrieve the classical Ornstein–Zernike asymptotic presented earlier in the introduction.

Chapter 2

The random-cluster model and its basic properties

This section is devoted to properly prove certain properties of the random-cluster measure that we shall use extensively in this dissertation. This chapter is mostly review; the classical references on general properties of random-cluster measures are [69, 70, 49]. We recall the definition of the random-cluster measure from the Introduction: fix two parameters $p \in [0, 1]$ and $q \geq 1$, a finite graph G, and a boundary condition η. If ω is a percolation configuration on G, we recall that the random-cluster measure with boundary conditions η is defined as:

$$\phi^{\eta}_{G,p,q}[\omega] = \frac{1}{Z^{\eta}_{G,p,q}} \left(\frac{p}{1-p}\right)^{o(\omega)} q^{k^{\eta}(\omega)},$$

where $Z^{\eta}_{G,p,q}$ is the partition function.

2.1 Monotonicity, positive association and couplings

2.1.1 Monotonicity via Glauber dynamics

In what follows, G is a fixed finite graph, and the parameters p, q are fixed. We start by computing the single-edge marginal of the random-cluster measure when the state of every edge but one is fixed.

Lemma 2.1.1. *For any boundary condition η, any edge $e = \{xy\} \in E(G)$ and any percolation configuration ξ on $G \setminus \{e\}$,*

$$\phi^{\eta}_{G,p,q}[\omega_e = 1 | \omega_{E\setminus\{e\}} = \xi] = \begin{cases} p & \text{if } x \longleftrightarrow y \text{ in } \xi^{\eta}, \\ \frac{p}{p+q(1-p)} & \text{else.} \end{cases}.$$

Proof. Observe that

$$\phi^{\eta}_{G,p,q}[\omega_e = 1|\omega_{E\setminus\{e\}} = \xi] = \frac{1}{1 + \frac{\phi^{\eta}_{G,p,q}[\omega_e=0,\omega_{E\setminus\{e\}}=\xi]}{\phi^{\eta}_{G,p,q}[\omega_e=1,\omega_{E\setminus\{e\}}=\xi]}}. \tag{2.1}$$

In the case in which $x \longleftrightarrow y$ in ξ^η, opening the edge e does not change the number of open clusters of ω. Therefore, the ratio appearing in the denominator of (2.1) is equal to $\frac{1-p}{p}$. This yields

$$\phi^{\eta}_{G,p,q}[\omega_e = 1|\omega_{E\setminus\{e\}} = \xi] = \frac{1}{1 + \frac{1-p}{p}} = p.$$

In the case in which $x \not\longleftrightarrow y$ in ξ^η, the ratio will be equal to $\frac{1}{1+\frac{q(1-p)}{p}}$, as the configuration in which $\omega_e = 0$ has one additional cluster than the configuration in which $\omega_e = 1$. Thus,

$$\phi^{\eta}_{G,p,q}[\omega_e = 1|\omega_{E\setminus\{e\}} = \xi] = \frac{1}{1 + \frac{q(1-p)}{p}} = \frac{p}{p + q(1-p)}.$$

□

This seemingly trivial observation has several consequences, amongst which is the celebrated *Domain Markov property*. Indeed, observe that the previous computation yields the formal equality

$$\phi^{\eta}_{G,p,q}[\omega_e = 1|\omega_{E\setminus\{e\}} = \xi] = \phi^{\xi^\eta}_{\{e\},p,q}[\omega_e = 1].$$

This identity extends to any finite domain as follows

Proposition 2.1.2 (Domain Markov property)**.** *Let $G' = (V', E')$ be a subgraph of G. Then for any boundary condition η and any percolation configuration $\xi \in \{0,1\}^{E\setminus E'}$,*

$$\phi^{\eta}_{G,p,q}[\cdot_{G'}|\omega_{E\setminus E'} = \xi] = \phi^{\xi^\eta}_{G',p,q}[\cdot]. \tag{DMP}$$

Proof. Use the fact that (DMP) was established in the case of E' consisting of a single edge and reason by induction on the cardinality of E'. □

Another easy yet important consequence of (2.1) is the so-called *finite energy property*.

Proposition 2.1.3 (Finite energy property)**.** *There exists a constant $c \in (0,1)$ such that for any finite graph G, any boundary condition η and any percolation configuration ξ on G,*

$$c^{|E|} \leq \phi^{\eta}_{G,p,q}[\omega = \xi] \leq (1-c)^{|E|}. \tag{FE}$$

We now introduce one of the most fundamental tools in percolation theory: the so-called *positive association* property of the random-cluster model. **For what follows, we fix $\mathbf{q \geq 1}$**[1]. The next proposition encompasses both the monotonicity of the model in p, in the boundary conditions and its positive association properties. We first recall the notion of *stochastic domination*.

Recall that we equipped the set of percolation configurations on G with the following partial order: $\omega \leq \omega' \Leftrightarrow \forall e \in E, \omega(e) \leq \omega'(e)$. An event $\mathcal{A}$ will be said to be *increasing* if for any $\omega \leq \omega', \omega \in \mathcal{A} \Rightarrow \omega' \in \mathcal{A}$. That is to say, opening some more edges in the percolation configuration can only favour the occurence of $\mathcal{A}$. A canonical instance of an increasing event is $\{\omega_e = 1\}$ for some fixed edge $e \in E$.

Definition 2.1.4. Let μ, μ' be two probability measures on $\{0,1\}^E$. We say that μ' *stochastically dominates* μ if for any increasing event $\mathcal{A}$,

$$\mu[\mathcal{A}] \leq \mu'[\mathcal{A}].$$

In that case, we will use the notation $\mu \preccurlyeq \mu'$. It is easy to check that $\preccurlyeq$ defines a partial order on the set of probability measures on $\{0,1\}^E$.

One method that is commonly used to prove stochastic domination between different percolation measures is the use of the *coupling method*. Assume that one can construct a probability measure Ψ on the set $\{0,1\}^E \times \{0,1\}^E$ satisfying the following three properties.

1. $\Psi(\cdot, \{0,1\}^E) = \mu[\cdot]$.
2. $\Psi(\{0,1\}^E, \cdot) = \mu'[\cdot]$.
3. $\Psi(\{(\omega,\omega') \in \{0,1\}^E, \omega \leq \omega'\}) = 1$.

In that case, it is straightforward that $\mu \preccurlyeq \mu'$. Actually, the converse of that statement is also true, and known as *Strassen's Theorem* — it is sometimes useful in the proof of certain statements in Statistical mechanics[2]. The proof of the next proposition, stating useful monotonicity properties of the random-cluster measure, crucially relies on the coupling method.

Proposition 2.1.5 (Monotonicity of the random-cluster measure)**.** *Let $p \leq p'$ and $1 \leq q \leq q'$. Let $\mathcal{A}$ be an increasing event of positive probability and let $\xi \leq \xi'$ be two boundary conditions on ∂G. The following monotonicity properties hold:*

[1]This is not for convenience; indeed, for $q < 1$, the random-cluster can be shown to exhibit negative association on certain graphs (see for instance the example following Theorem 3.8 of [70]). Furthermore, the model is known to converge towards the Uniform Spanning Tree when $p \to 0$ and $q/p \to 0$, which is known to be negatively associated.

[2]Especially in the theory of strong embedding of random variables in the Brownian motion, see the excellent survey [98].

1. ***Monotonicity in the boundary conditions.***

$$\phi^{\xi}_{G,p,q} \preccurlyeq \phi^{\xi'}_{G,p,q}. \tag{CBC}$$

2. ***Monotonicity in the parameters.***

$$\phi^{\xi}_{G,p,q'} \preccurlyeq \phi^{\xi}_{G,p',q}.$$

3. ***Positive association.***

$$\phi^{\xi}_{G,p,q} \preccurlyeq \phi^{\xi}_{G,p,q}[\cdot|\mathcal{A}].$$

Proof. The proofs of the three points follow the same pattern (with a slight subtlety for the last one that we will explain at the end of the proof). Let us focus on the proof of the first point. We shall construct an explicit coupling Ψ between $\phi^{\xi}_{G,p,q}$ and $\phi^{\xi'}_{G,p,q}$ satisfying the previously listed assumptions. The construction of the coupling itself is of crucial importance in percolation and spin models theory, and is known as *Glauber dynamics*: the measure $\phi^{\xi}_{G,p,q}$ is obtained as the stationnary measure of a Markov process in an enlarged probability space. Enlarge the probability space by assigning to each edge $e \in E$ a collection of independent and identically distributed exponential random variables $(\tau_{e,i})_{i\geq 1}$. The variables $T_{e,n} = \tau_{e,1} + \cdots + \tau_{e,n}$ will be seen as *clocks*: at each time $T_{e,n}$, the state of the edge e will be examined by the procedure and possibly resampled. We also equip the probability space with a collection $(U_i)_{i\geq 1}$ of independent and uniform random variables in $[0,1]$, also independent of the family $(\tau_{e,i})_{e\in E, i\geq 1}$. The dynamics is a continuous Markov chain $(\omega_t, \omega'_t)_{t\geq 0}$ on $\{0,1\}^E \times \{0,1\}^E$, evolving as follows. Inductively define the sequence (e_k, t_k) by setting $(e_0, t_0) = (\emptyset, 0)$, and (e_k, t_k) to be the location and the time associated to the first clock to ring after time t_{k-1} (this is well-defined as G is finite). Now define the configurations ω_{t_k} and ω'_{t_k} with the following rule. Initialize by setting $\omega_0 = \omega'_0 \equiv 0$. Then, set

$$\omega_{t_k}(e) = \begin{cases} \mathbb{1}[U_k \leq \phi^{\eta}_{G,p,q}[\omega(e) = 1 | \omega_{E\setminus\{e\}} = \omega_{t_k^-}]] & \text{if } e = e_k \\ \omega_{e,t_k^-} & \text{else,} \end{cases}$$

and

$$\omega'_{t_k}(e) = \begin{cases} \mathbb{1}[U_k \leq \phi^{\eta'}_{G,p,q}[\omega(e) = 1 | \omega'_{E\setminus\{e\}} = \omega'_{t_k^-}]] & \text{if } e = e_k \\ \omega'_{e,t_k^-} & \text{else,} \end{cases}$$

where $\omega_{t_k^-} = \lim_{s \nearrow t_k} \omega_s$. Two things now need to be argued. First of all, it is clear that this dynamic defines a continuous Markov chain on $\{0,1\}^E \times \{0,1\}^E$. Moreover, ω_t (resp. ω'_t) is irreducible, aperiodic and positive. Therefore, it converges towards its unique invariant measure, which clearly is $\phi^{\eta}_{G,p,q}$ (resp. $\phi^{\eta'}_{G,p,q}$) by construction. To conclude, it remains to argue that for any $t \geq 0$, one has $\omega_t \leq \omega'_t$. This can be proved inductively on k. Assume that $\omega_{t_k} \leq \omega'_{t_k}$. We argue that for a fixed edge e, one has that, thanks to (2.1),

$$\phi^{\eta}_{G,p,q}[\omega(e) = 1 | \omega_{E\setminus\{e\}} = \omega_{t_k^-}] \leq \phi^{\eta'}_{G,p,q}[\omega'(e) = 1 | \omega_{E\setminus\{e\}} = \omega'_{t_k^-}]. \tag{2.2}$$

Indeed, if the endpoints of some edge e are connected in $\omega^{\eta}_{t_k^-}$, then they need to be connected in $(\omega')^{\eta'}_{t_k^-}$ as $\omega^{\eta}_{t_k^-} \leq (\omega')^{\eta'}_{t_k^-}$ by hypothesis. As $q \geq 1$, this observation proves (2.2). Since $\omega_0 = \omega'_0$, this concludes the proof by setting $\Psi(\omega, \omega') = \lim_{t\to\infty} \mathbb{P}(\omega_t, \omega'_t)$, where $\mathbb{P}$ is the distribution of the configuration under the above-mentioned procedure.

The proof of the second item follows similarly.

For the proof of the third one, there is a slight subtlety, as the distribution of ω' (which eventually has the distribution $\phi[\cdot|\mathcal{A}]$) is not a positive Markov chain on $\{0,1\}^E$. However, as it remains positive, aperiodic and irreducible on $\mathcal{A}$, the proof carries on in the same fashion. □

Remark 2.1.6. Remark that in the previous proof, when resampling the state of an edge e, we decide to open it either with probability p or $\frac{p}{p+q(1-p)}$ depending on the state of the edges outside of $\{e\}$. When $q \geq 1$, the Glauber dynamics immediately yields the following stochastic domination by Bernoulli percolation, for any boundary condition η:

$$\phi_{G,\frac{p}{p+q(1-p)},1} \preccurlyeq \phi^{\eta}_{G,p,q} \preccurlyeq \phi_{G,p,1}.$$

This will help later in the proof of the non-triviality of the phase transition

The third item is often cast in a slightly different form, known as the *Fortuin–Kasteleyn–Ginibre* (FKG hereafter) inequality, which is the cornerstone of modern percolation theory and was first proved in [61].

Corollary 2.1.7 (FKG inequality)**.** *For any $q \geq 1$, any $p \in [0,1]$, any boundary condition η and any two increasing events $\mathcal{A}, \mathcal{B}$,*

$$\phi^{\eta}_{G,p,q}[\mathcal{A} \cap \mathcal{B}] \geq \phi^{\eta}_{G,p,q}[\mathcal{A}]\phi^{\eta}_{G,p,q}[\mathcal{B}]. \tag{FKG}$$

Proof. As $\mathcal{B}$ is increasing, the stochastic domination given by the positive association property immediately yields

$$\phi^{\eta}_{G,p,q}[\mathcal{B}] \leq \phi^{\eta}_{G,p,q}[\mathcal{B}|\mathcal{A}].$$

□

2.1.2 Increasing couplings by exploration

As seen previously, the use of the coupling method is of crucial importance in percolation theory, as it allows to derive very useful monotonicity properties of the random-cluster model. We described one important way of coupling two random-cluster measures, namely the Glauber dynamics. Here, we describe another family of couplings that also are of fundamental use in percolation theory (to the point that we challenge the reader to find a modern percolation paper that does not contain the word "exploration"!): the so-called *couplings by exploration*. The most general way to describe this procedure is by using the theory of *decision trees*.

Definition 2.1.8. Consider a finite graph $G = (V, E)$ such that $|E| = n$, equipped with a family of independent uniform random variables on $[0, 1]$, called $(U_e)_{e\in E}$. A decision tree $\mathbf{T}$ is a family $(e_1, (r_k)_{1\leq k\leq n})$, where e_1 is the starting edge of the tree (*i.e.*, the first edge whose state is going to be revealed) and $(r_k)_{k\in\{1,\dots,n\}}$ is a family of *decision rules*, *i.e.*, for each $k \in \{0, \dots, n-1\}$, r_{k+1} takes into input a k-tuple of edges $(e_1, \dots, e_k)$ and $(U_{e_1}, \dots U_{e_k})$ and returns an edge $e_{k+1} \in E \setminus \{e_1, \dots, e_k\}$. In other words, once the state of the edges $e_1, \dots, e_k$ has been revealed, r_{k+1} returns the next edge to be revealed according to the values of the randomness $(U_{e_1}, \dots U_{e_k})$. An important notion for decision trees, which — as the reader might guess — is very well suited to the Domain Markov property, is the notion of *stopping time*. A random variable $\tau \in \mathbb{N}^* \cup \{\infty\}$ shall be called a stopping time if the event $\{\tau \leq k\}$ is measurable with respect to $(e_1, \dots, e_k)$ and $(U_{e_1}, \dots, U_{e_k})$ for all $k \geq 0$.

The decision tree provides us a procedure to reveal the state the edges one by one in some predefined (possibly random, as it is allowed to depend on the family $(U_e)_{e\in E}$) order. As in the Glauber dynamics, there is a way of inductively sampling a random-cluster configuration, in the order given by the decision tree, if one choses the proper marginals for the revealed edges. Moreover, this way of sampling the random-cluster model has an important feature that we will use extensively later on: if one stops the revealment algorithm at a given stopping time the law of the unexplored configuration will simply be the random-cluster measure on a random graph, with random boundary conditions, due to (DMP). Finally, as in the Glauber dynamics, one can sample a pair of configurations using the same family of uniform variables $(U_e)_{e\in E}$. In particular conveniently choosing the revealment algorithm can be useful to couple two random-cluster configurations in a random subgraph of G in an increasing fashion. This discussion is summarized in the following

Lemma 2.1.9. *Let $\mathbf{T} = (e_1, (r_k)_{k\in\{0,1\}})$ be a decision tree on G. Let $\xi_1 \leq \xi_2$ be an ordered pair of boundary conditions. Let τ be a stopping time for $\mathbf{T}$. We sample inductively, edge by edge, two percolation configuration as follows. For any $k \in \{1, \dots, n\}$, set*

$$\omega^1_{r_k(e_1,\dots,e_k,U_{e_1},\dots,U_{e_k}))} = \mathbb{1}[U_{r_k(e_1,\dots,e_k,U_{e_1},\dots,U_{e_k})}] \leq \phi^{\xi_1}[\omega_{r_k(e_1,\dots,e_k,U_{e_1},\dots,U_{e_k})} = 1|\omega^1_{e_1}, \dots, \omega^1_{e_k}],$$

and

$$\omega^2_{r_k(e_1,\dots,e_k,U_{e_1},\dots,U_{e_k}))} = \mathbb{1}[U_{r_k(e_1,\dots,e_k,U_{e_1},\dots,U_{e_k})}] \leq \phi^{\xi_2}[\omega_{r_k(e_1,\dots,e_k,U_{e_1},\dots,U_{e_k})} = 1|\omega^2_{e_1}, \dots, \omega^2_{e_k}].$$

Denote by Ψ_τ the joint law of the revealment algorithm stopped at τ. Then, if G_τ is the (possibly random) subgraph induced by the set of edges that have been revealed before time τ and $(\omega^1_\tau, \omega^2_\tau)$ is the percolation configuration revealed on G_τ,

1.
$$\Psi_\tau(\cdot,\{0,1\}^{G_\tau}) = \phi^{(\omega^1_\tau)^\xi}_{G\setminus G_\tau}[\cdot] \textit{ and } \Psi_\tau(\{0,1\}^{G_\tau},\cdot) = \phi^{(\omega^2_\tau)^\xi}_{G\setminus G_\tau}[\cdot].$$

2.
$$\omega^1_\tau \le \omega^2_\tau \textit{ a.s.}$$

Proof. The proof is essentially clear, as the first item comes from the Domain Markov property (DMP) and the second item comes from a reasoning similar to the proof of the monotonicity in the boundary conditions using Glauber dynamics. □

Despite the generality of the latter statement, the reader should keep in mind that we will essentially use one special type of decision tree: the so-called exploration of the cluster of a given vertex. For instance, assume that one wants to explore the cluster of $x \in \Lambda_n$ inside Λ_n. One can construct the following decision tree. Fix an arbitrary order on the edges of G and take e_1 to be the minimal edge of $E(\Lambda_n)$ having x as one of its endpoints. At step k, let $e_1, \dots, e_k$ be the set of already revealed edges. Then

1. Either there exists an open edge of $E \setminus \{e_1, \dots, e_k\}$ that is adjacent to one of the edges of $\{e_1, \dots, e_k\}$. In that case, we reveal the minimal one.
2. Or there exists no such edge, and we stop the algorithm. This defines a stopping time τ.

The exploration procedure has the useful property that when $\{\tau < \infty\}$, then it induces free boundary conditions on the unexplored portion of the space. In particular, since $\omega^1_{\tau_2} \le \omega^2_{\tau_2}$, where τ_2 is the stopping time of the cluster exploration for ω^2, then the coupling also induces free boundary conditions for ω^1, and one can further couple ω^1 and ω^2 so that they agree on the unexplored region of the space.

Finally we note that this algorithm can be adapted to explore the cluster of a given connected set in Λ_n (we will use this several times later on). We refer the reader to the proof of Lemma 2.2.2 or of Proposition 2.2.5 for what might be the simplest example of the use of that type of exploration arguments

Equipped with those tools, we are now ready to prove the existence of infinite-volume limit measures on $\mathbb{Z}^d$, as announced in Lemma 1.1.1. From now, we do not work anymore on an abstract graph G, but rather on subgraphs of $\mathbb{Z}^d$, where the dimension $d \ge 1$ is fixed. For convenience, we recall the statement of Lemma 1.1.1.

Lemma 2.1.10. *Let $d \ge 1$, $q \ge 1$ and $p \in [0,1]$. There exist two measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ (not necessarily distinct) on $\{0,1\}^{E(\mathbb{Z}^d)}$ such that for any event $\mathcal{A}$ depending on a finite number of edges,*

$$\lim_{n\to\infty} \phi^0_{\Lambda_n,p,q}[\mathcal{A}] = \phi^0_{p,q}[A] \quad \textit{and} \quad \lim_{n\to\infty} \phi^1_{\Lambda_n,p,q}[\mathcal{A}] = \phi^1_{p,q}[A].$$

Proof. We focus on the case of ϕ^0, as the proof for ϕ^1 follows the same principle. First, we argue that it is sufficient to prove the existence of the limit for any *increasing* event $\mathcal{A}$ depending on a finite number of edges, as the inclusion-exclusion principle ensures that the probability of any event depending on a finite number of edges can be written as the combination of the probabilities of increasing events depending on a finite number of edges. Let $\mathcal{A}$ be an increasing event depending on a finite number of edges, and say that all those edges are contained in the box Λ_R, for some sufficiently large value of $R \geq 0$. We are going to argue that the sequence $\phi^0_{\Lambda_n,p,q}[\mathcal{A}]$ is non-decreasing in $n \geq R+1$. The argument relies on a very classical reasoning in percolation, called an "exploration argument". We first explain it in the percolation language, in order to make the reader more familiar with the percolation vocabulary, and then explain the reasoning rigorously. Let $n \geq R+1$, and consider the measure $\phi^0_{\Lambda_{n+1},p,q}$. Under that measure, explore the random boundary conditions induced by a percolation configuration on $\partial\Lambda_n$. As those boundary conditions dominate the free ones on $\partial\Lambda_n$, it follows that $\phi^0_{\Lambda_{n+1},p,q}[\mathcal{A}] \geq \phi^0_{\Lambda_n,p,q}[\mathcal{A}]$. Formally, the previous reasoning writes as follows. Use the Domain Markov property (DMP) to write

$$\phi^0_{\Lambda_{n+1},p,q}[\mathcal{A}] = \sum_{\xi \text{ b.c. on } \partial\Lambda_n} \phi^\xi_{\Lambda_n,p,q}[\mathcal{A}]\phi^0_{\Lambda_{n+1},p,q}[\omega \text{ induces } \xi \text{ on } \partial\Lambda_n].$$

Now by (CBC), observe that for any boundary condition ξ on $\partial\Lambda_n$, it is the case that $\phi^\xi_{\Lambda_n,p,q}[\mathcal{A}] \geq \phi^0_{\Lambda_n,p,q}[\mathcal{A}]$. Summing over all ξ, this proves that $\phi^0_{\Lambda_{n+1},p,q}[\mathcal{A}] \geq \phi^0_{\Lambda_n,p,q}[\mathcal{A}]$. Thus we proved that for any event depending on a finite number of edges, $\lim_{n\to\infty} \phi^0_{\Lambda_n,p,q}[\mathcal{A}] := \phi^0_{p,q}[\mathcal{A}]$ exists. By Kolmogorov's extension Theorem, $\phi^0_{p,q}$ can be extended to a probability measure on the σ-algebra generated by the cylinder events.

The proof of the existence of the limit of $\phi^1_{\Lambda_n,p,q}$ follows the same pattern, as one can show that if $\mathcal{A}$ is an increasing event depending on a finite number of edges, then $\phi^1_{\Lambda_n,p,q}[\mathcal{A}]$ forms a non-increasing sequence. □

From now on, we will not make reference to the underlying graph $\mathbb{Z}^d$, nor to the parameters $q \geq 1, p \in [0,1]$ when the context is clear. It is routine to check that the properties previously proved in finite volume, namely monotonicity in the parameters, in the boundary conditions and the FKG inequality extend to the measures ϕ^0 and ϕ^1. Moreover, it is standard that those measures are invariant by lattice translations and ergodic (in the sense that any event translationally invariant event must have probability either 0 or 1).

2.2 The phase transition of the random-cluster model

Equipped with the infinite-volume limit measures on $\mathbb{Z}^d$ and their monotonicity properties, we may define a *critical parameter* $p_c = p_c(q,d)$ as follows.

$$p_c := \inf\{p \geq 0, \phi^1_{p,q}[0 \longleftrightarrow \infty] > 0\}.$$

When $p > p_c$ we say that 0 has a positive probability to *percolate*, meaning that it has a positive probability to lie in an infinite cluster. The first result regarding the phase transition of the random-cluster model is that it is non-trivial, in the sense that $0 < p_c < 1$.

Proposition 2.2.1. *For any $q \geq 1$ and any $d \geq 2$, the critical parameter satisfies $p_c(q,d) \in (0,1)$.*

Proof. The proof follows a very classical perturbative argument, called the *Peierls argument*. One can safely say that this argument gave birth to the modern mathematical theory of percolation and spin systems. We start by reducing the problem to Bernoulli percolation by using Remark 2.1.6. Indeed, observe that when $q \geq 1$, the output of Remark 2.1.6 and a small computation yield that:

$$\frac{qp_c(1,d)}{1+p_c(1,d)(q-1)} \leq p_c(q,d) \leq p_c(1,d).$$

Thus, it is sufficient to prove that $p_c(1,d) \in (0,1)$. Abbreviate $p_c(1,d) := p_c(d)$. Moreover, observe that $p_c(d)$ is decreasing in d. It is then sufficient to prove that $p_c(2) < 1$, and that $0 < p_c(d)$.

Let us start with the first item. We will briefly use the duality of the model without formally referring to it, however the reader might refer to the next subsection to phrase the proof in a more natural way. Consider the (dual) lattice $(\mathbb{Z}^2)^* := (1/2,1/2) + \mathbb{Z}^2$, and observe that to any edge-circuit γ^* of $E((\mathbb{Z}^2)^*)$, one can associate the set of edges of $E(\mathbb{Z}^2)$ intersecting the edges of γ^*. We call such a set a *blocking section*. Now observe that one has the following graph-theoretical equivalence in dimension $d = 2$:

$$\text{The cluster of 0 is finite} \;\Leftrightarrow\; \text{There exists a closed blocking section surrounding } 0.$$

This implies that

$$\phi_{p,1}[|\mathcal{C}_0| < \infty] \leq \sum_{\gamma \text{ blocking section } \ni 0} \phi_{p,1}[\gamma \text{ is closed }].$$

Now observe that on the one hand, the probability for a blocking section with n edges to be closed is $(1-p)^n$. On the other hand, the set of blocking sections of length n surrounding 0 is in bijection with the set of edge-circuits of length n surrounding 0. An easy way to upper bound the cardinality of this set is for instance to choose a starting point somewhere along the x-axis (at most $2n+1$ choices) and say that the number of paths of length n starting from that point is at most 4^n (this is a very crude bound, but we do not seek for optimality in this proof). Thus we proved that

$$\phi_{p,1}[|\mathcal{C}_0| < \infty] \leq \sum_{n \geq 4} (2n+1)4^n(1-p)^n.$$

It is now clear that there exists a value of p sufficiently close to 1 such that this sum is strictly upper bounded by 1, and so that $p_c(2) < 1$.

Let us turn to the converse bound, which is even easier. Indeed, it suffices to notice that for any $n \geq 1$,

$$\begin{aligned}\phi_{p,1}[|\mathcal{C}_0| = \infty] &\leq \phi_{p,1}[\exists \gamma \text{ a self-avoiding path of open edges of length } n \text{ starting at } 0] \\ &\leq p^n |\{\gamma : \gamma \text{ is a self-avoiding path of length } n \text{ starting at } 0\}| \\ &\leq p^n (2d)^n.\end{aligned}$$

The estimation of the cardinality of the set of self-avoiding paths is far from being optimal, but once again we only need crude estimates. When $p < \frac{1}{2d}$, this sequence converges to 0, which proves that $p_c(1, d) \geq \frac{1}{2d} > 0$. □

The random-cluster thus enjoys a *phase transition* as follows: when $p < p_c$, there is almost surely no infinite cluster. However, when $p > p_c$ there exists almost surely an infinite cluster in $\mathbb{Z}^d$ for ϕ^1 (by ergodicity of the measure). As we shall see, the behaviour of the model drastically changes whether $p < p_c$ (the so-called *subcritical regime*), $p = p_c$ (the *critical regime*), or $p > p_c$ (the *supercritical regime*). We are going to put a special focus on properties of the subcritical regime in what follows.

2.2.1 The subcritical regime

We first observe that there exists a single infinite-volume limit in the subcritical regime.

Lemma 2.2.2. *When $d \geq 1$ and $p < p_c$, $\phi^0 = \phi^1$. Therefore there exists a unique infinite-volume limit measure.*

Proof. Fix $R > 0$, and $\mathcal{A}$ be an increasing event depending on the edges of the box Λ_R. We aim to prove that $\phi^1[\mathcal{A}] \leq \phi^0[\mathcal{A}]$, which will conclude the proof of the lemma since $\phi^0 \preccurlyeq \phi^1$. Fix $\varepsilon > 0$. We first argue that the definition of p_c implies that there exists a scale N satisfying that for any $n \geq N$, $\phi^1[\partial\Lambda_R \longleftrightarrow \partial\Lambda_n] < \varepsilon$. Indeed, observe that due to (CBC) and (DMP),

$$\phi^1[0 \longleftrightarrow \infty] \geq \phi^0_{\Lambda_R}[\omega \equiv 1]\phi^1[\partial\Lambda_R \longleftrightarrow \infty].$$

Then, since the quantity $\phi^0_{\Lambda_R}[\omega \equiv 1]$ is uniformly positive in n by the finite energy property (FE), $\phi^1[\partial\Lambda_R \longleftrightarrow \partial\Lambda_n]$ must tend to 0 as n tends to infinity. Define N as above. Now fix $n \geq N$, and explore the cluster of $\partial\Lambda_n$ in Λ_n. With probability at least $1 - \varepsilon$ this cluster does not reach $\partial\Lambda_R$, which means that it induces free boundary conditions in a random subset of Λ_n containing Λ_R. By comparison of boundary conditions, for any such set Γ, it is true that $\phi^0_\Gamma[\mathcal{A}] \leq \phi^0_{\Lambda_n}[\mathcal{A}]$. This yields:

$$\phi^1_{\Lambda_n}[\mathcal{A}] \leq \varepsilon + \phi^0_{\Lambda_n}[\mathcal{A}].$$

Letting n tend to infinity is sufficient to conclude.

□

Thus there is a unique infinite-volume limit measure in $\mathbb{Z}^d$ when $p < p_c$, and we shall refer to it as ϕ. Moreover, one knows that $\phi[0 \longleftrightarrow \partial\Lambda_n] \to 0$ when $n \to \infty$. One of the most celebrated results of the last decade is the following quantification of that rate of decay in a groundbreaking work by Duminil-Copin, Raoufi and Tassion [52] and known as the *sharpness* of the phase transition.

Theorem 2.2.3. *For any dimension $d \geq 1$, and any $q \geq 1$, $p < p_c$, there exists a constant $c > 0$ such that*

$$\phi^1_{\Lambda_n}[0 \longleftrightarrow \partial\Lambda_n] \leq \exp(-cn).$$

In particular,

$$\phi[0 \longleftrightarrow \partial\Lambda_n] \leq \exp(-cn).$$

Let us comment a bit on this result. It is easy to show that for very small values of p, this exponential decay of point-to-box probabilities holds: indeed, looking carefully at the Peierls argument implemented in Proposition 2.2.1, we established exponential decay for the *diameter* of the cluster when $p < \frac{1}{2d}$ for instance. However, there is no chance to push this reasoning all the way up to $p < p_c$, as it does not rely at all on the definition of p_c (but rather on an entropy/energy type argument). Thus, a proof of sharpness of the phase transition is much more of a challenge, and it usually involves much deeper ideas. The sharpness of the phase transition for Bernoulli percolation was proved independently by Menshikov [95] and Aizenman–Barsky [4]. Their methods, though different in nature, cannot be generalized to the random-cluster model as they rely heavily on independence. The next breakthrough came around 30 years later, with the proof by Vincent Beffara and Hugo Duminil-Copin of the sharpness of the phase transition in $\mathbb{Z}^2$ [13]. The method relies heavily on planarity of the model, as it builds on RSW-type constructions. Finally, the sharpness result was extended to a very general class of periodic graphs (including $\mathbb{Z}^d$ for $d \geq 1$) by Duminil-Copin, Raoufi and Tassion [52]. This is spectacular, as very little is known about percolation in dimensions $d \in \{3, \ldots, 6\}$[3]. The method relies on the use of a new inequality (the OSSS inequality) originally discovered in the context of theoretical computer science and randomized algorithms. This work is one the major achievements in the field of percolation during the last decade.

Remark 2.2.4. For the (very!) alert reader, it might appear contradictory that the Ornstein–Zernike Theorem described in the Introduction was proved in 2007, as the result stated is

[3]In particular, proving that $\mathbb{P}_{p_c}[0 \longleftrightarrow \infty] = 0$ for the three-dimensional Bernoulli percolation remains one the most challenging and mysterious problems in Statistical mechanics. This has been established rigorously when $d \geq 11$ [59] thanks to the so-called "mean-field theory" in which some robust tools, amongst which the so-called "lace expansion" are available (see [106]). However those methods should not work below the so-called upper critical dimension which is $d = 6$ for the Bernoulli percolation

stronger than the sharpness of the phase transition which was established in 2017. However, this is no paradox, as the Ornstein–Zernike was proved *under the assumption of exponential decay* which was not yet known to hold for any subcritical temperature.

From a physical perspective, the subcritical random-cluster model thus appears to be "trivial" in the sense that the large scale geometry of percolation clusters is degenerate: as $\mathbb{Z}^d$ is of polynomial growth, union bounds suffice to show that the volume or the diameter of the clusters decay exponentially fast in n. From a renormalization group perspective[4], a physicist would say that the model converges towards a *trivial* fixed point of the renormalization map. However, we hope to have convinced the reader in the introduction of this thesis that some interesting problems remain in the subcritical phase, amongst which are the computations of the corrections to the exponential decay of the connection probabilities and the phase separation problem. From sharpness, further properties of the subcritical regime can be derived, such as the so-called *ratio mixing property* of the model. Several ways of stating the results are possible, we chose the following one.

Proposition 2.2.5 (Exponential mixing for the subcritical random-cluster model)**.** *Let $\mathcal{A}$ be an event depending on the edges of Λ_r, for some $r \geq 0$. For any $q \geq 1$, any $p < p_c$, there exists a constant $c > 0$ such that for any boundary conditions ξ^1, ξ^2 on $\partial\Lambda_R$ with $R \geq r$ large enough,*

$$\Big|1 - \frac{\phi^{\xi^1}_{\Lambda_R}[\mathcal{A}]}{\phi^{\xi^2}_{\Lambda_R}[\mathcal{A}]}\Big| \leq \exp(-cR).$$

Several proofs of the statement are possible; in particular it has been recognized by [6] that exponential decay of the connectivity probabilities together with the Domain Markov property implies the strong ratio mixing property. However, we here closely follow the lines of [51] as the statement is slightly stronger and as it illuminates the use of increasing couplings described in Subsection 2.1.2.

Proof. Let D be a domain in Λ_R such that $\Lambda_{R/2} \subset D$. The proof uses the increasing coupling exploring the cluster of ∂D in D described in Subsection 2.1.2. Let us call Ψ the increasing coupling under which $\omega^1 \sim \phi^0_D, \omega^2 \sim \phi^{\xi^1}_D$ and let us call τ the stopping time associated to the fact that the exploration procedure stops for ω^2 before revealing an edge of Λ_r. As previously described, if $\tau < \infty$, then one can couple ω^1 and ω^2 in such a way that

[4]We will not emphasize on this notion but refer to [58] for a comprehensive review

they agree on Λ_r. As a consequence,

$$\begin{aligned}\phi_D^{\xi^1}[\mathcal{A}] - \phi_D^0[\mathcal{A}] &= \Psi[\omega_2 \in \mathcal{A}, \omega_1 \notin \mathcal{A}] \\ &\leq \Psi[\omega_2 \in \mathcal{A}, \tau = \infty] \\ &= \phi_D^{\xi^1}[\partial D \longleftrightarrow \partial \Lambda_r, \mathcal{A}] \\ &\leq \phi^1_{D\setminus\Lambda_r}[\partial D \longleftrightarrow \partial\Lambda_r]\phi_D^{\xi^1}[\mathcal{A}] \\ &\leq \exp(-c(R/2-r))\phi_D^{\xi^1}[\mathcal{A}],\end{aligned}$$

where the third inequality holds by comparison of boundary conditions and Domain Markov property, and the last one is a consequence of the sharpness result stated in Theorem 2.2.3. We obtain that $\phi_D^{\xi^1}[\mathcal{A}] \leq (1-\mathrm{e}^{-c(R/2-r)})^{-1}\phi_D^0[\mathcal{A}]$. In particular, this holds for $D = \Lambda_R$. Now by exploring the boundary conditions induced by $\phi^0_{\Lambda_R}$ on some $D \supset \Lambda_{R/2}$, it is easy to show that

$$\phi^0_{\Lambda_R}[\mathcal{A}] \leq (1-\mathrm{e}^{-c(R/2-r)})^{-1}\phi_D^0[\mathcal{A}]. \tag{2.3}$$

We proved that $\phi^{\xi^1}_{\Lambda_R}[\mathcal{A}] - \phi^0_{\Lambda_R}[\mathcal{A}] \leq \phi^{\xi^1}_{\Lambda_R}[\mathcal{A}]\mathrm{e}^{-c(R/2-r)}$ and are now aiming for a converse bound. Consider once again the increasing coupling Ψ exploring the cluster of $\partial\Lambda_R$ both in $\omega^1 \sim \phi^0_{\Lambda_R}$ and in $\omega^2 \sim \phi^{\xi^1}_{\Lambda_R}$ in an increasing fashion, and let now $\tilde{\tau}$ be the stopping time associated to the event that the exploration stops in ω^2 before reaching $\partial\Lambda_{R/2}$. In the case that $\{\tau < \infty\}$, observe that it induces free boundary conditions on a random subset $\Omega \supset \Lambda_{R/2}$, and we can use (2.3) to argue that

$$\begin{aligned}\phi^{\xi^1}_{\Lambda_R}[\mathcal{A}] &= \sum_{D\supset\Lambda_{R/2}} \phi^0_D[\mathcal{A}]\Psi[\Omega = D] + \Psi[\tilde{\tau} = \infty, \omega_2 \in \mathcal{A}] \\ &\geq (1-\mathrm{e}^{-c(R/2-r)})\phi^0_{\Lambda_R}[\mathcal{A}]\Psi[\tilde{\tau} < \infty] \\ &= (1-\mathrm{e}^{-c(R/2-r)})\phi^0_{\Lambda_R}[\mathcal{A}](1-\phi^{\xi^1}_{\Lambda_R}[\partial\Lambda_R \longleftrightarrow \partial\Lambda_{R/2}]) \\ &\geq (1-\mathrm{e}^{-cR/2})(1-\mathrm{e}^{-c(R/2-r)})\phi^0_{\Lambda_R}[\mathcal{A}].\end{aligned}$$

Thus we got that

$$\phi^{\xi^1}_{\Lambda_R}[\mathcal{A}] \geq (1-\mathrm{e}^{-c(R/2-r)})^2\phi^0_{\Lambda_R}[\mathcal{A}].$$

We used that $\phi^{\xi^1}_{\Lambda_R}[\partial\Lambda_R \longleftrightarrow \partial\Lambda_{R/2}] \leq \phi^1_{\Lambda_R}[\partial\Lambda_R \longleftrightarrow \partial\Lambda_{R/2}] \leq \mathrm{e}^{-cR/2}$ by the sharpness Theorem. This shows that — up to altering the constant $c > 0$,

$$\phi^0_{\Lambda_R}[\mathcal{A}] - \phi^{\xi^1}_{\Lambda_R}[\mathcal{A}] \leq \mathrm{e}^{-cR}\phi^0_{\Lambda_R}[\mathcal{A}]$$

Gathering the pieces together, we proved that

$$|\phi^{\xi^1}_{\Lambda_R}[\mathcal{A}] - \phi^0_{\Lambda_R}[\mathcal{A}]| \leq \exp(-cR)\phi^0_{\Lambda_R}[\mathcal{A}].$$

The desired claim follows by the triangle inequality. Indeed,

$$|\phi^{\xi^1}_{\Lambda_R}[\mathcal{A}] - \phi^{\xi^2}_{\Lambda_R}[\mathcal{A}]| \leq 2\exp(-cR)\phi^{0}_{\Lambda_R}[\mathcal{A}] \leq 2\exp(-cR)(1-\exp(-cR))^{-2}\phi^{\xi^1}_{\Lambda_R}[\mathcal{A}],$$

and the conclusion holds altering once again the value of $c > 0$. □

As a side remark, observe that the only place in which we use the subcriticality of the model is when invoking the sharpness result. That means that good upper bounds on the box-to-box connection probabilities can automatically be converted in good mixing statements. In particular, in [51], the authors use the polynomial decay of the 1-arm probabilities at criticality to derive a polynomial ratio mixing for the *planar* random-cluster measures at *any* value of the parameter p.

2.2.2 The phase transition of the random-cluster model on a planar graph

As previously mentioned, the phase transition of the random-cluster is quite poorly understood in dimensions $d \geq 3$. In particular, the properties of the *critical* and *supercritical* phases remain mysterious (for the subcritical phase however, one can safely say that the sharpness statement provides a very good understanding of the geometry of the regime). However, things are special in dimension 2 as $\mathbb{Z}^2$ is a *planar* graph, and a special tool is available in that setting: the so-called *planar duality*. In that case, we are going to see that one is able to compute the exact value of the critical point, and that the question of the continuity or discontinuity of the phase transition is solved.

Planar duality

Consider the translated lattice $(\mathbb{Z}^2)^* := (1/2, 1/2) + \mathbb{Z}^2$. Observe that each edge e of the original lattice $\mathbb{Z}^2$ intersects a unique edge of $(\mathbb{Z}^2)^*$, that we call e^*. To any percolation configuration $\omega \in \{0,1\}^{\mathbb{Z}^2}$ we associate a percolation configuration ω^* called the *dual configuration* by setting

$$\forall e^* \in E((\mathbb{Z}^2)^*), \omega^*(e^*) = 1 - \omega(e).$$

Assume for a minute that $\omega \sim \phi_{p,1}$ (*i.e.*, that we are interested in Bernoulli percolation). Then it is immediate that $\omega^* \sim \phi_{1-p,1}$. This is called a *duality relation*. It hints that 1/2 plays a special role in the model, as in that case $\omega \sim \omega^*$ (this point is called the self-dual point). Such a duality result is also available for the random-cluster model, but the role of boundary conditions cannot be ignored, and duality only works for so-called *planar boundary conditions*.

Let G be a finite subgraph of $\mathbb{Z}^2$ and ξ be a boundary condition on ∂G. ξ will be said to be *planar* if it can be induced by some percolation configuration η on $\mathbb{Z}^2 \setminus G$. In that case, we define the dual boundary condition ξ^* as the boundary condition induced by η^* on ∂G^*. The duality relation then takes the following form

Lemma 2.2.6. *For ξ a planar boundary condition on ∂G and $\omega \sim \phi^{\xi}_{G,p,q}$, the law of its dual configuration is given by $\omega^* \sim \phi^{\xi^*}_{G^*,p^*,q}$, where p^* is uniquely defined by the equation*

$$pp^* = q(1-p)(1-p^*). \tag{2.4}$$

Proof. We only treat the case of free boundary conditions and G connected. The proof relies on the famous *Euler formula*. Indeed, recall that we call $o(\omega)$ the number of open edges of a configuration ω and $k(\omega)$ the number of its open clusters. Let us also call $f(\omega)$ the number of faces of the configuration ω (observe that due to the fact that we are looking at free boundary conditions for ω, ω^* possesses a unique infinite face). Now note that $f(\omega^*) = k(\omega)$. Thus, Euler's relation becomes

$$k(\omega) = |V| + f(\omega) - o(\omega) - 1$$

We will also use the fact that $o(\omega) + o(\omega^*) = |E|$ to obtain that

$$\begin{aligned}\phi^0_{G,p,q}(\omega) &\propto \left(\frac{p}{1-p}\right)^{o(\omega)} q^{f(\omega^*)} \\ &\propto \left(\frac{1-p}{p}\right)^{o(\omega^*)} q^{o(\omega^*)+k(\omega^*)} \\ &\propto \left(\frac{(1-p)q}{p}\right)^{o(\omega^*)} q^{k(\omega^*)}.\end{aligned}$$

Thus, when $\frac{p^*}{(1-p^*)} = q\frac{1-p}{p}$ we obtain the desired identity. □

Observe that the unique point such that $p = p^*$ (the so-called *self-dual* point) can be shown to be equal to

$$p_{\mathsf{sd}} = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

It is a famous theorem of Vincent Beffara and Hugo Duminil-Copin [13] that in $\mathbb{Z}^2$, when $q \geq 1$, the critical point and the self-dual point coincide.

Theorem 2.2.7. *When $q \geq 1$, for the planar random-cluster model,*

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$$

We mention another very important result proved in the last decades regarding the *continuity* of the phase transition *i.e.*, the question of determining whether there is a unique infinite volume measure at criticality, or equivalently to decipher wehther $\phi^1_{p_c,q}[0 \longleftrightarrow \infty]$ is positive. Surprisingly, it turns out that the answer to that question depends on the value of q. Indeed, one has the following result.

Theorem 2.2.8 ([53, 50])**.** *Consider the critical random-cluster model at $p = p_c$.*

1. *For $q \in [1, 4]$, $\phi^0 = \phi^1$. Equivalently, $\phi^1[0 \longleftrightarrow \infty] = 0$.*
2. *For $q > 4$, $\phi^0 \neq \phi^1$. Furthermore, $\phi^1[0 \longleftrightarrow \infty] > 0$.*

We conclude this list of results by mentioning a recent theorem by Alexander Glazman and Ioan Manolescu [65] providing a total classification of the infinite-volume limit measures of the random-cluster model. Recall that this set is convex, and is thus by the Krein–Milman Theorem the convex hull of its extremal points. It is then sufficient to identify the latter to totally understand the set of Gibbs states of the model. Due to the previous result, the only case in which there is more than a single infinite-volume limit measure is when $q > 4$ and $p = p_c(q)$.

Theorem 2.2.9. *Let $q > 4$ and $p = p_c(q)$. The measures ϕ^0 and ϕ^1 are distinct and extremal. Moreover, for any infinite-volume limit measure ϕ of the planar random-cluster model, there exists some $\lambda \in [0, 1]$ such that*

$$\phi = \lambda\phi^0 + (1 - \lambda)\phi^1.$$

Chapter 3

The phase separation at a mesoscopic scale: a random walk example

3.1 Introduction

The phase separation problem is concerned with the study of the boundary appearing between two different phases of a statistical mechanics model, in a regime where those two phases can coexist. In his seminal work [113], Wulff proposed that such a boundary should macroscopically adopt a deterministic limit shape given by the solution of a variational problem involving thermodynamic quantities such as the surface tension. This prediction has been an object of intense study, and has by now been made rigorous in a very wide variety of settings, see for instance the monographs [47, 15, 29].

While this macroscopic shape is dependent on the model, the *fluctuations* of the random phase boundary are widely believed to behave in an universal way. A first natural candidate to measure these fluctuations is the deviations from the limit shape. These have been shown to be Gaussian in [45] in the context of area-constrained random walks, and later on in [46] for the 2D Ising phase boundary at low temperatures. Two other quantities of interest are given by the *maximal facet length* and the *maximal local roughness* of the interface, that are respectively the length of the largest segment of the convex hull of the interface and the maximal deviation of the interface from this convex hull. In a seminal paper [5], Alexander conjectured that the exponent governing the scaling of the maximal local roughness should be $1/3$. In the context of percolation models in the phase separation setting, he derived upper bounds for an averaged version of the local roughness. Alexander and Usun [108] then complemented this work by providing lower bounds for the local roughness in Bernoulli percolation. Later on, in a remarkable series of papers, Hammond [73, 74, 75] was able to identify the exact scale of the maximal facet length and the maximal local roughness of a droplet of volume N^2 in the planar subcritical random-cluster model. Indeed, he proved that the former is of order $N^{2/3}(\log N)^{1/3}$, while the latter is of order $N^{1/3}(\log N)^{2/3}$, validating the exponent derived in [5]. These results are built on the identification of $N^{2/3}$ as the scale at which the curvature effect enforced by the conditioning has the same order of magnitude as the Gaussian fluctuations of the interface. Finally, let us mention that in an earlier paper, Hammond and Peres [78] introduced a continuous and Brownian version of the phase separation problem, studying a two-dimensional Brownian loop conditioned to enclose a large area. They proved results in favour of the appearance of the cube-root fluctuations in this setting.

In this paper, we study a model of random walks, with geometrically randomised length, conditioned to enclose an area at least equal to N^2. This model was suggested by Hammond in [74, Section 1.0.4] and later in [77]. We prove that, as predicted by the author, his techniques can successfully be applied to this setting, enabling us to establish the above-mentioned polylogarithmic corrections. However, the main innovation of this paper is the identification of the scaling of the *typical* facet length and local roughness (rather than their maximal values). Call MeanFL (resp. MeanLR) the length (resp. the local roughness) of the facet intersecting a given line. We prove that $\mathsf{MeanFL} \asymp N^{2/3}$ and $\mathsf{MeanLR} \asymp N^{1/3}$, see Theorem 3.1.3.

The scaling exponents $1/3$ and $2/3$ have been shown to arise in various related contexts in statistical mechanics. An important example is the *critical prewetting* in the Ising model, which has been extensively studied in a beautiful series of papers [80, 109, 86, 64, 83] to mention a few of them. A remarkable aspect of these works is that the results are derived without the help of any integrable feature. Let us also note that [26, 18] provided strong evidence for a similar behaviour in the context of the SOS model above a wall in $(2 + 1)$ dimensions.

To the best of our knowledge, it is the first time that those exponents are identified in a context such as ours. Indeed, in the above-mentioned works, the interface lies above a facet of length much longer than $N^{2/3}$ — this facet being deterministic and artificially created by looking at the interface along a side of a large box for instance. However, in our work, the facets are not deterministic, and themselves reflect the competition between the randomness and the curvature induced by the conditioning.

Finally, we strongly believe (supported by [74]) that our approach is robust and should allow to derive the same result for a large variety of models, including the fluctuations of the outermost circuit in a subcritical random-cluster model conditioned to enclose a large area, which itself is a good toy model for the boundary of a droplet in a supercritical Potts model. Note that this suggests that at scale $N^{2/3}$, our model and more general phase boundaries models should lie in the same universality class. This is discussed in Section 3.5.

3.1.1 Definition of the model and statement of the main results

Let Λ be the set of finite paths in the first quadrant of $\mathbb{Z}^2$ which start on the y-axis and end on the x-axis, and which are oriented in the sense that they only take rightward and downward steps, see Figure 3.1. For $\gamma \in \Lambda$, we define $|\gamma|$ to be its length, i.e. its number of steps (which is also the sum of the y-coordinate of its starting point and the x-coordinate of its ending point). It will be convenient to identify an element $\gamma \in \Lambda$ with the set of points of $\mathbb{N}^2$ it passes by, i.e $\gamma = (\gamma(k))_{0 \leq k \leq |\gamma|}$. We set Λ_n to be the subset of Λ of such oriented paths of length n. It is clear that $|\Lambda_n| = 2^n$. If $a, b \in \mathbb{N}^2$, we denote by $\Lambda^{a \to b}$ the set of downright paths from a to b.

Let $0 < \lambda < \frac{1}{2}$. We define a probability measure $\mathbb{P}_\lambda$ on Λ by requiring that, for $\gamma \in \Lambda$,

$$\mathbb{P}_\lambda[\gamma] = \frac{1}{Z_\lambda} \lambda^{|\gamma|},$$

where Z_λ is the normalisation constant given by $Z_\lambda = (1 - 2\lambda)^{-1}$.

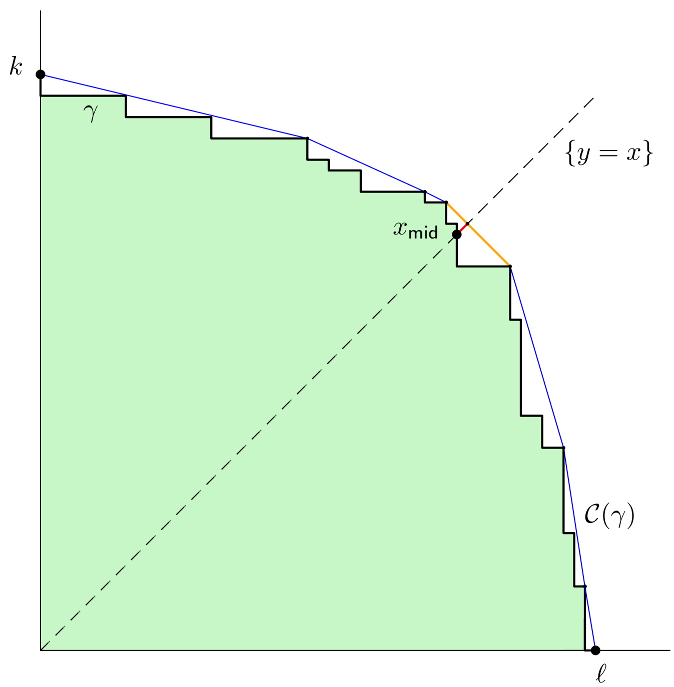

Figure 3.1: An example of path γ (in bold black) with $|\gamma| = k + \ell$, and its least concave majorant $\mathcal{C}(\gamma)$ (in blue). The mean facet of γ is coloured in orange, while the mean local roughness is the length of the red segment. The area $\mathcal{A}(\gamma)$ is the area of the light green shaded region.

The probability measure $\mathbb{P}_\lambda$ is to be seen as a *background* measure which imposes an exponential decay of the probability of sampling a long path. We now enforce a competing constraint of enclosing a large area. For $\gamma \in \Lambda$, we define the area of γ, denoted by $\mathbb{A}(\gamma)$, to be the area enclosed by the graph of γ and the two coordinate axes (see Figure 3.1). Let $N \in \mathbb{N}$ and let Γ be a sample of $\mathbb{P}_\lambda$. We define a new probability measure $\mathbb{P}_\lambda^{N^2}$ on Λ by

$$\mathbb{P}_\lambda^{N^2}[\,\cdot\,] := \mathbb{P}_\lambda\big[\,\cdot \mid \mathbb{A}(\Gamma) \geq N^2\big].$$

We also introduce $\Lambda^{N^2} \subset \Lambda$, the set of all paths of Λ that capture an area greater than N^2 (this set is exactly the support of $\mathbb{P}_\lambda^{N^2}$). Samples of $\mathbb{P}_\lambda^{N^2}$ typically experience a competition between capturing a large area and minimizing the length of the paths. As explained before, it suggests that for large values of N, a typical sample of $\mathbb{P}_\lambda^{N^2}$ will have a global curvature imposed by the area condition and Gaussian fluctuations imposed by the background measure (once the length is fixed, the background measure is the uniform law). The object of interest in our work, which tracks the competition between these two phenomena, is defined below.

Definition 3.1.1 (Least concave majorant, facets)**.** Let $\gamma \in \Lambda$. We set $\mathsf{Ext}(\gamma)$ to be the set of non-zero extremal points of the convex hull of $\gamma \cup \{(0,0)\}$. Let $\mathcal{C}(\gamma)$ be the *least concave majorant* of γ, that is, the graph of the piecewise affine function passing through all the points of $\mathsf{Ext}(\gamma)$. Observe that with this definition, $\mathcal{C}(\gamma)$ becomes a finite union of line segments of $\mathbb{R}^2$ which are called *facets* (see Figure 3.1), the endpoints of these facets precisely being the elements of $\mathsf{Ext}(\gamma)$. For $x \in \gamma$, we define its *local roughness* to be its Euclidean distance to $\mathcal{C}(\gamma)$,

$$\mathsf{LR}(x) := \mathrm{d}(x, \mathcal{C}(\gamma)).$$

Our main result concerns the tightness of the length of a typical facet intercepting a given ray emanating from the origin (resp. the fluctuations of Γ along the facet) at scale $N^{2/3}$ (resp. $N^{1/3}$). Let us first define the quantities of interest.

Definition 3.1.2 (MeanFL, MeanLR)**.** Let $\gamma \in \Lambda$. We define $\mathsf{MeanFac}(\gamma)$ to be the unique facet intersecting the line $\{y = x\}$ (if we are in the case where the ray $\{y = x\}$ intercepts the endpoint of two consecutive facets, we arbitrarily choose the leftmost one for $\mathsf{MeanFac}(\gamma)$). We call $x_{\mathsf{mid}}(\gamma)$ the point of $\mathbb{N}^2 \cap \gamma$ that is the closest to the line $\{y = x\}$ (as previously, in case of conflict, we arbitrarily choose the leftmost one). Assume that $\mathsf{MeanFac} = [a,b](= \{at + (1-t)b,\ t \in [0,1]\})$, with $a, b \in \mathbb{N}^2$. Define the following random variables,

$$\mathsf{MeanFL}(\gamma) := \mathrm{d}(a,b), \qquad \mathsf{MeanLR}(\gamma) := \mathsf{LR}(x_{\mathsf{mid}}).$$

In words, MeanFL is the length of $\mathsf{MeanFac}$ and MeanLR is the local roughness of the "mean vertex" of γ.

The following result is the main contribution of the paper.

Theorem 3.1.3 (Tightness of MeanFL and MeanLR at scales $N^{2/3}$ and $N^{1/3}$)**.** *Let* $0 < \lambda < \frac{1}{2}$. *For any* $\varepsilon > 0$, *there exist* $c, C > 0$ *and* $N_0 \in \mathbb{N}$ *such that for any* $N \geq N_0$,

$$\mathbb{P}_\lambda^{N^2}\big[cN^{\frac{2}{3}} < \mathsf{MeanFL}(\Gamma) < CN^{\frac{2}{3}}\big] > 1 - \varepsilon,$$

and,

$$\mathbb{P}_\lambda^{N^2}\left[cN^{\frac{1}{3}} < \mathsf{MeanLR}(\Gamma) < CN^{\frac{1}{3}}\right] > 1-\varepsilon.$$

We chose for convenience to formulate Theorem 3.1.3 in terms of the statistics of MeanFac. However, the result is not specific to the choice of the line $\{y = x\}$: for any $\alpha > 0$, the result holds for $\mathsf{MeanFac}_\alpha$, defined to be the unique facet intercepting the line $\{y = \alpha x\}$ (however the constants c, C might now depend on α). Indeed, the proof never uses the symmetry of the model around the line $\{y = x\}$.

Remark 3.1.4. In the proof of Theorem 3.1.3, we actually derive a slightly stronger statement. Indeed, we obtain stretch-exponential upper tails for MeanFL and MeanLR (see Propositions 3.3.1 and 3.3.2)

Following [74] and [75], our second result identifies the logarithmic corrections to the *maximal* facet length and to the *maximal* local roughness along $\mathcal{C}(\gamma)$. We call these quantities $\mathsf{MaxFL}(\gamma)$ (resp. $\mathsf{MaxLR}(\gamma)$).

Theorem 3.1.5. *Let* $0 < \lambda < \frac{1}{2}$. *There exist* $c, C > 0$ *such that,*

$$\mathbb{P}_\lambda^{N^2}\left[cN^{\frac{2}{3}}(\log N)^{\frac{1}{3}} < \mathsf{MaxFL}(\Gamma) < CN^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\right] \xrightarrow[N\to\infty]{} 1, \tag{3.1}$$

and

$$\mathbb{P}_\lambda^{N^2}[cN^{\frac{1}{3}}(\log N)^{\frac{2}{3}} < \mathsf{MaxLR}(\Gamma) < CN^{\frac{1}{3}}(\log N)^{\frac{2}{3}}] \xrightarrow[N\to\infty]{} 1.$$

3.1.2 Related works and known results

As pointed out in the introduction, similar models have been studied quite extensively in the literature, especially in [45], where they are introduced as toy models for the study of a low-temperature interface of a (1+1)-dimensional SOS model. In this work, the authors investigate the behaviour of the model at the macroscopic scale N and at the mesoscopic scale $N^{1/2}$. It is possible to extend their result to our setup. To properly state it, we introduce the following parametrisation of Γ: let $\Gamma(t)$ be the linear interpolation between the points $\Gamma(k)$ for $0 \le k \le |\Gamma|$. Using the discussion of [45, Section 1], together with the basic estimates given by Lemma 3.2.1 and Proposition 3.2.4, one can obtain the following result.

Theorem 3.1.6. *Let* $0 < \lambda < \frac{1}{2}$. *There exists a deterministic, concave and continuous function* $f_\lambda : [0,1] \to \mathbb{R}^+$ *such that for any* $\varepsilon > 0$,

$$\mathbb{P}_\lambda^{N^2}\left[\sup_{t\in[0,1]} \left|N^{-1}\Gamma_N(|\Gamma_N|t) - f_\lambda(t)\right| > \varepsilon\right] \xrightarrow[N\to\infty]{} 0. \tag{3.2}$$

Remark 3.1.7. Using Lemma 3.2.1, the time scaling in (3.1.6) is linear in N.

Such a phenomenon is by now very well known under the name of a *limit shape phenomenon*, and is known to arise in a large variety of situations (see for instance [100, 90]). In (1+1) dimensions such as in our setting, two different points of view can be adopted to prove a statement such as Theorem 3.1.6: the first one is the classical theory of sample path large deviations culminating with the celebrated Mogulski'i Theorem (see [42, Section 5]). The second possible point of view has a more statistical mechanics flavour and is known as *Wulff theory* (see the reference monograph [47]). In the work [45], it is shown that both approaches can be implemented and yield the same result. In both cases, the function f_λ is identified as the minimiser of a deterministic variational problem. Let us conclude this discussion by noticing that actually much stronger statements than (3.1.6) can be obtained, and in particular large deviations principles for the sample path Γ, though we will not focus on results of this type.

The main result of [45, Theorem 2.1] focuses on the fluctuations of Γ_N around f_λ. Again, minor modifications of their proof lead to the following result.

Theorem 3.1.8. *Let $0 < \lambda < \frac{1}{2}$. There exists a (centered) Gaussian process ξ_λ on the space $\mathcal{C}([0,1])$ (equipped with the topology of the uniform convergence) such that under the measure $\mathbb{P}_\lambda^{N^2}$,*

$$\Big(\frac{1}{\sqrt{N}}(\Gamma_N(|\Gamma_N|t) - Nf_\lambda(t))\Big)_{t\in[0,1]} \xrightarrow[N\to\infty]{(d)} \xi_\lambda,$$

where the convergence holds in distribution.

These two results can be stated heuristically as follows: at the macroscopic scale N, the conditioning on the event $\{\mathcal{A}(\Gamma) \geq N^2\}$ enforces a deterministic and global curvature, whereas at the mesoscopic scale $N^{1/2}$ the conditioning has no effect on $\mathbb{P}_\lambda$ and the Gaussian nature of the measure $\mathbb{P}_\lambda$ is unchanged. As explained above, Theorem 3.1.3 identifies $N^{2/3}$ as being the scale at which those two competing effects are of the same order.

3.1.3 A RESAMPLING STRATEGY

The proofs below heavily relies on a particularly simple (yet crucial) feature of the model: the so-called *Brownian Gibbs property* (see [37]). It can be stated as follows: start from a sample Γ of $\mathbb{P}_\lambda^{N^2}$, choose two points $a, b \in \Gamma$ with any random procedure "explorable from the exterior[1] of the path" (by this we mean that the event $\{a = x, b = y\}$ is measurable with respect to $\Gamma \setminus \Gamma_{x,y}$). Then, conditionally on (a, b) and $\Gamma \setminus \Gamma_{a,b}$, the distribution of the random variable $\Gamma_{a,b}$ is the uniform distribution on $\Lambda^{a\to b}$ **conditionally on the fact that the resulting path of Λ encloses an area greater than N^2**. This apparently naive observation allows one to implement a strategy of *resampling*. Indeed, one can construct several Markovian dynamics on Λ leaving the distribution $\mathbb{P}_\lambda^{N^2}$ invariant in a quite general fashion: start from a sample Γ of $\mathbb{P}_\lambda^{N^2}$, choose two points $a, b \in \Gamma$ according to the above procedure and replace $\Gamma_{a,b}$ by a sample of the uniform distribution of $\Gamma_{a,b}$ subject to the

[1] This terminology is directly inspired by the notion of explorable set in percolation theory.

above-mentioned conditioning. Then, it is clear, thanks to the preceding observation, that the distribution of the output is $\mathbb{P}_\lambda^{N^2}$. In what follows, we shall call such a dynamic on Λ a *resampling procedure*.

This point of view will be used several times in the proofs to analyse marginals of $\mathbb{P}_\lambda^{N^2}$ in well-chosen regions of the first quadrant. A simple illustration is given in the proof of Proposition 3.2.10.

3.1.4 Open problems

In light of the preceding discussions, two natural questions arise. We now describe them.

As explained above, we expect the methods developed in this article to help the analysis of the droplet of a subcritical planar random-cluster model. We strongly believe that our strategy could complement the results obtained by Hammond in [73, 74, 75]. We provide strong heuristics towards this result in Section 3.5.

Open Problem A (Droplet boundary in the subcritical planar random-cluster model)**.** Extend Theorem 3.1.3 to the study of a typical facet in the setup of the droplet in the subcritical planar random-cluster model.

Once tightness is obtained in Theorem 3.1.3, it is quite natural to try to identify the candidate for the scaling limit of the excursion below a typical facet. A similar question has been answered in [83] where the authors obtained convergence (in the bulk phase) of the object of study to the so-called *Ferrari–Spohn diffusion*. In our setup, the excursion below the mean facet should belong to the same universality class, the only difference being that the process is now an excursion. We intend to study this question in the future.

Open Problem B (Scaling limit of the excursion below a facet)**.** Construct the Ferrari–Spohn excursion (FSE). Prove that the piece of path lying below the mean facet, after a proper rescaling, converges to the FSE under $\mathbb{P}_\lambda^{N^2}$ in the limit $N \to \infty$.

Compared to [83], the study of Open Problem B requires two additional ingredients. Indeed, one needs a fine understanding of the way that the $O(1)N^{1/3}$ excursions of size $N^{2/3}$ interact together along the convex hull of the droplet. Moreover, the construction of FSE and the proof of the convergence of the mean excursion towards it are non-trivial as the pinning condition takes place *at the scale of the correlation length* of the system.

3.1.5 Organisation of the paper

The paper is organised as follows: in Section 3.2, we gather some preliminary results that are going to be our toolbox for the proofs of the main results. Section 3.3 is then devoted to the proof of Theorem 3.1.3, while Section 3.4 consists in an adaptation of the arguments of [74, 75] to prove Theorem 3.1.5. Finally, Section 3.5 is devoted to a discussion regarding the extension of the results to other statistical mechanics models in the Wulff setting.

Notations and conventions. We shall adopt Landau formalism for real valued-sequences. Namely, whenever (a_n) and (b_n) are two real-valued sequences, we will write $a_n = o(b_n)$ when $|a_n|/|b_n| \xrightarrow[n\to\infty]{} 0$. We will also use the notation $a_n = O(b_n)$ when there exists some constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all n large enough. If $a_n = O(b_n)$ and $b_n = O(a_n)$, we shall write that $a_n \asymp b_n$. Finally we shall write $a_n \sim b_n$ when $a_n/b_n \xrightarrow[n\to\infty]{} 1$.

If A is a set, we denote by $\mathcal{P}(A)$ the power set of A. For $x = (x_1, x_2) \in \mathbb{R}^2$, $\|x\| := \sqrt{x_1^2 + x_2^2}$ denotes the Euclidean norm of x. If $S \subset \mathbb{R}^2$ is a Borel set, we denote by $|S|$ its Lebesgue measure. Moreover, for $x \in \mathbb{Z}^2$ we will write $\arg(x) \in [0, 2\pi)$ to denote the complex argument of x seen as an element of $\mathbb{C}$. For $t \in \mathbb{R}$, $\lfloor t \rfloor$ will denote the integer part of t and $\lceil t \rceil := \inf\{k \in \mathbb{Z},\ k \geq t\}$.

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3.2 Preliminary results

In the rest of this work, we fix $0 < \lambda < \frac{1}{2}$.

3.2.1 Basic statistics of typical samples of $\mathbb{P}_\lambda^{N^2}$

In this subsection, we study some basic properties of a typical sample of $\mathbb{P}_\lambda^{N^2}$.

Lemma 3.2.1 (Tail estimates for the length of a sample of $\mathbb{P}_\lambda^{N^2}$)**.** *There exist $c, C > 0$ such that for any $N \geq 1$, any $t \geq 0$,*

$$\mathbb{P}_\lambda^{N^2}\left[|\Gamma| \geq tN\right] \leq C\mathrm{e}^{-cN(t-2\sqrt{2})}.$$

Proof. It is clear that

$$\mathbb{P}_\lambda^{N^2}\left[|\Gamma| \geq tN\right] \leq \frac{\mathbb{P}_\lambda\left[|\Gamma| \geq tN\right]}{\mathbb{P}_\lambda\left[\mathcal{A}(\Gamma) \geq N^2\right]}.$$

An easy computation yields

$$\mathbb{P}_\lambda\left[|\Gamma| \geq tN\right] = \frac{1}{Z_\lambda} \sum_{k \geq \lfloor tN \rfloor} (2\lambda)^k = (2\lambda)^{\lfloor tN \rfloor}.$$

It remains to lower bound $\mathbb{P}_\lambda\left[\mathcal{A}(\Gamma) \geq N^2\right]$. Let $\mathsf{y}(\Gamma)$ (resp. $\mathsf{x}(\Gamma)$) be the y-coordinate (resp. x-coordinate) of the first (resp. last) vertex of Γ. The measure $\mathbb{P}_\lambda$ conditioned on $(\mathsf{y}(\Gamma), \mathsf{x}(\Gamma))$ is exactly the uniform measure over path starting at $(0, \mathsf{y}(\Gamma))$ and ending at $(\mathsf{x}(\Gamma), 0)$. We claim that

$$\mathbb{P}_\lambda\big[\mathcal{A}(\Gamma) \geq N^2 \mid (\mathsf{y}(\Gamma), \mathsf{x}(\Gamma)) = (\lceil \sqrt{2}N \rceil, \lceil \sqrt{2}N \rceil)\big] \geq \frac{1}{2}.$$

Indeed, the square formed by the vertices $(0, \lceil \sqrt{2}N \rceil), (\lceil \sqrt{2}N \rceil, \lceil \sqrt{2}N \rceil), (\lceil \sqrt{2}N \rceil, 0)$ and $(0,0)$ has an area at least equal to $2N^2$. Hence, a symmetry argument shows that the proportion of oriented paths starting at $(0, \lceil \sqrt{2}N \rceil)$ and ending at $(\lceil \sqrt{2}N \rceil, 0)$ that fulfill the requirement $\left\{\mathcal{A}(\Gamma) \geq N^2\right\}$ is at least $1/2$. By a standard computation, we find $c_1 = c_1(\lambda) > 0$ such that, for all $N \geq 1$,

$$\mathbb{P}_\lambda\left[(\mathsf{y}(\Gamma), \mathsf{x}(\Gamma)) = (\lceil \sqrt{2}N \rceil, \lceil \sqrt{2}N \rceil)\right] = \frac{1}{Z_\lambda} \lambda^{2\lceil \sqrt{2}N \rceil} \binom{2\lceil \sqrt{2}N \rceil}{\lceil \sqrt{2}N \rceil} \geq c_1 (2\lambda)^{2\sqrt{2}N} N^{-1/2}.$$

Putting all the pieces together

$$\mathbb{P}_\lambda^{N^2}[|\Gamma| \geq tN] \leq c_1^{-1} \sqrt{N} (2\lambda)^{tN} (2\lambda)^{-2\sqrt{2}N}.$$

The proof follows readily. □

This tail estimate allows us to argue that a typical sample of $\mathbb{P}_\lambda^{N^2}$ stays confined between two balls of linear radii with very high probability. For $K \geq 0$, let B_K be the Euclidean ball of radius K centered at 0.

Lemma 3.2.2 (Confinement lemma)**.** *There exist $K_1, K_2, c, C > 0$ such that for all $N \geq 1$,*

$$\mathbb{P}_\lambda^{N^2}[\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}] \geq 1 - C\mathrm{e}^{-cN}.$$

Proof. Using Lemma 3.2.1 and a simple geometric observation, we get that for $K_1 \in \mathbb{N}_{>0}$,

$$\begin{aligned} \mathbb{P}_\lambda^{N^2}\left[\Gamma \cap (B_{K_1 N})^c \neq \emptyset\right] &\leq \mathbb{P}_\lambda^{N^2}[|\Gamma| \geq K_1 N] \\ &\leq C\mathrm{e}^{-c(K_1 - 2\sqrt{2})N}, \end{aligned}$$

which gives that $\Gamma \subset B_{K_1 N}$ with high probability for $K_1 > 2\sqrt{2}$.

For the second part of the statement, notice that the area of a path that is contained in $B_{K_1 N}$ and intersects B_{KN} (for $K < K_1$) is necessarily smaller than $2K_1 K N^2$. Choosing $K_2 < 1/(2K_1)$ so that the preceding quantity is smaller than N^2 yields that

$$\mathbb{P}_\lambda^{N^2}\left[\Gamma \cap B_{K_2 N} \neq \emptyset, \Gamma \cap (B_{K_1 N})^c = \emptyset\right] = 0.$$

Hence,

$$\begin{aligned}\mathbb{P}_\lambda^{N^2}[\Gamma \cap B_{K_2N} \neq \emptyset] &= \mathbb{P}_\lambda^{N^2}\left[\Gamma \cap B_{K_2N} \neq \emptyset, \Gamma \cap (B_{K_1N})^c \neq \emptyset\right] + 0 \\ &\leq \mathbb{P}_\lambda^{N^2}\left[\Gamma \cap (B_{K_1N})^c \neq \emptyset\right],\end{aligned}$$

and the result follows. □

In what follows, K_1, K_2 will always denote the constants given by Lemma 3.2.2.

The following quantity will be of particular interest for the rest of this work.

Definition 3.2.3 (Excess area)**.** The *excess area* of a path $\gamma \in \Lambda^{N^2}$ is the quantity $\mathsf{ExcessArea}(\gamma)$ defined by

$$\mathsf{ExcessArea}(\gamma) := \mathcal{A}(\gamma) - N^2.$$

Since $\mathbb{P}_\lambda^{N^2}$exponentially penalizes long paths, we may expect the typical area of a path to be close to N^2. The following result quantifies this observation and will be very useful later.

Proposition 3.2.4 (Tail behaviour of $\mathsf{ExcessArea}$)**.** *There exists $c > 0$ such that for all $0 \leq t \leq N$,*

$$\mathbb{P}_\lambda^{N^2}[\mathsf{ExcessArea}(\Gamma) \geq tN] \leq 2\mathrm{e}^{-ct}.$$

The idea of the proof is simple. By Lemma 3.2.1, a sample Γ of $\mathbb{P}_\lambda^{N^2}$ has a length of order N. Assume that this path satisfies $\mathbb{A}(\Gamma) > N^2 + tN$. If we remove its first t steps we obtain a path of area (roughly) at least N^2 and which is exponentially (in t) favoured by $\mathbb{P}_\lambda^{N^2}$.

Proof. If A is any subset of Λ^{N^2}, we shall write $Z_\lambda^{N^2}[A] = \sum_{\gamma \in A} \lambda^{|\gamma|}$ for the partition function of A. We will also write $Z_\lambda^{N^2} := Z_\lambda^{N^2}[\Lambda^{N^2}]$. By definition,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{ExcessArea}(\Gamma) \geq tN,\ \Gamma \subset B_{K_1N} \backslash B_{K_2N}] = \frac{1}{Z_\lambda^{N^2}} \sum_{K_2N \leq a,b \leq K_1N} \lambda^{a+b} \sum_{\substack{\gamma:(0,a)\to(b,0) \\ \mathcal{A}(\gamma) \geq N^2+tN}} 1.$$

If $\gamma : (0,a) \to (b,0)$ with $K_2N \leq a, b \leq K_1N$, call $\mathsf{c} = \mathsf{c}(\gamma)$ the y-coordinate of the point of γ of x-coordinate $\lfloor t/K_1 \rfloor$. Splitting the path γ at the point of coordinates $(\lfloor t/K_1 \rfloor, \mathsf{c})$ splits γ into a pair of elements of Λ (after translation) that we denote by (γ_1, γ_2), where γ_1 is a path from $(0, a - \mathsf{c})$ to $(\lfloor t/K_1 \rfloor, 0)$ and γ_2 is an element of Λ^{N^2} from $(0, \mathsf{c})$ to $(b - \lfloor t/K_1 \rfloor, 0)$. As a result,

$$\sum_{\substack{\gamma:(0,a)\to(b,0) \\ \mathcal{A}(\gamma) \geq N^2+tN}} 1 \leq \sum_{c=0}^{a} \left|\Lambda^{(0,a-c)\to(\lfloor t/K_1 \rfloor, 0)}\right| \cdot \left|\Lambda_{N^2}^{(0,c)\to(b-\lfloor t/K_1 \rfloor),0)}\right|,$$

where $\Lambda^{x\to y}$ is the set of path of starting at x and ending at y, and the subscript N^2 accounts for the condition of enclosing an area of at least N^2. Now, notice that

$$\sum_{b=K_2N}^{K_1N} \lambda^{c+b-\lfloor t/K_1\rfloor}\left|\Lambda_{N^2}^{(0,c)\to(b-\lfloor t/K_1\rfloor,0)}\right| \leq Z_\lambda^{N^2}[\mathsf{y}(\Gamma)=c],$$

where we recall that $\mathsf{y}(\gamma)$ is the y-coordinate of the starting point of γ. Recall also that $\mathsf{x}(\gamma)$ is the x-coordinate of the last point of γ. Hence,

$$\begin{aligned}\sum_{K_2N\leq a,b\leq K_1N} \lambda^{a+b} &\sum_{\substack{\gamma:(0,a)\to(b,0)\\ \mathcal{A}(\gamma)\geq N^2+tN}} 1 \\ &\leq \sum_{a=K_2N}^{K_1N}\sum_{c=0}^{a}\left|\Lambda^{(0,a-c)\to(\lfloor t/K_1\rfloor,0)}\right|\lambda^{a-c+\lfloor t/K_1\rfloor} Z_\lambda^{N^2}[\mathsf{y}(\Gamma)=c] \\ &\leq \sum_{c=0}^{K_1N} Z_\lambda^{N^2}[\mathsf{y}(\Gamma)=c]\sum_{a\geq c}\left|\Lambda^{(0,a-c)\to(\lfloor t/K_1\rfloor,0)}\right|\lambda^{a-c+\lfloor t/K_1\rfloor} \\ &\leq \sum_{c=0}^{K_1N} Z_\lambda^{N^2}[\mathsf{y}(\Gamma)=c] Z_\lambda[\mathsf{x}(\Gamma)=\lfloor t/K_1\rfloor] \\ &\leq Z_\lambda^{N^2} Z_\lambda[|\Gamma|\geq\lfloor t/K_1\rfloor] \\ &\leq Z_\lambda^{N^2}(2\lambda)^{\lfloor t/K_1\rfloor}.\end{aligned}$$

The proof follows readily. □

3.2.2 Non-existence of large flat sections of a typical sample of $\mathbb{P}_\lambda^{N^2}$

In the analysis of samples of $\mathbb{P}_\lambda^{N^2}$ in given angular sectors, we will sometimes need to ensure that the marginal is not "degenerate" in the sense that it is not supported on "flat" paths (i.e almost horizontal / vertical paths). Because of the oriented feature of the model, this is an additional difficulty in comparison to the setup of subcritical statistical mechanics models, where it is often known thanks to the Ornstein–Zernike theory that the surface tension is analytic and bounded away from 0 and infinity (see for instance [25, Theorem A]).

Let $\mathbf{A}$ be a cone of apex the origin in the first quadrant of angle $\Theta_{\mathbf{A}}\in(0,\pi/2]$. For $\gamma\in\Lambda^{N^2}$, let $x_{\mathbf{A}}=x_{\mathbf{A}}(\gamma)$ (resp $y_{\mathbf{A}}=y_{\mathbf{A}}(\gamma)$) be the left-most (resp. right-most) point of $\gamma\cap\mathbf{A}$. Also, recall that the set $\Lambda^{x\to y}\subset\Lambda$ consists of all the oriented path going from x to y. Observe that if γ is a path of Λ containing x and y, then there is a natural notion of restriction of γ between x and y: it is the only element of $\Lambda^{x\to y}$ which coincides with γ between x and y. We also denote by $\theta(x_{\mathbf{A}},y_{\mathbf{A}})\in[0,\pi/2]$ the angle formed by the segment $[x_{\mathbf{A}},y_{\mathbf{A}}]$ and the horizontal line going trough $x_{\mathbf{A}}$. For $\varepsilon > 0$, we define the following events

$$\mathsf{Bad}^+_{\varepsilon,\mathbf{A}} := \{\gamma \in \Lambda^{N^2}, \ \theta(x_{\mathbf{A}}, y_{\mathbf{A}}) \in [0,\varepsilon]\},$$

and,

$$\mathsf{Bad}^-_{\varepsilon,\mathbf{A}} := \{\gamma \in \Lambda^{N^2}, \ \theta(x_{\mathbf{A}}, y_{\mathbf{A}}) \in [\tfrac{\pi}{2}-\varepsilon, \tfrac{\pi}{2}]\}.$$

Proposition 3.2.5. *There exist* $\varepsilon > 0$ *and* $c = c(\varepsilon), C = C(\varepsilon) > 0$ *such that, for any cone* $\mathbf{A}$ *as above,*

$$\mathbb{P}^{N^2}_{\lambda}\left[\Gamma \in \mathsf{Bad}^+_{\varepsilon,\mathbf{A}} \cup \mathsf{Bad}^-_{\varepsilon,\mathbf{A}}\right] \leq C\mathbb{E}^{N^2}_{\lambda}\left[\mathrm{e}^{-c\|x_{\mathbf{A}}(\Gamma)-y_{\mathbf{A}}(\Gamma)\|}\right].$$

For the proof of Proposition 3.2.5, we will use a probabilistic version of the multi-valued map principle, stated and proved in the appendix (Lemma 3.5.3).

Proof of Proposition 3.2.5. We are going to create an appropriate multi-valued map T, transforming a path belonging to $\mathsf{Bad}^+_{\varepsilon,\mathbf{A}}$ to another path of $\Lambda^{N^2} \setminus \mathsf{Bad}^+_{\varepsilon,\mathbf{A}}$, in such a way that this map has a lot of possible images but very few pre-images, and such that the probability of one element of the image of a path $\gamma \in \mathsf{Bad}^+_{\varepsilon,\mathbf{A}}$ by T is not too small compared to $\mathbb{P}^{N^2}_{\lambda}[\gamma]$.

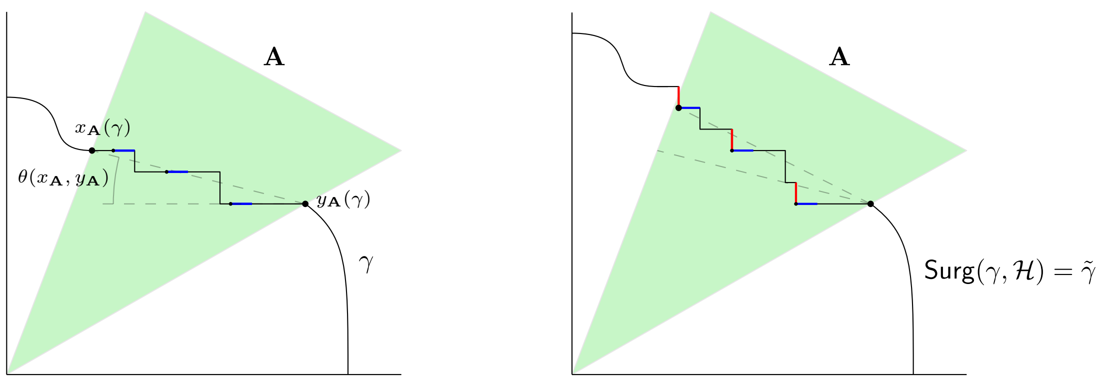

Figure 3.2: An illustration of the effect of the surgery map $\mathsf{Surg}(\gamma,\mathcal{H})$. The cone $\mathbf{A}$ is the light green region. The ηk edges of $\mathcal{H}$ that have been selected are the blue ones, and the vertical edges that have been added are depicted in red. We observe that the angle $\theta(x_{\mathbf{A}}, y_{\mathbf{A}})$ increases in $\mathsf{Surg}(\gamma,\mathcal{H})$.

Let $\eta > 0$ be a small parameter to be fixed later. For $\gamma \in \Lambda^{N^2}$, we denote by $\ell_{\mathbf{A}} = \ell_{\mathbf{A}}(\gamma)$ the length of the section of the path enclosed by $\mathbf{A}$, that is, $\ell_{\mathbf{A}}(\gamma) = |\gamma_{x_{\mathbf{A}},y_{\mathbf{A}}}|$. We first define a surgical procedure modifying γ: if $\mathcal{H}$ is some subset of the set of the horizontal steps of γ inside $\mathbf{A}$, we set $\mathsf{Surg}\,(\gamma,\mathcal{H})$ to be the new path obtained by adding a vertical step just before each step belonging to $\mathcal{H}$ (see Figure 3.2). We are going to apply this procedure for $\mathcal{H}$ being a proportion η of the horizontal steps that γ makes in $\mathbf{A}$. Of course to be able to do so we need to ensure that the paths γ we apply the map to already contain a proportion at least η of horizontal steps in $\mathbf{A}$. Since we will apply the map to elements of $\mathsf{Bad}^+_{\varepsilon,\mathbf{A}}$, we expect the proportion of horizontal steps to be quite large. Indeed, if $\gamma \in \mathsf{Bad}^+_{\varepsilon,\mathbf{A}}$, and if $\alpha(\gamma)$ is the proportion of vertical steps in the piece of path of γ in $\mathbf{A}$, then one has that

$$\varepsilon \geq \arctan\left(\frac{\alpha(\gamma)}{1-\alpha(\gamma)}\right), \tag{3.3}$$

so that $\alpha(\gamma) \leq \tan(\varepsilon) \leq 2\varepsilon$ (if ε is small enough). For technical reasons, it will also be convenient to work on a set of paths for which $\ell_{\mathbf{A}}$ is fixed. Introduce, for $k \geq 1$,

$$A_k := \mathsf{Bad}^+_{\varepsilon,\mathbf{A}} \cap \{\ell_{\mathbf{A}} = k\} \cap \Lambda^{N^2},$$

and

$$B_k := \{k \leq \ell_{\mathbf{A}} \leq (1+\eta)k\} \cap \Lambda^{N^2}.$$

Define the multi-valued map $T_{\eta,k} : A_k \to \mathcal{P}(B_k)$ by setting for any $\gamma \in A_k$,

$$T_{\eta,k}(\gamma) = \{\mathsf{Surg}(\gamma, \mathcal{H}), |\mathcal{H}| = \eta k\}.$$

We notice that for $\gamma \in A_k$, $T_{\eta,k}(\gamma) \subset \Lambda^{N^2}$. Indeed, the area can only increase under the action of $\mathsf{Surg}(\gamma, \mathcal{H})$, for any $\mathcal{H}$ being a subset of the horizontal steps of γ. Moreover, after the surgery, the length of a path inside $\mathbf{A}$ has increased by at most ηk. Thus, the map $T_{\eta,k}$ is well-defined. We will consider the probability measures $\mathbb{P}^{N^2}_\lambda$ on A_k and B_k. This way, we are exactly in the setting of Lemma 3.5.3. Introducing

$$\varphi(T_{\eta,k}) := \max_{a\in A_k} \max_{b \in T_{\eta,k}(a)} \frac{\mathbb{P}^{N^2}_\lambda[a]}{\mathbb{P}^{N^2}_\lambda[b]}, \qquad \psi(T_{\eta,k}) := \frac{\max_{b\in\tilde{B}_k} \left|T^{-1}_{\eta,k}(b)\right|}{\min_{a\in A_k}\left|T_{\eta,k}(a)\right|},$$

where $\tilde{B}_k := \mathrm{Im}(T_{\eta,k})$. A direct application of Lemma 3.5.3 yields that

$$\mathbb{P}^{N^2}_\lambda[A_k] \leq \varphi(T_{\eta,k})\psi(T_{\eta,k})\mathbb{P}^{N^2}_\lambda[B_k]. \tag{3.4}$$

We are now left with estimating $\varphi(T_{\eta,k})$ and $\psi(T_{\eta,k})$.

We start by observing that $\mathsf{Surg}(\gamma, \mathcal{H})$ increases the length of a path γ by $|\mathcal{H}|$ units, so that for any $\gamma \in A_k$ and $\tilde{\gamma} \in T_{\eta,k}(\gamma)$, the ratio defining φ is constant, and given by

$$\varphi(T_{\eta,k}) = \lambda^{-\eta k}.$$

Now, our task is to bound $\psi(T_{\eta,k})$. Let $\gamma \in A_k$. Then, it is clear that

$$\left|T_{\eta,k}(\gamma)\right| \geq \binom{(1-\alpha(\gamma))k}{\eta k} \geq \binom{(1-2\varepsilon)k}{\eta k},$$

where $\alpha(\gamma)$ is the proportion of vertical steps in the piece of path of γ in $\mathbf{A}$ and where we used (3.2.2) to get that $\alpha(\gamma) \leq 2\varepsilon$ (if ε is small enough). Next, we have to bound $\left|T_{\eta,k}^{-1}(\widetilde{\gamma})\right|$, for $\widetilde{\gamma} \in \tilde{B}_k$. Some vertical steps may have been pushed outside the sector $\mathbf{A}$ by the effect of $T_{\eta,k}$, and we must explore them to reconstruct γ knowing $T_{\eta,k}(\gamma)$. Since at most ηk of these steps disappeared in $\mathbf{A}$, we obtain

$$\left|T_{\eta,k}^{-1}(\widetilde{\gamma})\right| \leq \binom{(2\varepsilon+2\eta)k}{\eta k}.$$

Putting all the pieces together and using (3.2.2), we obtain

$$\mathbb{P}_\lambda^{N^2}[A_k] \leq \lambda^{-\eta k} \frac{\binom{(2\varepsilon+2\eta)k}{\eta k}}{\binom{(1-2\varepsilon)k}{\eta k}} \mathbb{P}_\lambda^{N^2}[B_k].$$

Standard estimates[2] yield that, as $k \to \infty$,

$$\lambda^{-\eta k} \frac{\binom{(2\varepsilon+2\eta)k}{\eta k}}{\binom{(1-2\varepsilon)k}{\eta k}} \sim \exp(-k[(1-2\varepsilon)I\left(\frac{\eta}{1-2\varepsilon}\right) - (2\varepsilon+2\eta)I\left(\frac{\eta}{2\varepsilon+2\eta}\right) + \eta\log\lambda]),$$

where for $x \in (0,1)$, $I(x) := -x\log x - (1-x)\log(1-x)$.

Set

$$c_{\varepsilon,\eta} = (1-2\varepsilon)I\left(\frac{\eta}{1-2\varepsilon}\right) - (2\varepsilon+2\eta)I\left(\frac{\eta}{2\varepsilon+2\eta}\right) + \eta\log\lambda.$$

It is easy to check that one can find a pair $\varepsilon, \eta(\varepsilon) > 0$ satisfying $c_{\varepsilon,\eta(\varepsilon)} > 0$. Let us fix such an $\varepsilon > 0$ and drop η from the notations. We obtained that there exists $C = C(\varepsilon) > 0$ such that for $k \geq 1$,

$$\lambda^{-\eta k} \frac{\binom{(2\varepsilon+2\eta)k}{\eta k}}{\binom{(1-2\varepsilon)k}{\eta k}} \leq C\mathrm{e}^{-c_\varepsilon k}.$$

Hence,

$$\mathbb{P}_\lambda^{N^2}[\Gamma \in \mathsf{Bad}_{\varepsilon,\mathbf{A}}^+,\ \ell_A = k] \leq C\mathrm{e}^{-c_\varepsilon k}\mathbb{P}_\lambda^{N^2}[k \leq \ell_{\mathbf{A}} \leq (1+\eta)k].$$

Summing over k we finally obtain the existence of $c_1 = c_1(\varepsilon), C_1 = C_1(\varepsilon)$ such that

$$\mathbb{P}_\lambda^{N^2}\left[\Gamma \in \mathsf{Bad}_{\varepsilon,\mathbf{A}}^+\right] \leq C_1\mathbb{E}_\lambda^{N^2}\left[e^{-c_1\ell_{\mathbf{A}}}\right] \leq C_1\mathbb{E}_\lambda^{N^2}\left[\mathrm{e}^{-c_1\|x_{\mathbf{A}}(\Gamma)-y_{\mathbf{A}}(\Gamma)\|}\right].$$

A similar argument holds for $\mathsf{Bad}_{\varepsilon,\mathbf{A}}^-$. This concludes the proof. □

[2] For any $c_1 < c_2$, as $k \to \infty$,

$$\binom{c_2 k}{c_1 k} \sim \mathrm{e}^{c_2 k I\left(\frac{c_1}{c_2}\right)}.$$

3.2.3 Couplings between $\mathbb{P}_\lambda^{N^2}$ and a conditioned random walk

This subsection is devoted to the description of two couplings between the measure $\mathbb{P}_\lambda^{N^2}$ conditioned on a pinning event and the law of a random walk bridge. We first describe and prove the existence of these couplings and then explain what they will be useful for.

Let $a, b \in \mathbb{N}^2$ such that $\arg(a) > \arg(b)$. We introduce the event $\mathsf{Fac}(a,b)$ defined by

$$\mathsf{Fac}(a,b) := \{\gamma \in \Lambda, \text{ The segment } [a,b] \text{ is a facet of } \gamma\}.$$

Recall that the set $\Lambda^{a\to b} \subset \Lambda$ consists of all the oriented path going from a to b. We introduce the following partial order in the set $\Lambda^{a\to b}$: for $\gamma^1, \gamma^2 \in \Lambda^{a\to b}$, we shall say that $\gamma^1 \geq \gamma^2$ when

$$\forall (x_1,y_1) \in \gamma^1, (x_2,y_2) \in \gamma^2, x_1 = x_2 \Rightarrow y_1 \geq y_2.$$

Finally, we denote by $\mathbb{P}_{a,b}^-$ the uniform measure over all the paths of $\Lambda^{a\to b}$ that remain under the segment $[a,b]$.

With these definitions in hand, we are now able to describe the first coupling.

Proposition 3.2.6. *Let a and b be two vertices of $\mathbb{N}^2$ such that $\arg(a) > \arg(b)$. There exists a probability measure Ψ on the space $\Lambda^{a\to b} \times \Lambda^{a\to b}$ such that:*

(i) the marginal of Ψ on its first coordinate has the law of $\Gamma_{a,b}$ where Γ is sampled according to $\mathbb{P}_\lambda^{N^2}[\,\cdot \mid \mathsf{Fac}(a,b)]$,

(ii) the marginal of Ψ on its second coordinate has law $\mathbb{P}_{a,b}^-$,

(iii) $\Psi\big[\{(\gamma^1,\gamma^2) \in \Lambda^{a\to b} \times \Lambda^{a\to b}, \gamma^1 \geq \gamma^2\}\big] = 1.$

Proof. The proof relies on a dynamical Markov chain argument and is, for percolation *aficionados*, very similar to that of Holley's inequality (see for instance [71]).

We shall start from the restriction of a sample of $\mathbb{P}_\lambda^{N^2}[\,\cdot \mid \mathsf{Fac}(a,b)]$ to $\Lambda^{a\to b}$ that we denote $\Gamma_{a,b}$, and then describe two Markovian dynamics $(\Gamma_k^1)_{k\geq 0}$ and $(\Gamma_k^2)_{k\geq 0}$ on the state space $\Lambda^{a\to b}$ with the following properties:

$(\mathbf{P1})$ $\Gamma_0^1 = \Gamma_0^2 = \Gamma_{a,b}$,

$(\mathbf{P2})$ $\forall k \geq 0, \Gamma_k^1 \geq \Gamma_k^2$,

$(\mathbf{P3})$ the law of $\Gamma_{a,b}$ is the stationary measure of the Markov chain $(\Gamma_k^1)_{k\geq 0}$,

$(\mathbf{P4})$ $\mathbb{P}_{a,b}^-$ is the stationary measure of the Markov chain $(\Gamma_k^2)_{k\geq 0}$.

Now, let us describe the Markovian dynamics. Let $p = \|a-b\|_1$ be the length of any path of $\Lambda^{a\to b}$. Let $(\ell_k)_{k\geq 0}$ be a family of independent random variables all having a uniform distribution on $\{1,\dots,p-1\}$, and $(X_k)_{k\geq 0}$ a family of independent random variables all having a uniform distribution on $\{0, 1\}$. At step k, the random variable ℓ_k is the position of the vertex on which we will proceed a random modification.

Looking at the two edges incident to the vertex $\Gamma_k^i(\ell_k)$ for $i \in \{1, 2\}$, three different configurations can arise:

1) If these two edges are co-linear, we do not modify anything: $\Gamma_{k+1}^i = \Gamma_k^i$ for $i \in \{1, 2\}$.
2) If these edges form a corner of the type $\llcorner$
   - (i) For Γ^1 we flip the corner into the opposite corner $\urcorner$ if and only if we have $X_k = 1$ **and** the modified path does not go above the facet $[a, b]$. If any of these two conditions is not verified, we do not modify Γ_k^1.
   - (ii) We apply the same procedure for Γ^2.
3) If these edges form a corner of the type $\urcorner$
   - (i) For Γ^1, we flip this corner into the opposite corner $\llcorner$ if and only if $X_k = 1$ **and** the area enclosed by this new path is still greater than N^2. If any of these two conditions is not verified we do not modify Γ_k^1.
   - (ii) For Γ^2, we flip the corner into the opposite one if and only if $X_k = 1$.

It is straightforward that these Markov dynamics are irreducible, aperiodic, and reversible. Then, Γ^1 (resp. Γ^2) admits a stationnary measure and converges towards it: it is straightforward to see that this stationary measure is the law of $\Gamma_{a,b}$ (resp $\mathbb{P}_{a,b}^-$). Moreover, by construction, one always has $\Gamma^1 \geq \Gamma^2$. This yields the desired stochastic domination. □

This coupling will be very efficient when computing local statistics of a sample of $\mathbb{P}_\lambda^{N^2}$ along a facet (e.g. the local roughness along a facet) as it allows us to compare them to the ones of a random walk excursion on which we know much more information. More precisely, Proposition 3.2.6 has the following consequence: an upper bound on the length of a facet automatically converts into an upper bound on the local roughness under this facet.

Lemma 3.2.7. *There exist $c, C, N_0 > 0$ such that the following holds. For any $a, b \in \mathbb{N}^2$ with $\arg(a) > \arg(b)$ and $\|b - a\| \geq N_0$ and any $t \geq 0$,*

$$\mathbb{P}_\lambda^{N^2}\big[\max_{x \in \Gamma_{a,b}} \mathsf{LR}(x) > t\|b-a\|^{1/2} \,\big|\, \mathsf{Fac}(a,b)\big] \leq C\exp(-ct^2).$$

Proof. Observe that thanks to Proposition 3.2.6, the random variable $\max_{x\in\Gamma_{a,b}} \mathsf{LR}(x)$ where Γ is sampled according to $\mathbb{P}_\lambda^{N^2}[\,\cdot \mid \mathsf{Fac}(a,b)]$ is stochastically dominated by the random variable $\max_{x\in\omega_{a,b}} \mathsf{LR}(x)$ where $\omega_{a,b}$ is sampled according to $\mathbb{P}_{a,b}^-$.

The statement then follows by classical random walks arguments. Indeed, the results of [28] imply that a lattice random walk bridge conditioned to stay below a line segment with increments having exponential tails, after subtraction of the linear term corresponding to the equation of that segment, converges towards the standard Brownian excursion. As a consequence, knowing the fluctuations of the Brownian excursion (see [36]) on an interval of size $\|b-a\|$ leads to the bound

$$\mathbb{P}^{-}_{a,b}\big[\max_{x\in\omega_{a,b}} \mathsf{LR}(x) \geq t\|b-a\|^{1/2}\big] \leq C\exp(-ct^2),$$

when $\|b-a\|$ is large enough. □

Remark 3.2.8. In Section 3.4, we will use the preceding result in the regime $t = (\log\|b-a\|)^{\alpha}$ for some power $0 < \alpha < 1$. We claim that the result still holds true and is a consequence of classical moderate deviations estimates for random walks (see [41]).

The second coupling we shall need is closely related to the first one with the only exception that we do not assume that $[a,b]$ is a facet anymore. This time, we will build an "increasing coupling" (in the sense of the partial order of $\Lambda^{a\to b}$) between the law of $\Gamma_{a,b}$ where Γ is sampled according to $\mathbb{P}^{N^2}_{\lambda}[\,\cdot\mid a,b\in\Gamma]$ and the distribution of the random walk bridge between a and b. We denote by $\mathbb{P}_{a,b}$ the uniform distribution on $\Lambda^{a\to b}$.

Proposition 3.2.9. *Let a and b be two vertices of $\mathbb{N}^2$ such that $\arg(a) > \arg(b)$. There exists a probability measure $\overline{\Psi}$ on the space $\Lambda^{a\to b}\times\Lambda^{a\to b}$ such that:*

(i) the marginal of $\overline{\Psi}$ on its first coordinate has the law of $\Gamma_{a,b}$ where Γ is sampled according to $\mathbb{P}^{N^2}_{\lambda}[\,\cdot\mid a,b\in\Gamma]$,

(ii) the marginal of $\overline{\Psi}$ on its second coordinate has law $\mathbb{P}_{a,b}$,

(iii) $\overline{\Psi}\big[\{(\gamma^1,\gamma^2)\in\Lambda^{a\to b}\times\Lambda^{a\to b}, \gamma^1\geq\gamma^2\}\big] = 1.$

Proof. The proof follows the strategy used to obtain Proposition 3.2.6 except that we do not have the condition of not going above the segment $[a,b]$ anymore. □

3.2.4 A rough upper bound on the length of a typical facet

The goal of this section is to give an illustration of the resampling strategy which will be used throughout the paper. The result proved in this section will also be useful for the proof of the upper bounds of Theorem 3.1.3.

Proposition 3.2.10. *Let $\varepsilon > 0$. There exist $c = c(\varepsilon), C = C(\varepsilon) > 0$ such that, for all $1 \leq t \leq N^{2/3-\varepsilon}$,*

$$\mathbb{P}^{N^2}_{\lambda}\left[\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{2}{3}+\varepsilon}\right] \leq C\mathrm{e}^{-ct^{3/2}N^{3\varepsilon/2}}. \tag{3.5}$$

Moreover, there exists $c', C' > 0$ such that, for all $t \geq 1$

$$\mathbb{P}^{N^2}_{\lambda}\left[\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{4}{3}}\right] \leq C'\mathrm{e}^{-c'tN^{4/3}}.$$

Before proving this result, we need to define an event which we will ask the resamplings to satisfy in order to ensure they capture a large enough area.

Let us first recall and introduce some notations. For $x, y \in \mathbb{N}^2$, let $\mathbf{A}_{x,y}$ be the cone of apex 0 bounded by x and y and let $\mathbf{T}_{0,x,y}$ be the triangle of apexes $0, x$ and y. For a cone $\mathbf{A}_{x,y}$ and $\gamma \in \Lambda$, we define $\text{Enclose}(\gamma \cap \mathbf{A}_{x,y})$ to be the region delimited by γ and the two boundary axes of $\mathbf{A}_{x,y}$. The following event, illustrated in Figure 3.3, will be the one we will require our resampling to satisfy.

Definition 3.2.11 (Good area capture)**.** Let $\gamma \in \Lambda$ be a path, $\eta > 0$ and $x, y \in \mathbb{N}^2$. We say that γ realises the event $\mathsf{GAC}\,(x, y, \eta)$ (meaning "good area capture") if

(i) $\gamma \cap \mathbf{A}_{x,y} \in \Lambda^{x \to y}$, or in words, γ connects x and y by a oriented path,

(ii) $|\text{Enclose}(\gamma \cap \mathbf{A}_{x,y})| - |\mathbf{T}_{0,x,y}| \geq \eta \parallel x - y \parallel^{\frac{3}{2}}$.

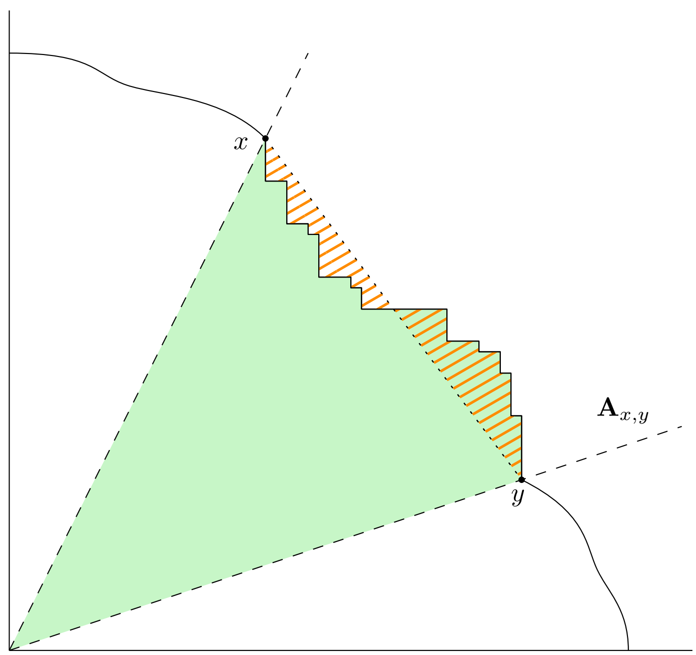

Figure 3.3: An illustration of the event $\mathsf{GAC}\,(x, y, \eta)$. The light green shaded region corresponds to $\text{Enclose}(\gamma \cap \mathbf{A}_{x,y})$. The event that the path γ satisfies $\mathsf{GAC}\,(x, y, \eta)$ means that the (algebraic) area of the orange hatched region exceeds $\eta \|x - y\|^{3/2}$.

Proof of Proposition 3.2.10. Let $1 \leq t \leq N^{2/3-\varepsilon}$. Introduce the event

$$\mathsf{BigMeanFL}(t) := \left\{\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{2}{3}+\varepsilon}\right\}.$$

We pick uniformly at random, and independently of Γ, two points $\mathbf{x}, \mathbf{y}$ (with $\arg(\mathbf{x}) > \arg(\mathbf{y})$)

in $B_{K_1 N}$ and call $\mathsf{GoodHit}$ the event that these two points coincide with the extremities of $\mathsf{MeanFac}$. The measure associated with this procedure is denoted $\mathbb{P}$.

Let us call $\mathsf{Bad}_\varepsilon^{+,-}$ the event that $\theta(\mathbf{x}, \mathbf{y}) \in [0, \varepsilon) \cup (\pi/2 - \varepsilon, \pi/2]$ for the $\varepsilon > 0$ given by Proposition 3.2.5. Let $\mathcal{E} := \mathsf{GoodHit} \cap (\mathsf{Bad}_\varepsilon^{+,-})^c \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}$. Resample Γ between $\mathbf{x}$ and $\mathbf{y}$ according to the uniform law among paths $\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}}$ such that $(\Gamma \setminus \Gamma_{\mathbf{x},\mathbf{y}}) \cup \gamma$ satisfies the area condition and call $\tilde{\Gamma}$ the resampling. Note that $\tilde{\Gamma}$ has law $\mathbb{P}_\lambda^{N^2}$.

Since we are resampling along a facet, if $(\Gamma, (\mathbf{x}, \mathbf{y}))$ realises $\mathcal{E}$, then $\Gamma_{\mathbf{x},\mathbf{y}}$ lies underneath the segment joining $\mathbf{x}$ and $\mathbf{y}$. Thus, if $(\Gamma, (\mathbf{x}, \mathbf{y})) \in \mathcal{E}$, replacing $\Gamma_{\mathbf{x},\mathbf{y}}$ by any path satisfying $\mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)$ still satisfies the area condition and[3]

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\tilde{\Gamma} \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta) \mid (\Gamma, (\mathbf{x}, \mathbf{y})) \in \mathsf{BigMeanFL}(t) \cap \mathcal{E}] \geq \mathbb{P}_{\mathbf{x},\mathbf{y}}[\mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)]. \tag{3.6}$$

where we recall that $\mathbb{P}_{a,b}$ is the uniform law on $\Lambda^{a \to b}$. Indeed, conditioning on $(\Gamma, (\mathbf{x}, \mathbf{y})) \in \mathsf{BigMeanFL}(t) \cap \mathcal{E}$, the above inequality is equivalent to

$$\frac{|\{\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}},\ \gamma \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)\}|}{|\{\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}},\ (\Gamma \setminus \Gamma_{\mathbf{x},\mathbf{y}}) \cup \gamma \in \Lambda^{N^2}\}|} \geq \frac{|\{\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}},\ \gamma \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)\}|}{|\Lambda^{\mathbf{x} \to \mathbf{y}}|},$$

which is trivially true. Now, if $\theta(\mathbf{x}, \mathbf{y}) \in [\varepsilon, \pi/2 - \varepsilon]$, standard computations on the simple random walk bridge (see Lemma 3.3.4 for more details) ensure that for $\eta > 0$ sufficiently small there exists $c_1 = c_1(\eta, \varepsilon) > 0$ such that,

$$\mathbb{P}_{\mathbf{x},\mathbf{y}}[\mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)] \geq c_1.$$

Notice also that

$$\{\tilde{\Gamma} \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta), (\Gamma, (\mathbf{x}, \mathbf{y})) \in \mathsf{BigMeanFL}(t) \cap \mathcal{E}\} \subset \left\{\mathsf{ExcessArea}(\tilde{\Gamma}) \geq \eta t^{\frac{3}{2}} N^{1+\frac{3}{2}\varepsilon}\right\}.$$

Putting all the pieces together, and using the fact that $\tilde{\Gamma}$ has law $\mathbb{P}_\lambda^{N^2}$,

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t) \cap \mathcal{E}] \leq c_1^{-1} \mathbb{P}_\lambda^{N^2}\left[\mathsf{ExcessArea}(\Gamma) \geq \eta t^{\frac{3}{2}} N^{1+\frac{3}{2}\varepsilon}\right].$$

Finally, notice that for some $c_2 > 0$,

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}\left[\mathsf{GoodHit} \mid \mathsf{BigMeanFL}(t) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}\right] \geq \frac{c_2}{N^4}.$$

Putting all the pieces together,

$$\begin{aligned}
\mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t)] &\leq \frac{N^4}{c_2} \cdot \mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t) \cap \mathcal{E}] \\
&+ \frac{N^4}{c_2} \cdot \mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t) \cap \mathsf{GoodHit} \cap (\mathsf{Bad}_\varepsilon^{+,-}) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}] \\
&+ \mathbb{P}_\lambda^{N^2}\left[\mathsf{BigMeanFL}(t) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}^c\right].
\end{aligned}$$

[3]We use the fact that $\mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)$ can be seen as an event on $\Lambda^{\mathbf{x} \to \mathbf{y}}$ and for convenience we keep the same notation when looking at this event under the two measures of interest.

To bound the second term we use Proposition 3.2.5 above. Notice that,

$$\mathsf{BigMeanFL}(t) \cap \mathsf{GoodHit} \cap (\mathsf{Bad}_\varepsilon^{+,-}) \cap \{\Gamma \subset B_{K_1N} \setminus B_{K_2N}\} \subset \bigcup_{\substack{u,v \in B_{K_1N}\setminus B_{K_2N} \\ \|u-v\| \geq tN^{\frac{2}{3}+\varepsilon}}} \{\Gamma \in \mathsf{Bad}^+_{\varepsilon,\mathbf{A}_{u,v}} \cup \mathsf{Bad}^+_{\varepsilon,\mathbf{A}_{u,v}}\},$$

where $\mathbf{A}_{u,v}$ is the cone of apex 0 bounded by u and v. Finally, apply Lemma 3.2.2, Proposition 3.2.4, and Proposition 3.2.5 to get that for some $c = c(\varepsilon) > 0$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t)] \leq \frac{N^4}{c_2} \cdot c_1^{-1} \cdot \exp(-ct^{\frac{3}{2}} N^{\frac{3}{2}\varepsilon}) + \frac{N^4}{c_2} \cdot (K_1 N)^4 \cdot 2\exp(-ctN^{\frac{2}{3}+\varepsilon}) + \exp(-cN),$$

and thus the first inequality (recall that $t \leq N^{2/3-\varepsilon}$).

For $t \geq 1$[4] notice that

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{4}{3}}\right] \leq \mathbb{P}_\lambda^{N^2}\left[|\Gamma| \geq (tN^{1/3}) \cdot N\right].$$

Using Lemma 3.2.1 we get the result. □

3.3 Analysis of the mean facet length and the mean local roughness

3.3.1 Upper bounds

In the following subsection, we actually prove a stronger statement than the upper bounds of Theorem 3.1.3. Recall that MeanFL and MeanLR where defined in Definition 3.1.2. The goal of this subsection is to prove upper tail estimates on MeanFL and MeanLR. Our first result is a refinement of Proposition 3.2.10.

Proposition 3.3.1 (Upper tail of MeanFL)**.** *There exist $\tilde{c}, c, C > 0$ such that, for any $\tilde{c} \leq t \leq N^{2/3}$,*

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{2}{3}}\right] \leq C\mathrm{e}^{-ct^{3/2}}.$$

Proposition 3.3.2 (Upper tail of MeanLR)**.** *There exist $\tilde{c}, c, C > 0$ such that for any $\tilde{c} \leq t \leq N^{5/6}$,*

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanLR}(\Gamma) \geq tN^{\frac{1}{3}}\right] \leq C\mathrm{e}^{-ct^{6/5}}.$$

Remark 3.3.3. For larger values of t, we can use Lemma 3.2.1 and Proposition 3.2.10 to get explicit tails for MeanFL and MeanLR. In particular, we obtain that

$$\limsup_{N\to\infty} \mathbb{E}_\lambda^{N^2}\left[\frac{\mathsf{MeanFL}(\Gamma)}{N^{2/3}}\right] < \infty, \qquad \limsup_{N\to\infty} \mathbb{E}_\lambda^{N^2}\left[\frac{\mathsf{MeanLR}(\Gamma)}{N^{1/3}}\right] < \infty.$$

[4] In fact, one can take $t \geq 3\sqrt{2}N^{-1/3}$ here.

We keep the notations of Section 3.2.4. We will again use the event $\mathsf{GAC}\,(x, y, \eta)$ introduced in Definition 3.2.11. While it is difficult to estimate the probability of the event $\mathsf{GAC}\,(x, y, \eta)$ under $\mathbb{P}_\lambda^{N^2}$, it is much simpler to estimate it under $\mathbb{P}_{x,y}$ which is the uniform law over $\Lambda^{x \to y}$ (assuming that we chose $x, y \in \mathbb{N}^2$ such that $\Lambda^{x \to y} \neq \emptyset$). The following result, whose proof is postponed to the appendix, gives us a lower bound on $\mathbb{P}_{x,y}[\mathsf{GAC}\,(x, y, \eta)]$. Denote by $\theta(x, y) \in [0, \pi/2]$ the angle formed by the horizontal axis and the segment joining x and y.

Lemma 3.3.4. *Let $\varepsilon, \eta > 0$. There exist $c = c(\varepsilon, \eta) > 0$ and $N_0 = N_0(\varepsilon, \eta) > 0$ such that for any $x, y \in \mathbb{N}^2$ satisfying $\Lambda^{x \to y} \neq \emptyset$, $\varepsilon \leq \theta(x, y) \leq \pi/2 - \varepsilon$ and $\|x - y\| \geq N_0$,*

$$\mathbb{P}_{x,y}[\mathsf{GAC}\,(x, y, \eta)] \geq c.$$

Remark 3.3.5.

1. The constant $c(\varepsilon, \eta)$ in the result above degenerates as $\varepsilon \to 0$. However, as we saw in Proposition 3.2.5, up to an event of small probability, we will be able to assume that θ stays bounded away from 0 and $\pi/2$.
2. The constant $\tilde{c}$ in the statement of Propositions 3.3.1 and 3.3.2 will be chosen in terms of $N_0 = N_0(\varepsilon, \eta)$ for ε given by Proposition 3.2.5 and a properly chosen $\eta > 0$.

Proof of Proposition 3.3.1

As in the proof of Proposition 3.2.10, we would like to resample along the mean facet. However, as we saw, to pick the extremities of the mean facet without revealing any information about the path Γ comes with a cost of $O(N^4)$. This cost was previously handled by the requirement that $t \geq N^\varepsilon$, but now we allow t to be of smaller order. To tackle this difficulty, we will not resample between the exact extremities of the mean facet but between points which approximate them (see Figure 3.4).

Proof of Proposition 3.3.1. Let $t > 0$ be fixed, and let $\delta, \varepsilon > 0$ be two small parameters that will be fixed later. Thanks to Proposition 3.2.10, we can additionally assume that $t \leq N^\varepsilon$ for some $\varepsilon \in (0, 1/3)$. Indeed, if $N^\varepsilon \leq t \leq N^{2/3}$, using (3.2.10), we get

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanFL}(\Gamma) \geq (tN^{-\varepsilon})N^{2/3+\varepsilon}\right] \leq C\mathrm{e}^{-ct^{3/2}}.$$

We now assume $t \leq N^\varepsilon$. Let us introduce the following event

$$\mathsf{BigMeanFL}(t) = \left\{\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{2}{3}}\right\}.$$

We also introduce, for any integer $k \geq 1$,

$$\mathsf{BigMeanFL}_k(t) = \left\{ktN^{\frac{2}{3}} \leq \mathsf{MeanFL}(\Gamma) < (k+1)tN^{\frac{2}{3}}\right\},$$

so that the family $(\mathsf{BigMeanFL}_k(t))_{k \geq 1}$ partitions the event $\mathsf{BigMeanFL}$.

Notice that for large values of k, we can use Proposition 3.2.10 again to conclude. If $k_N(\varepsilon, t)$ is the smallest $k \geq 1$ such that $tk \geq N^{\varepsilon}$, using (3.2.10) again,

$$\mathbb{P}_{\lambda}^{N^2}\left[\mathsf{MeanFL}(\Gamma) \geq tk_N(\varepsilon, t)N^{\frac{2}{3}}\right] \leq Ce^{-c(tk_N)^{3/2}}.$$

In particular it suffices to control the probabilities of $\mathsf{BigMeanFL}_k(t)$ for $1 \leq k \leq k_N(\varepsilon, t)$. In fact, it is sufficient to control the probabilities of $\mathsf{BigMeanFL}_k(t/3)$ for $3 \leq k \leq k_N(\varepsilon, t/3) = 3k_N(\varepsilon, t)$. We now let $s = t/3$ and $k_N = k_N(\varepsilon, s)$.

Fix $3 \leq k \leq k_N$. For $j \geq 0$, define the angles

$$\theta_j^+ = \theta_j^+(\delta, k) := \frac{\pi}{4} + \arctan\left(\frac{j\delta ks}{N^{1/3}}\right), \qquad \theta_j^- = \theta_j^-(\delta, k) := \frac{\pi}{4} - \arctan\left(\frac{j\delta ks}{N^{1/3}}\right),$$

and denote $\ell_j^+ = \ell_j^+(\delta, k)$ (resp. $\ell_j^- = \ell_j^-(\delta, k)$) the half-line rooted at the origin and of argument θ_j^+ (resp. θ_j^-).

When $\mathsf{BigMeanFL}_k(s)$ occurs, there are at most $2/\delta$ indices j such that ℓ_j^+ (resp. ℓ_j^-) intersects the mean facet. For a sample Γ and $0 \leq j \leq 2/\delta$, we call $a_j = a_j(\Gamma, \delta, k)$ (resp. $b_j = b_j(\Gamma, \delta, k)$) the points of $\mathbb{N}^2$ that are the closest[5] to the intersection between Γ and ℓ_j^+ (resp. ℓ_j^-). Observe that these points are defined from Γ by a deterministic procedure which means that resampling a sample Γ between two of these points yields an output distributed according to $\mathbb{P}_{\lambda}^{N^2}$. In particular, we can define $\mathbf{a} = \mathbf{a}(\Gamma, \delta, k)$ (resp. $\mathbf{b} = \mathbf{b}(\Gamma, \delta, k)$) to be the point of Γ defined as follows: if j_0 (resp. j_1) is the largest j such that ℓ_j^+ (resp. ℓ_j^-) intersects the mean facet, then $\mathbf{a} = a_{j_0}$ (resp. $\mathbf{b} = b_{j_1}$).

In order to be able to pick $\mathbf{a}$ and $\mathbf{b}$ and resample between them without revealing the portion $\Gamma_{\mathbf{a},\mathbf{b}}$ of Γ between $\mathbf{a}$ and $\mathbf{b}$, we introduce some extra randomness. We pick uniformly, a pair of points $(\mathbf{x}, \mathbf{y})$ in $\{a_j,\ 0 \leq j \leq 2/\delta\} \times \{b_j,\ 0 \leq j \leq 2/\delta\}$ and call $\mathbb{P}$ the measure associated with this random procedure. Let $\mathsf{GoodHit}_k$ be the event that $(\mathbf{x}, \mathbf{y}) = (\mathbf{a}, \mathbf{b})$. We are going to show that with uniformly positive probability, if we resample between $\mathbf{a}$ and $\mathbf{b}$, the resampled path captures a linear excess of area, an event which is exponentially unlikely by Proposition 3.2.4.

As in the proof of Proposition 3.2.10, let us call $\mathsf{Bad}_{\varepsilon}^{+,-}$ the event that $\theta(\mathbf{x}, \mathbf{y}) \in [0, \varepsilon) \cup (\pi/2 - \varepsilon, \pi/2]$ for the $\varepsilon > 0$ given by Proposition 3.2.5. Write $\mathcal{E}_k := \mathsf{GoodHit}_k \cap (\mathsf{Bad}_{\varepsilon}^{+,-})^c \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}$. Resample Γ between $\mathbf{x}$ and $\mathbf{y}$ according to the uniform law among paths $\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}}$ such that $(\Gamma \setminus \Gamma_{\mathbf{x},\mathbf{y}}) \cup \gamma$ satisfies the area condition and call $\tilde{\Gamma}$ the resampling. Note that $\tilde{\Gamma}$ has law $\mathbb{P}_{\lambda}^{N^2}$. In the proof of Proposition 3.2.10, since we were resampling along the mean facet, it was clear that replacing the path below the facet by any path having a positive area (above the facet) kept the total area condition. However, in our case, the

[5] There might be ambiguity in the choice of such points and we solve this issue by asking a_j (resp. b_j) to lie on the right (resp. on the left) of ℓ_j^+ (resp. ℓ_j^-).

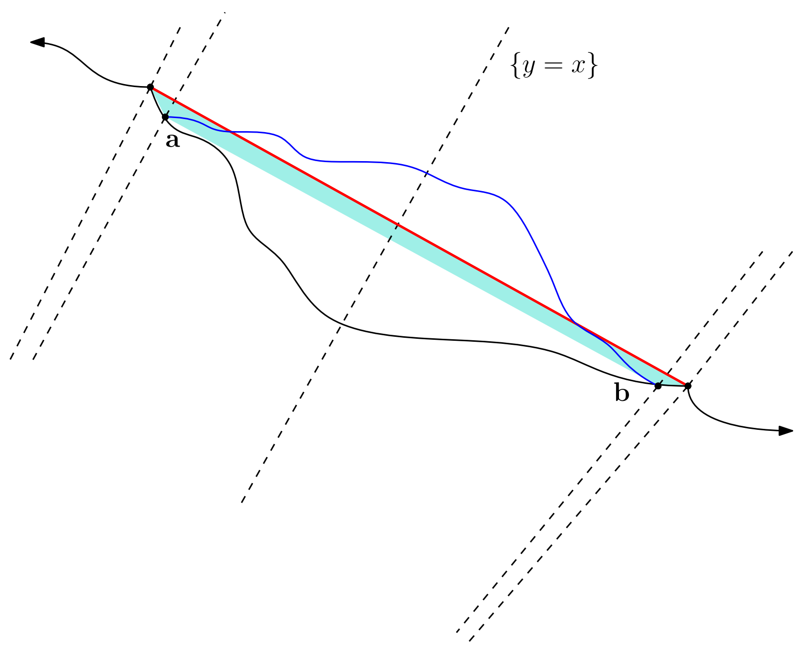

Figure 3.4: Illustration of the proof of Proposition 3.3.1. The original is path depicted in black. With large probability, the new resampled blue path captures an amount of area sufficient to compensate the area that could have been lost (in turquoise) due to the fact that $\mathbf{a}$ and $\mathbf{b}$ do not exactly coincide with the endpoints of the red facet. The event SD ensures that $\mathbf{a}$ and $\mathbf{b}$ are not atypically far from these two endpoints, allowing us to control the area of the turquoise region.

approximate "mean facet resampling" comes with a potential area loss. This possibility is ruled out for sufficiently small values of δ. For $\eta > 0$, introduce the (small deviation) event

$$\mathsf{SD}(\eta) := \big\{ \max(\mathsf{LR}(\mathbf{a}), \mathsf{LR}(\mathbf{b})) < \eta(\delta s(k+1))^{1/2} N^{1/3} \big\}.$$

Claim 3.3.6. There exist $\delta_0 > 0$ sufficiently small and $c_0 = c_0(\delta_0, \eta) > 0$ such that the following holds: if $((\mathbf{x}, \mathbf{y}), \Gamma) \in \mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)$, then, if we replace $\Gamma_{\mathbf{x},\mathbf{y}}$ by a path $\gamma \in \mathsf{GAC}\,(\mathbf{x}, \mathbf{y}, \eta)$, the excess area of the resulting path is at least $c_0(sk)^{3/2}N$. In particular, the new path lies in Λ^{N^2}.

Proof of Claim 3.3.6. Notice that by definition $(\mathbf{x}, \mathbf{y}) = (\mathbf{a}, \mathbf{b})$ are located below the mean facet. Moreover, these points are at distance at most $\delta(k+1)tN^{2/3}$ from the endpoints of the mean facet. Changing Γ between $\mathbf{a}$ and $\mathbf{b}$ by an element of $\mathsf{GAC}\,(\mathbf{a}, \mathbf{b}, \eta)$ gives a path $\tilde{\Gamma}$ which gained an area at least $\eta\|\mathbf{a} - \mathbf{b}\|^{3/2} - \mathcal{A}(\mathbf{a}, \mathbf{b}, \Gamma)$ where $\mathcal{A}(\mathbf{a}, \mathbf{b}, \Gamma)$ is the area of the quadrilateral delimited by the extremities of the mean facet of Γ and the pair $(\mathbf{a}, \mathbf{b})$, see Figure 3.4. Given the hypothesis on Γ, we find that

$$\mathcal{A}(\mathbf{a}, \mathbf{b}, \Gamma) \leq \eta\sqrt{\delta}(s(k+1))^{3/2} N.$$

Hence, for $\delta = \delta_0 > 0$ sufficiently small, there exists $c_0 = c_0(\delta_0, \eta) > 0$ such that (recall that $k \geq 3$)

$$\mathsf{ExcessArea}(\tilde{\Gamma}) \geq \eta s^{3/2}(k-2)^{3/2}N - \eta\sqrt{\delta_0}(s(k+1))^{3/2}N \geq c_0(sk)^{3/2}N.$$

□

We now fix $\delta = \delta_0$ and let $\eta > 0$ to be fixed later. Using the same reasoning as in (3.2.4) we get that for $\|\mathbf{a} - \mathbf{b}\| \geq N_0$,

$$\begin{aligned}\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\tilde{\Gamma} \in \mathsf{GAC}\,(\mathbf{x}, \mathbf{y}, \eta) \mid ((\mathbf{x}, \mathbf{y}), \Gamma) \in \mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)]\\ \geq \mathbb{P}_{\mathbf{a},\mathbf{b}}[\mathsf{GAC}\,(\mathbf{a}, \mathbf{b}, \eta)] \geq c_1,\end{aligned}$$

where $c_1, N_0 > 0$ are given by Lemma 3.3.4. Since $\|\mathbf{a} - \mathbf{b}\| \geq s(k-2)N \geq sN$, the above bound always hold as soon as $s \geq N_0/N$. Assume that $s \geq N_0$. By the claim above, one has

$$\big\{\tilde{\Gamma} \in \mathsf{GAC}\,(\mathbf{x}, \mathbf{y}, \eta), \Gamma \in \mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)\big\} \subset \big\{\mathsf{ExcessArea}(\tilde{\Gamma}) \geq c_0(tk)^{3/2}N\big\}.$$

Putting all the pieces together, and using the fact that $\tilde{\Gamma}$ has law $\mathbb{P}_\lambda^{N^2}$,

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)] \leq c_1^{-1}\mathbb{P}_\lambda^{N^2}[\mathsf{ExcessArea}(\Gamma) \geq c_0(tk)^{3/2}N]. \quad (3.7)$$

We now argue that $\mathsf{SD}(\eta)$ occurs with uniform non-zero probability under $\mathbb{P}_\lambda^{N^2}$.

Claim 3.3.7. There exist $\eta > 0$ and $c_2 = c_2(\eta) > 0$ such that

$$\mathbb{P}_\lambda^{N^2}[\mathsf{SD}(\eta) \mid \mathsf{BigMeanFL}_k(s)] \geq c_2.$$

Proof of Claim 4.4.12. The claim is a consequence of the coupling obtained in Proposition 3.2.6. Indeed, $\mathbf{a}$ and $\mathbf{b}$ belong to the excursion of Γ under the mean facet. Thus, conditioning on the extremities of the mean facet, which we call $\mathbf{a}_{\mathsf{fac}}$ and $\mathbf{b}_{\mathsf{fac}}$, the local roughness of $\mathbf{a}$ (resp. $\mathbf{b}$) is dominated by the height of a negative random walk excursion between $\mathbf{a}_{\mathsf{fac}}$ and $\mathbf{b}_{\mathsf{fac}}$ after $\mathrm{d}_{\mathbf{a}}$ steps, where $\mathrm{d}_{\mathbf{a}}$ is the number of steps from $\mathbf{a}_{\mathsf{fac}}$ to $\mathbf{a}$ (resp $\mathrm{d}_{\mathbf{b}}$ steps, where $\mathrm{d}_{\mathbf{b}}$ is the number of steps from $\mathbf{b}$ to $\mathbf{b}_{\mathsf{fac}}$) which is of order $\sqrt{\mathrm{d}_{\mathbf{a}}}$ (resp. $\sqrt{\mathrm{d}_{\mathbf{b}}}$). Since by choice of $\mathbf{a}$ and $\mathbf{b}$ we can deterministically bound $\mathrm{d}_{\mathbf{a}}$ and $\mathrm{d}_{\mathbf{b}}$ by $O(\delta s k N^{2/3})$, we get the claim choosing η large enough. □

We are now able to conclude. Let $\eta > 0$ be given by the claim above. Notice that

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}\left[\mathsf{Goodhit}_k \,\middle|\, \mathsf{BigMeanFL}_k(s) \cap \mathsf{SD}(\eta)\right] \geq \left(\tfrac{\delta}{2}\right)^2.$$

Write

$$\begin{aligned}\mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s)] &\leq \left(\tfrac{2}{\delta}\right)^2 \cdot c_2^{-1} \cdot \mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)] \\ &+ \left(\tfrac{2}{\delta}\right)^2 \cdot \mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s) \cap \mathsf{GoodHit}_k \cap (\mathsf{Bad}_\varepsilon^{+,-}) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}] \\ &+ \mathbb{P}_\lambda^{N^2}\left[\mathsf{BigMeanFL}_k(s) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}^c\right].\end{aligned}$$

To bound the first term we use (3.3.1) together with Proposition 3.2.4. To bound the second term we use Proposition 3.2.5 and proceed as in the proof of Proposition 3.2.10. Finally, the last term is bounded using Lemma 3.2.2. Putting all the pieces together, we get the existence of $c, C > 0$ such that for $3 \leq k \leq k_N$ (recall that $s = t/3$),

$$\mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s)] \leq C e^{-c(tk)^{3/2}}.$$

As a consequence, setting $\tilde{c} = 3N_0$, there exists some constants $c', C' > 0$ such that, for all $\tilde{c} \leq t \leq N^\varepsilon$,

$$\begin{aligned}\mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) \geq t N^{\frac{2}{3}}] &\leq \sum_{k=3}^{k_N} \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(t/3)] + \mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanFL}(\Gamma) \geq (t/3) k_N N^{\frac{2}{3}}\right] \\ &\leq C' e^{-c' t^{3/2}}.\end{aligned}$$

□

Proof of Proposition 3.3.2

From Proposition 3.3.1, it is easy to deduce the bound of Proposition 3.3.2 using the coupling introduced in Subsection 3.2.3.

Proof of Proposition 3.3.2. Let $t > 0$ and $\delta > 0$ be a (small) constant to be fixed later . We split the event $\{\mathsf{MeanLR} > tN^{1/3}\}$ according to the value of MeanFL,

$$\mathbb{P}_\lambda^{N^2}\big[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}\big] = \mathbb{P}_\lambda^{N^2}\big[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}, \mathsf{MeanFL}(\Gamma) \leq t^{2-\delta}N^{\frac{2}{3}}\big] + \mathbb{P}_\lambda^{N^2}\big[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}, \mathsf{MeanFL}(\Gamma) \geq t^{2-\delta}N^{\frac{2}{3}}\big].$$

By Proposition 3.3.1, we know that, for $\tilde{c} \leq t^{2-\delta} \leq N^{2/3}$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}, \mathsf{MeanFL}(\Gamma) \geq t^{2-\delta}N^{\frac{2}{3}}] \leq C\exp(-ct^{\frac{3}{2}(2-\delta)}).$$

Now for $j \geq 0$, let A_j be the following event:

$$A_j := \{\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}, \mathsf{MeanFL}(\Gamma) = j\}.$$

We can now make use of Lemma 3.2.7 to argue that for every $j \leq t^{2-\delta}N^{2/3}$,

$$\begin{aligned}\mathbb{P}_\lambda^{N^2}[A_j] &= \sum_{\substack{\|a-b\|=j \\ \arg(a) > \frac{\pi}{4} \geq \arg(b)}} \mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) \geq tN^{\frac{1}{3}} \mid \mathsf{Fac}_{a,b}]\mathbb{P}_\lambda^{N^2}[\mathsf{Fac}_{a,b}] \\ &\leq C\exp\big(-c\frac{t^2N^{2/3}}{j}\big)\mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) = j] \\ &\leq C\exp(-ct^{\delta})\mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) = j].\end{aligned}$$

The first equality comes from the fact that there exists exactly one pair a, b such that $\arg(a) > \frac{\pi}{4} \geq \arg(b)$ and $\mathsf{Fac}(a, b)$ occurs. Moreover, if $\mathsf{Fac}(a, b)$ occurs with such a choice of a and b, then $[a, b]$ will automatically be the mean facet, so Lemma 3.2.7 holds for the second line. Thus, we obtained,

$$\begin{aligned}\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}] &\leq C\exp(-ct^{\frac{3}{2}(2-\delta)}) + \sum_{j=0}^{t^{2-\delta}N^{2/3}} C\exp(-ct^{\delta})\mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) = j] \\ &\leq C\exp(-ct^{\frac{3}{2}(2-\delta)}) + C\exp(-ct^{\delta}).\end{aligned}$$

Equating the exponents yields an optimal value of $\delta = \frac{6}{5}$ and the result. $\square$

3.3.2 Lower bounds

Recall the setting of Theorem 3.1.3. Our target estimates are the following.

Proposition 3.3.8. *There exists a function $F_1 : \mathbb{R}^+ \to [0,1]$ satisfying $\lim_{t\to 0^+} F_1(t) = 0$, such that for any $t > 0$ small enough,*

$$\limsup_{N\to\infty} \mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanLR}(\Gamma) < tN^{\frac{1}{3}}\right] \leq F_1(t).$$

Proposition 3.3.9. *There exists a function $F_2 : \mathbb{R}^+ \to [0,1]$ satisfying $\lim_{t\to 0^+} F_2(t) = 0$, such that for any $t > 0$ small enough,*

$$\limsup_{N\to\infty} \mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanFL}(\Gamma) < tN^{\frac{2}{3}}\right] \leq F_2(t).$$

Remark 3.3.10. As before, we obtain

$$\liminf_{N\to\infty} \mathbb{E}_\lambda^{N^2}\left[\frac{\mathsf{MeanFL}(\Gamma)}{N^{2/3}}\right] > 0, \qquad \liminf_{N\to\infty} \mathbb{E}_\lambda^{N^2}\left[\frac{\mathsf{MeanLR}(\Gamma)}{N^{1/3}}\right] > 0.$$

Let us briefly describe the strategy of the proof. We carefully analyze the marginal of $\mathbb{P}_\lambda^{N^2}$ in the cone $\mathbf{A}_N$ of apex 0 and angular opening $N^{-1/3}$. Due to the Brownian Gibbs property, conditionally on the outside of the cone, this marginal is the law of a random walk bridge conditioned on capturing a random amount of area. We first prove in Lemma 3.3.12 that this random area is of order N, with Gaussian tails. Hence, the marginal of $\mathbb{P}_\lambda^{N^2}$ in $\mathbf{A}_N$ is absolutely continuous with respect to the law of a random walk bridge without the area conditioning, in the large N limit. The strategy of proof is thus clear: we prove the corresponding statements for the unconditioned random walk bridge (see Lemmas 3.3.16 and 3.3.17), and transmit then to $\mathbb{P}_\lambda^{N^2}$ thanks to the latter observation.

Let us introduce a few quantities that we will be useful in the remainder of this section.

Definition 3.3.11. For $N \geq 1$, we define $\theta_N := N^{-1/3}$. We set $\mathbf{A}_N$ to be the cone of apex 0 and angular opening of θ_N centered around the line $\{y = x\}$. If $\gamma \in \Lambda^{N^2}$, let us define $\mathbf{x}_N(\gamma)$ (resp. $\mathbf{y}_N(\gamma)$) to be point of γ that lies in $\mathbf{A}_N$ and that is the closest to the line constituting the left (resp. right) boundary of $\mathbf{A}_N$. Finally, we define

$$\mathcal{A}_N(\gamma) := \left|\mathrm{Enclose}(\gamma_{\mathbf{x}_N(\gamma),\mathbf{y}_N(\gamma)} \cap \mathbf{A}_{\mathbf{x}_N(\gamma),\mathbf{y}_N(\gamma)})\right| - \left|\mathbf{T}_{0,\mathbf{x}_N(\gamma),\mathbf{y}_N(\gamma)}\right|.$$

We shall also make a slight abuse of notation in considering $\mathcal{A}_N(\widetilde{\gamma})$ for some oriented path $\widetilde{\gamma}$ linking $\mathbf{x}_N(\gamma)$ to $\mathbf{y}_N(\gamma)$ (for some $\gamma \in \Lambda^{N^2}$). In that case, the meaning of $\mathcal{A}_N(\widetilde{\gamma})$ is clear: it is the corresponding $\mathcal{A}_N(\gamma')$ for any $\gamma' \in \Lambda^{N^2}$ which extends $\widetilde{\gamma}$ outside the cone $\mathbf{A}_{\mathbf{x}_N(\gamma),\mathbf{y}_N(\gamma)}$.

The following lemma will play a central role in the proofs of Propositions 3.3.8 and 3.3.9.

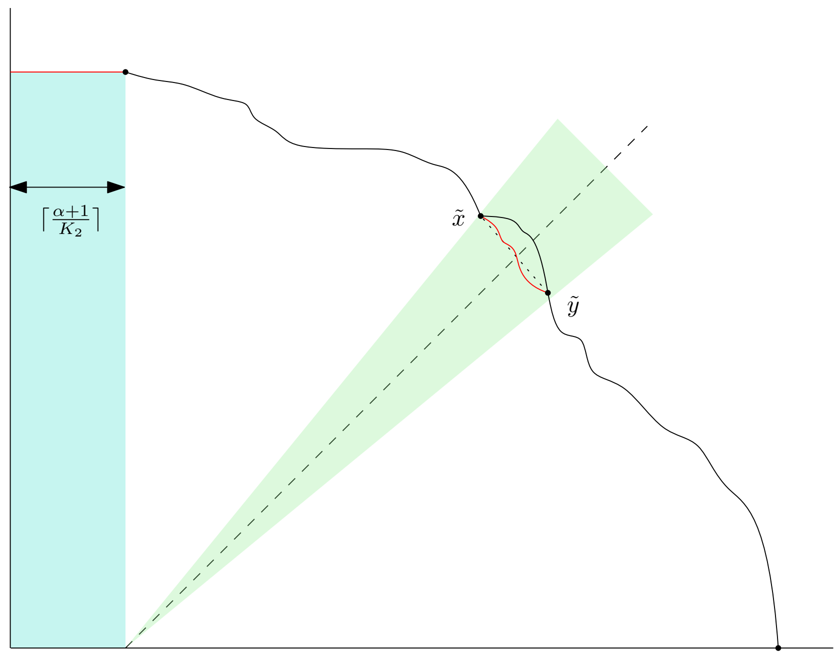

Figure 3.5: One possible output of the map $T_{x,y}$. The shift of (x, y) is denoted $(\tilde{x}, \tilde{y})$. The initial path is the black path (after a shift to the left). The two modifications involved in the procedure described above are represented in red. The red path in the light green shaded region has been sampled uniformly amongst the paths that capture a non-negative area (computed with respect to the black dashed line). The area captured by the turquoise region is sufficient to compensate the possible area loss due to the resampling of the path between x and y.

Lemma 3.3.12. *There exist two constants* $c, C > 0$ *such that for any* $\beta > 0$*, and for* N *large enough,*

$$\mathbb{P}_\lambda^{N^2}\left[\mathcal{A}_N(\Gamma) \geq \beta N\right] \leq C\mathrm{e}^{-c\beta^2}.$$

Proof. The proof of this statement heavily relies on the probabilistic multi-valued map principle stated in Lemma 3.5.3. For $\alpha > 0$, we introduce

$$\mathsf{BadArea}_\alpha^N := \left\{\gamma \in \Lambda^{N^2},\ \mathcal{A}_N(\gamma) \in [\alpha N, (\alpha+1)N]\right\}.$$

Define for $x, y \in \mathbb{N}^2$,

$$A_{x,y} := \mathsf{BadArea}_\alpha^N \cap \left\{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\right\} \cap \left\{\mathbf{x}_N(\Gamma) = x,\ \mathbf{y}_N(\Gamma) = y\right\} \\ \cap \left\{\Gamma \notin \mathsf{Bad}^{+,-}_{\varepsilon, \mathbf{x}_N(\Gamma), \mathbf{y}_N(\Gamma)}\right\},$$

where $\mathsf{Bad}^{+,-}_{\varepsilon,\mathbf{x}_N(\Gamma),\mathbf{y}_N(\Gamma)} := \mathsf{Bad}^{+}_{\varepsilon,\mathbf{A}_{\mathbf{x}_N(\Gamma),\mathbf{y}_N(\Gamma)}} \cup \mathsf{Bad}^{-}_{\varepsilon,\mathbf{A}_{\mathbf{x}_N(\Gamma),\mathbf{y}_N(\Gamma)}}$, and ε is given by Proposition 3.2.5.

Fix $x, y \in \mathbb{N}^2$ such that $\mathbb{P}_\lambda^{N^2}[A_{x,y}] \neq 0$.

Let us define a multi-valued map $T_{x,y} : A_{x,y} \to \mathcal{P}(\Lambda^{N^2})$ via the following two-step procedure (see Figure 3.5 for an illustration of the procedure). From some path $\gamma \in \mathsf{BadArea}_\alpha^N$, we can create a new oriented path of the first quadrant by erasing the portion of γ lying between x and y (previously called $\gamma_{x,y}$) and replacing it by any oriented path $\widetilde{\gamma}_{x,y}$ linking x to y thus obtaining an intermediate path $\widetilde{\gamma}$ which we require to satisfy $\mathcal{A}_N(\widetilde{\gamma}) \geq 0$, and finally by adding $\lceil \frac{\alpha+1}{K_2} \rceil$ horizontal steps at the beginning of $\widetilde{\gamma}$ (K_2 is the constant defined in 3.2.2). We define $T_{x,y}(\gamma)$ to be the subset of Λ of the paths that can be obtained from γ by this procedure. Notice that T is well defined since if $\gamma \in A_{x,y}$, the intermediate path $\widetilde{\gamma}$ defined above satisfies $\mathcal{A}(\widetilde{\gamma}) \geq N^2 - (\alpha+1)N$, and it is easy to check that the first $\lceil \frac{\alpha+1}{K_2} \rceil$ horizontal steps that have been added to $\widetilde{\gamma}$ capture an area at least equal to $(\alpha+1)N$ (this comes from the fact that $\gamma \in \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}$). Hence, for any $\gamma \in A_{x,y}$, $T_{x,y}(\gamma) \subset \Lambda^{N^2}$.

We are going to apply Lemma 3.5.3 to $T_{x,y}$. Before doing so, let us notice that we may co-restrict the map defined above to the set

$$B_{x,y} := \{\mathbf{x}_N(\Gamma^{\text{cut}}) = x, \mathbf{y}_N(\Gamma^{\text{cut}}) = y\} \cap \{\text{The first } \lceil \tfrac{\alpha+1}{K_2} \rceil \text{ steps are horizontal}\} \subset \Lambda^{N^2},$$

where Γ^{cut} is the path Γ minus the first $\lceil \frac{\alpha+1}{K_2} \rceil$ steps, translated of $\lceil \frac{\alpha+1}{K_2} \rceil$ towards the left direction, see Figure 3.5. Now, applying Lemma 3.5.3 yields

$$\mathbb{P}_\lambda^{N^2}[A_{x,y}] \leq \varphi(T_{x,y})\psi(T_{x,y})\mathbb{P}_\lambda^{N^2}[B_{x,y}], \tag{3.8}$$

where the quantities $\varphi(T_{x,y})$ and $\psi(T_{x,y})$ are defined in Lemma 3.5.3. It is very easy to check that

$$\varphi(T_{x,y}) = \left(\frac{1}{\lambda}\right)^{\lceil \frac{\alpha+1}{K_2} \rceil}, \qquad \psi(T_{x,y}) = \frac{|\{\gamma \in \Lambda^{x\to y}, \mathcal{A}_N(\gamma) \in [\alpha N, (\alpha+1)N]\}|}{|\{\gamma \in \Lambda^{x\to y}, \mathcal{A}_N(\gamma) \geq 0\}|}.$$

Recall that $\mathbb{P}_{x,y}$ is the uniform measure on $\Lambda^{x\to y}$. We can write

$$\psi(T_{x,y}) = \mathbb{P}_{x,y}\big[\mathcal{A}_N(\gamma) \in [\alpha N, (\alpha+1)N] \,\big|\, \mathcal{A}_N(\gamma) \geq 0\big].$$

Let us call $\theta(x,y)$ the positive angle formed by the segment $[x,y]$ and the horizontal line passing through x. Since we assumed that $\mathbb{P}_\lambda^{N^2}[A_{x,y}] \neq 0$, we have $\theta(x,y) \in [\varepsilon, \pi/2 - \varepsilon]$. Let $h_N := \|y - x\|$. As it turns out, h_N is of order $N^{2/3}$.

Claim 3.3.13. There exist two constants $c = c(\varepsilon), C = C(\varepsilon) > 0$ such that

$$h_N \in [cN^{2/3}, CN^{2/3}].$$

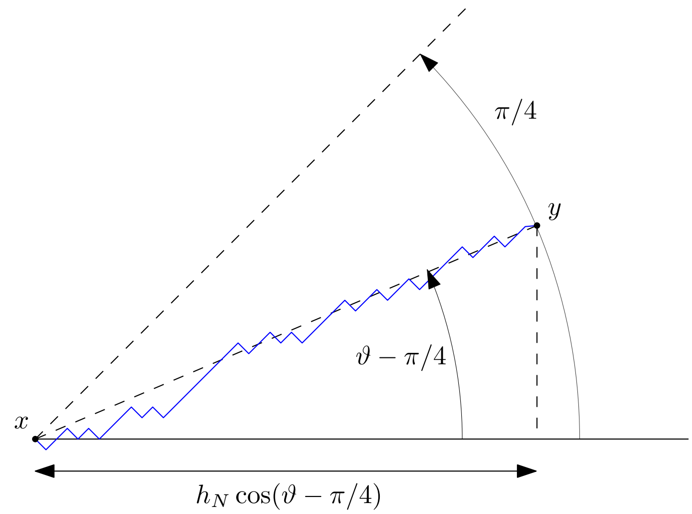

Figure 3.6: An illustration of a sample $X^{(\vartheta)}$ of $\mathbb{P}^{\vartheta}_{h_N}$ (in blue). This path corresponds to a symmetric random walk that does $\pm 2^{-1/2}$ jumps at times $k2^{-1/2}$ for $1 \leq k \leq \sqrt{2} h_N \cos(\vartheta - \pi/4)$.

Proof of Claim 3.3.13. The constraint $\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}$ enforces that for N large enough $h_N \geq \frac{1}{2} K_2 N \theta_N = \frac{1}{2} K_2 N^{2/3}$. Then, since we restricted ourselves to the case $\varepsilon \leq \theta(x, y) \leq \pi/2 - \varepsilon$, there exists $C_1 > 0$ such that

$$h_N \leq \frac{C_1}{\sin \varepsilon} N \theta_N = \frac{C_1}{\sin \varepsilon} N^{2/3}.$$

□

By translation invariance, $\mathbb{P}_{x,y}$ only depends on $\theta = \theta(x, y)$ and h_N. Since these parameters are more relevant we write $\mathbb{P}^{\theta}_{h_N} := \mathbb{P}_{x,y}$. Moreover, with this observation and to the cost of rotating the picture by $\pi/4$, we can always assume that samples of $\mathbb{P}^{\theta}_{h_N}$ start at 0 and end at $e^{i\pi/4}(y - x)$, see Figure 3.6. We will denote by $X^{(\theta)}$ a sample of $\mathbb{P}^{\theta}_{h_N}$.

Now,

$$\begin{aligned}
\mathbb{P}^{\theta}_{h_N}\left[\mathcal{A}_N(X^{(\theta)}) \in [\alpha N, (\alpha+1)N]\right] &= \mathbb{P}^{\theta}_{h_N}\left[h_N^{-3/2}\mathcal{A}_N(X^{(\theta)}) \in [\alpha N h_N^{-3/2}, (\alpha+1) N h_N^{-3/2}]\right] \\
&\leq \mathbb{P}^{\theta}_{h_N}\left[h_N^{-3/2}\mathcal{A}_N(X^{(\theta)}) \in [C^{-3/2}\alpha, c^{-3/2}(\alpha+1)]\right] \\
&\leq \sup_{\vartheta \in [\varepsilon, \frac{\pi}{2} - \varepsilon]} \mathbb{P}^{\vartheta}_{h_N}\left[h_N^{-3/2}\mathcal{A}_N(X^{(\vartheta)}) \in [C^{-3/2}\alpha, c^{-3/2}(\alpha+1)]\right],
\end{aligned}$$

where the second inequality follows by Claim 3.3.13.

Classical invariance theorems for random walks (started at 0) conditioned to reach αk after k steps (see [93] or [28, Remark 2.6]) yield the following convergence in distribution[6]: calling $(X^{(\pi/4)}_{th_N})_{0\leq t\leq 1}$ the trajectory[7] of a sample of $\mathbb{P}^{\pi/4}_{h_N}$, one has that under $\mathbb{P}^{\pi/4}_{h_N}$

$$\lim_{N\to\infty}\left(\frac{X^{(\pi/4)}_{th_N}}{\sigma\sqrt{2^{1/2}h_N}}\right)_{t\in[0,1]} = (\mathsf{BB}_t)_{t\in[0,1]},$$

where $(\mathsf{BB}_t)_{t\in[0,1]}$ is the standard Brownian bridge on $[0,1]$, and $\sigma = 2^{-1/2}$. This observation, together with the strategy described in [28, Remark 2.6], yields a similar invariance principle for the trajectory $(X^{(\vartheta)}_{th_N\cos(\vartheta-\pi/4)})_{0\leq t\leq 1}$ of a sample of $\mathbb{P}^{\vartheta}_{h_N}$. For $\vartheta\in(0,\pi/2)$ and $t\in[0,1]$, denote

$$f^{(\vartheta)}_N(t) := \frac{1}{\sigma(\vartheta)\sqrt{2^{1/2}h_N\cos(\vartheta-\pi/4)}}(X^{(\vartheta)}_{th_N\cos(\vartheta-\pi/4)} - th_N\sin(\vartheta-\pi/4)), \quad (3.9)$$

where $\sigma(\vartheta) := 2^{-1/2}\sqrt{1-\tan^2(\vartheta-\pi/4)}$. Then, for $\vartheta\in(0,\pi/2)$, under $\mathbb{P}^{\vartheta}_{h_N}$,

$$\lim_{N\to\infty}\left(f^{(\vartheta)}_N(t)\right)_{t\in[0,1]} = (\mathsf{BB}_t)_{t\in[0,1]}, \quad (3.10)$$

where the convergence holds in distribution. In particular, one has that

$$\begin{aligned} h_N^{-3/2}\mathcal{A}_N(X^{(\vartheta)}) &= h_N^{-3/2}\int_0^{h_N\cos(\vartheta-\pi/4)}(X^{(\vartheta)}_t - \tan(\vartheta-\pi/4)t)\mathrm{d}t \\ &= 2^{1/4}\sigma(\vartheta)[\cos(\vartheta-\pi/4)]^{3/2}\int_0^1 f^{(\vartheta)}_N(t)\mathrm{d}t. \end{aligned}$$

Hence, (3.3.2) implies the following convergence in distribution under $\mathbb{P}^{\vartheta}_{h_N}$:

$$\lim_{N\to\infty} h_N^{-3/2}\mathcal{A}_N(X^{(\vartheta)}) = c_0(\vartheta)\int_0^1 \mathsf{BB}_t\mathrm{d}t,$$

where $c_0(\vartheta) := 2^{1/4}\sigma(\vartheta)[\cos(\vartheta-\pi/4)]^{3/2}$.

Remark 3.3.14. It follows from [28, Remark 2.6] that the latter convergence is uniform (for instance at the level of the convergence of cumulative distribution function) in $\vartheta\in[\varepsilon,\pi/2-\varepsilon]$. Indeed, when the underlying random walk has increments with exponential moments, the authors identify the law of the random walk bridge conditioned to stay above a line of slope ϑ with a unconditioned random walk whose increments are given by a suitable exponential tilt

[6]The convergence holds in the space $\mathcal{C}([0,1])$ of continuous functions on $[0,1]$ equipped with the topology of uniform convergence.

[7]The sample is a priori only defined for the discrete times $k/\sqrt{2}$ for $0\leq k\leq\sqrt{2}h_N$ but we extend it to $[0,h_N]$ by linear interpolation.

of the increments of the former walk. Hence the Radon-Nikodym derivative of the ϑ-tilted walk with respect to the $\pi/4$-tilted walk is explicit and is a continuous function of ϑ. In particular, it is uniformly bounded on a compact interval such as $[\varepsilon, \pi/2-\varepsilon]$. We shall use this observation several times in what follows.

Call P the law of the Brownian bridge $(\mathsf{BB}_t)_{t\in[0,1]}$. The above observations yield that for N large enough,

$$\mathbb{P}^{\theta}_{h_N}\big[\mathcal{A}_N(X^\theta)\in[\alpha N,(\alpha+1)N]\big]\le 2\mathsf{P}\Big[\int_0^1 \mathsf{BB}_t \mathrm{d}t \ge c_0(\varepsilon)^{-1}C^{-3/2}\alpha\Big]. \tag{3.11}$$

A similar reasoning yields the lower bound, for N large enough,

$$\mathbb{P}^{\theta}_{h_N}[\mathcal{A}_N(X^\theta)\ge 0]\ge \frac{1}{2}\mathsf{P}\left[\int_0^1 \mathsf{BB}_t \mathrm{d}t\ge 0\right]=:\eta. \tag{3.12}$$

Gathering (3.3.2) and (3.3.2), we get

$$\psi(T_{x,y})\le 2\eta^{-1}\mathsf{P}\Big[\int_0^1 \mathsf{BB}_t \mathrm{d}t\ge c_0(\varepsilon)^{-1}C^{-3/2}\alpha\Big].$$

Using the following estimate on the tail area of a Brownian bridge (see [94, Theorem 1.2]), we obtain $c_1=c_1(\varepsilon), C_1=C_1(\varepsilon)>0$ such that

$$\mathsf{P}\Big[\int_0^1 \mathsf{BB}_t \mathrm{d}t\ge c_0(\varepsilon)^{-1}C^{-3/2}\alpha\Big]\le C_1\mathrm{e}^{-c_1\alpha^2}. \tag{3.13}$$

Using (3.3.2), (3.3.2), and (3.3.2), we get that for N large enough,

$$\begin{aligned}\mathbb{P}^{N^2}_{\lambda}[A_{x,y}] &\le 2C_1\eta^{-1}\mathrm{e}^{-c_1\alpha^2}\lambda^{-\lceil\frac{\alpha+1}{K_2}\rceil}\mathbb{P}^{N^2}_{\lambda}[B_{x,y}]\\ &\le C_2\mathrm{e}^{-c_2\alpha^2}\mathbb{P}^{N^2}_{\lambda}[B_{x,y}].\end{aligned}$$

To obtain the lemma, it remains to sum over $\alpha\ge\beta$ and over all possible values for x and y,

$$\mathbb{P}^{N^2}_{\lambda}[\mathcal{A}_N(\Gamma)\ge\beta N,\ \Gamma\subset B_{K_1N}\setminus B_{K_2N},\ \Gamma\notin \mathsf{Bad}^{+,-}_{\varepsilon,\mathbf{x}_N(\Gamma),\mathbf{y}_N(\Gamma)}]\le C_2\sum_{\alpha=\lfloor\beta\rfloor}^{+\infty}\exp(-c_2\alpha^2).$$

Then, for some $C_3>0$,

$$\mathbb{P}^{N^2}_{\lambda}[\mathcal{A}_N(\Gamma)\ge\beta N,\ \Gamma\subset B_{K_1N}\setminus B_{K_2N},\ \Gamma\notin \mathsf{Bad}^{+,-}_{\varepsilon,\mathbf{x}_N(\Gamma),\mathbf{y}_N(\Gamma)}]\le C_3\mathrm{e}^{-c_2\beta^2}.$$

Using Lemma 3.2.2 and Proposition 3.2.5, there exists $C_4>0$ such that for N large enough (namely $N\ge\beta^3$),

$$\mathbb{P}^{N^2}_{\lambda}[\mathcal{A}_N(\Gamma)\ge\beta N]\le C_4\mathrm{e}^{-c_2\beta^2}.$$

□

Definition 3.3.15. If $\gamma_{u,v} \in \Lambda^{u \to v}$ and $z \in \gamma_{u,v}$, we define $\mathsf{LR}(z)(\gamma_{u,v})$, *the local roughness of the vertex* $z \in \gamma_{u,v}$, to be the local roughness of z in the path $\gamma_{u,v}$ extended to the left with only horizontal steps to up to the y-axis and to the right with only vertical steps up to the x-axis. We define the mean facet length of $\gamma_{u,v}$ similarly.

The two next lemmas study the typical behaviours of the mean facet length and of the local roughness of a uniform path between $\mathbf{x}_N(\Gamma)$ and $\mathbf{y}_N(\Gamma)$ when Γ is sampled according to $\mathbb{P}_\lambda^{N^2}$. For simplicity, in the rest of the section, when Γ is a sample of $\mathbb{P}_\lambda^{N^2}$, we shall abbreviate $\mathbf{x}_N(\Gamma)$ (resp. $\mathbf{y}_N(\Gamma)$) by $\mathbf{x}_N$ (resp. $\mathbf{y}_N$). The proofs of these lemmas rely on Gaussian computations and are not specific to the considered model. They are deferred to the end of the section.

We will write $\mathcal{G}_N := \{\Gamma \in B_{K_1 N} \setminus B_{K_2 N}\} \cap (\mathsf{Bad}^{+;-}_{\varepsilon,\mathbf{x}_N,\mathbf{y}_N})^c$.

Lemma 3.3.16. *There exists a function* Ψ *independent of every parameter of the problem with* $\Psi(t) \xrightarrow[t \to 0^+]{} 0$ *such that, for* $t > 0$*, if* N *is large enough,*

$$\mathbb{E}_\lambda^{N^2}\left[\mathbb{1}_{\mathcal{G}_N} \mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3}]\right] \le \Psi(t).$$

The corresponding statement for the mean facet length is the following

Lemma 3.3.17. *There exists a function* Φ *independent of every parameter of the problem with* $\Phi(t) \xrightarrow[t \to 0^+]{} 0$ *such that, for* $t > 0$*, if* N *is large enough,*

$$\mathbb{E}_\lambda^{N^2}\left[\mathbb{1}_{\mathcal{G}_N} \mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}[\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{2/3}]\right] \le \Phi(t).$$

These two lemmas in hand, we can now prove the lower bound of Proposition 3.3.8. Proposition 3.3.9 will be proved by the same method.

Proof of Proposition 3.3.8. Fix some $t > 0$, and some $\beta > 0$ that is going to be chosen later as a function of t. We implement the following resampling procedure: let us start with a sample Γ of $\mathbb{P}_\lambda^{N^2}$. As usual we condition on the configuration outside of the cone $\mathbf{A}_{\mathbf{x}_N,\mathbf{y}_N}$ that we name $\Gamma_{\mathsf{ext}} := \Gamma \setminus \Gamma_{\mathbf{x}_N,\mathbf{y}_N}$. By the Brownian Gibbs property, the distribution of $\Gamma_{\mathbf{x}_N,\mathbf{y}_N}$ is uniform among the oriented paths linking $\mathbf{x}_N$ and $\mathbf{y}_N$ that enclose at least a certain amount of area which is measurable with respect to the exterior configuration. Let us call A_{ext} the area already enclosed by Γ_{ext} (namely, $A_{\mathsf{ext}} = \mathrm{Enclose}(\Gamma \cap \mathbf{A}^c_{\mathbf{x}_N,\mathbf{y}_N} \cap \mathbb{N}^2)$), so that the conditional distribution of $\Gamma_{\mathbf{x}_N,\mathbf{y}_N}$ is nothing but $\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\left[\,\cdot \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \ge N^2 - A_{\mathsf{ext}}\right]$.

Before starting, let us point out a crucial fact. With our definition of $\mathsf{LR}(z)(\Gamma_{u,v})$, it is a deterministic fact that for any $u, v \in \Gamma$, $z \in \Gamma_{u,v}$,

$$\mathsf{LR}(z)\,(\Gamma_{u,v}) \le \mathsf{LR}(z)\,(\Gamma) \quad \text{and thus} \quad \mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \le \mathsf{MeanLR}(\Gamma).$$

Thus, we have

$$\mathbb{P}_\lambda^{N^2}\big[\mathsf{MeanLR}(\Gamma) < tN^{1/3}\big] \\ \leq \mathbb{E}_\lambda^{N^2}\big[\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3} \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}\big]\big].$$

Introduce

$$\mathcal{F}_N := \{N^2 - A_{\mathsf{ext}} \leq \beta N\} \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\} \cap (\mathsf{Bad}^{+;-}_{\varepsilon,\mathbf{x}_N,\mathbf{y}_N})^c,$$

where $\mathsf{Bad}^{+;-}_{\varepsilon,\mathbf{x}_N,\mathbf{y}_N}$ was introduced in the proof of Lemma 3.3.12. Note that by Lemma 3.2.2, Proposition 3.2.5 and Lemma 3.3.12, one has that for $N \geq \beta^3$,

$$\mathbb{P}_\lambda^{N^2}[\mathcal{F}_N^c] \leq Ce^{-c\beta^2}.$$

Hence,

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3} \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}]\big] \\ \leq \mathbb{E}_\lambda^{N^2}\big[\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3} \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}\big]\mathbb{1}_{\mathcal{F}_N}\big] \\ + Ce^{-c\beta^2}.$$

Now, note that

$$\mathbb{1}_{\mathcal{F}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3} \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}\big] \\ \leq \mathbb{1}_{\mathcal{F}_N}\frac{\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3}]}{\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}[\mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq \beta N]}.$$

Using the methods developed above[8] we could also show that for some $c', C' > 0$, if N is large enough,

$$\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\left[\mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq \beta N\right] \geq C'\mathrm{e}^{-c'\beta^2}.$$

By Lemma 3.3.16, we obtain that, integrating over Γ_{ext}, and putting everything together,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) < tN^{1/3}] \leq C\mathrm{e}^{-c\beta^2} + \frac{\Psi(t)}{C'\mathrm{e}^{-c'\beta^2}}.$$

It remains to carefully chose β in terms of t. We optimize the above equation by setting

$$\beta(t) = \sqrt{(c+c')^{-1}\log(\tfrac{c}{c'}\tfrac{CC'}{\Psi(t)})}.$$

Observe that $\beta(t) \xrightarrow[t\to 0]{} +\infty$. Choosing N large enough, we obtain that for some $C'' > 0$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) < tN^{1/3}] \leq C''(\Psi(t))^{\frac{c}{c+c'}}$$

which is the announced result. □

[8] This is again an application of Donsker's Theorem and the use of the asymptotic results of [94], we do not write the full proof but rather refer to the proof of Lemmas 3.3.16 and 3.3.17 for details.

Proposition 3.3.9 is a consequence of the exact same argument.

Proof of Proposition 3.3.9. We proceed exactly as in the proof of Proposition 3.3.8. Indeed, observe that as previously,

$$\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \leq \mathsf{MeanFL}(\Gamma).$$

Hence, the proof can be reproduced *mutatis mutandi* using this time Lemma 3.3.17. □

We now turn to the proofs of the two Gaussian lemmas, namely Lemma 3.3.16 and Lemma 3.3.17. The process of the concave majorant of a Brownian bridge has been extensively studied in [72, 107, 12]. Let us briefly summarise their results.

- Groeneboom showed in [72] that conditionally on the convex majorant of a Brownian motion, the difference process was a succession of independent Brownian excursions between the extremal points of the concave majorant. The result was later extended to the standard Brownian bridge in [12] via a Doob tranformation.
- Groeneboom also gave an explicit representation of the distribution of the process of the slopes of the concave majorant. Later, Suidan [107] was able to derive the joint law of the ordered lengths of the segments of $[0, 1]$ in the partition of $[0, 1]$ induced by the extremal points of the concave majorant of a Brownian bridge.

We refer to the introduction of [12] for a clear and detailed presentation of these results. In particular, we import the following statement from the three cited articles.

Theorem 3.3.18. *Let* $(\mathsf{BB}_t)_{0\leq t\leq 1}$ *be the standard Brownian bridge and let* $(\mathcal{C}(t))_{0\leq t\leq 1}$ *be its smallest concave majorant. Let* $s \in (0, 1)$*. Let us define the two following random variables:*

- $L(s)$ *is the length of the almost surely unique facet of* $\mathcal{C}$ *intersecting the line* $\{x = s\}$*,*
- $R(s)$ *is the difference process defined by* $R(s) := \mathcal{C}(s) - B(s)$*.*

Fix $\varepsilon > 0$*. Then, one has*

$$\sup_{s\in[\varepsilon,1-\varepsilon]} \mathbb{P}\left[L(s) < r\right] \xrightarrow[r\to 0^+]{} 0,$$

and

$$\sup_{s\in[\varepsilon,1-\varepsilon]} \mathbb{P}\left[R(s) < r\right] \xrightarrow[r\to 0^+]{} 0.$$

We briefly sketch the proof of Theorem 3.3.18 and refer to [72, 107, 12] for more details.

Sketch of proof of Theorem 3.3.18. Fix some $s \in [\varepsilon, 1-\varepsilon]$. For $L(s)$, we remark that the results of [72] imply that $\mathcal{C}$ almost surely has a finite number of slope changes (or extremal points) in $[\varepsilon, 1-\varepsilon]$, and [12] gives an explicit description of their lengths in terms of a size-biased uniform stick-breaking process. In particular, it implies that the distribution of $L(s)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+$, which implies that $\mathbb{P}\left[L(s) < r\right] \xrightarrow[r\to 0]{} 0$. The statement follows by the observation that $s \in (0,1) \mapsto \mathbb{P}\left[L(s) < r\right]$ is continuous.

For $R(s)$, let us denote by $a \leq s \leq b$ the horizontal coordinates of the endpoints of the facet intersecting the line $\{x = s\}$. The exact same argument as above implies that $\mathbb{P}\left[\min(s-a, b-s) < \varepsilon\right]$ goes to 0 when ε goes to 0. On the complementary of the latter event, by [72], $R(s)$ is the height of a Brownian excursion at a positive distance of its endpoints, and is indeed continuous with respect to the Lebesgue measure on $\mathbb{R}^+$. We conclude as previously, observing that $s \in (0,1) \mapsto \mathbb{P}\left[R(s) < r\right]$ is continuous. □

Now observe that in Lemmas 3.3.16 and 3.3.17, $\|\mathbf{y}_N - \mathbf{x}_N\|$ is of order $N^{2/3}$ so that the scaling is Gaussian. Hence, Donsker's Theorem suggests that we may use the above-mentioned results to obtain lower tails for the random variables $N^{-1/3}\mathsf{MeanLR}$ and $N^{-2/3}\mathsf{MeanFL}$.

Proof of Lemma 3.3.16. As in the proof of Lemma 3.3.12, we write $h_N = \|\mathbf{y}_N - \mathbf{x}_N\|$. Moreover, let us call $\mathbf{t}_N^0$ the intersection between the line segment $[\mathbf{x}_N, \mathbf{y}_N]$ and the line $\{y = x\}$. Remember that we work under the event $\{\Gamma \in B_{K_1 N} \setminus B_{K_2 N}\} \cap (\mathsf{Bad}^{+,-}_{\varepsilon,\mathbf{x}_N,\mathbf{y}_N})^c$, which allows us to reuse Claim 3.3.13, namely

$$cN^{2/3} \leq h_N \leq CN^{2/3}. \tag{3.14}$$

Moreover, there exists a constant $c_\varepsilon > 0$ such that

$$\min(\|\mathbf{t}_N^0 - \mathbf{x}_N\|_2, \|\mathbf{y}_N - \mathbf{t}_N^0\|) \geq c_\varepsilon h_N. \tag{3.15}$$

Recall the parametrisation of the probability measures $\mathbb{P}^\theta_{h_N} = \mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}$ introduced in the proof of Lemma 3.3.12. Similarly as before,

$$\begin{aligned}
&\mathbb{E}_\lambda^{N^2}\left[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\left[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3}\right]\right] \\
&\qquad = \mathbb{E}_\lambda^{N^2}\left[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}^\theta_{h_N}\left[h_N^{-1/2}\mathsf{LR}(\mathbf{t}_N^0) < t h_N^{-1/2} N^{1/3}\right]\right] \\
&\qquad \leq \mathbb{E}_\lambda^{N^2}\left[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}^\theta_{h_N}\left[h_N^{-1/2}\mathsf{LR}(\mathbf{t}_N^0) < c^{-1/2} t\right]\right] \\
&\qquad \leq \mathbb{E}_\lambda^{N^2}\left[\mathbb{1}_{\mathcal{G}_N}\sup_{\vartheta\in[\varepsilon,\pi/2-\varepsilon]}\mathbb{P}^\vartheta_{h_N}\left[h_N^{-1/2}\mathsf{LR}(\mathbf{t}_N^0) < c^{-1/2} t\right]\right],
\end{aligned}$$

where c is the constant appearing in (3.3.2). Now, observe that $\mathbf{t}_0^N$ is independent of the random path sampled according to $\mathbb{P}^\vartheta_{h_N}$, and we can write

$$\mathbb{P}^\vartheta_{h_N}\left[h_N^{-1/2}\mathsf{LR}(\mathbf{t}_N^0) < c^{-1/2}t\right] \leq \sup_{s\in[c_\varepsilon, 1-c_\varepsilon]} \mathbb{P}^\vartheta_{h_N}\left[h_N^{-1/2}\mathsf{LR}(h_N s) < c^{-1/2}t\right],$$

Thus, we obtained that

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3}\big]\big] \\ \leq \mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\sup_{\vartheta\in[\varepsilon,\pi/2-\varepsilon]}\sup_{s\in[c_\varepsilon,1-c_\varepsilon]}\mathbb{P}_{h_N}^{\vartheta}\big[h_N^{-1/2}\mathsf{LR}(sh_N) < c^{-1/2}t\big]\big], \quad (3.16)$$

where c_ε is the constant appearing in (3.3.2). Reasoning as above, one has the following convergence in distribution (the random variable R was introduced in Theorem 3.3.18), under $\mathbb{P}_{h_N}^{\vartheta}$ and assuming $\mathcal{G}_N$ occurs, for $s \in [c_\varepsilon, 1-c_\varepsilon]$,

$$\lim_{N\to\infty} h_N^{-1/2}\mathsf{LR}(sh_N) = \gamma(\vartheta)R(s),$$

where $\gamma(\vartheta) = \sigma(\vartheta)\sqrt{2^{1/2}\cos(\vartheta-\pi/4)}$ and $\sigma(\vartheta)$ was defined below (3.3.2). Once again, we claim that this convergence is uniform in $\vartheta \in [\varepsilon, \pi/2-\varepsilon]$ (see Remark 3.3.14). Hence, the limsup of the right-hand side of (3.3.2) is bounded by

$$\sup_{s\in[c_\varepsilon,1-c_\varepsilon]} \mathsf{P}[R(s) < tc^{-1/2}\gamma(\varepsilon)^{-1}].$$

We argue as in the proof of Lemma 3.3.12 to get for N large enough

$$\begin{aligned}\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3}\big]\big] &\leq C_1\sup_{s\in[c_\varepsilon,1-c_\varepsilon]}\mathsf{P}\big[R(s) < tc^{-1/2}\gamma(\varepsilon)^{-1}\big] \\ &=: \Psi(t).\end{aligned}$$

The fact that Ψ goes to 0 is a consequence of Theorem 3.3.18. □

We turn to the proof of Lemma 3.3.17.

Proof of Lemma 3.3.17. The proof follows the exact same strategy as above. Indeed, keeping the notations of the proof of Lemma 3.3.16, we may write

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{2/3}\big]\big] \\ \leq \mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\sup_{\vartheta\in[\varepsilon,\pi/2-\varepsilon]}\mathbb{P}_{h_N}^{\vartheta}\big[h_N^{-1}\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tc^{-1}\big]\big].$$

We are going to use the same method as in the proof of Lemma 3.3.16. We refer to [107, Theorems 4 and 5] to observe that the facet length scales as the time h_N in a random walk bridge. Some extra care is needed due to the fact that the function $\phi \in \mathcal{C}([0,1],\mathbb{R}) \mapsto \mathsf{MeanFL}(\phi) \in \mathbb{R}^+$ is not continuous in ϕ; however it is easy to check that its points of discontinuity are included in the set of continuous functions having at least three aligned local maxima. Using the fact that the set of such functions has measure 0 under P and the above-mentioned uniformity in ϑ, we obtain from arguments similar to that of proof of Lemma 3.3.16 that there exists a constant $\delta(\varepsilon) > 0$ such that, for N large enough,

$$\begin{aligned}\mathbb{E}_{\lambda}^{N^2}\left[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\left[\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{2/3}\right]\right] &\leq C_1' \sup_{s\in[c_\varepsilon,1-c_\varepsilon]} \mathsf{P}[L(s) < t\delta(\varepsilon)c^{-1}] \\ &=: \Phi(t)\,.\end{aligned}$$

We conclude using Theorem 3.3.18. □

3.4 Analysis of the maximal facet length and of the maximal local roughness

This section is devoted to the proof of Theorem 3.1.5. The corresponding statements in the context of the random-cluster have been obtained by Hammond in [74, 75] (see Section 3.5 for a detailed discussion). We will essentially adapt the robust arguments developed by Hammond in these papers although some new difficulties, coming from the oriented feature of the model, will emerge.

The proof of the upper bound of (3.1.5) is in spirit the same as the one of Proposition 3.2.10 observing that (with care) the proof can be reproduced replacing the N^{ε} by $C(\log N)^{1/3}$ with a large C.

The proof of the lower bound is more technical and follows the lines of [75]. We use a different strategy of resampling consisting in successively resampling Γ in deterministic angular sectors. We then show that each one of these resampling has a sufficiently large probability of producing a favourable output (meaning with a large local roughness)— and thus that the final output has a large local roughness with very high probability. However, we will encounter several technical issues, the main one being the fact that a favourable ouptut could be destroyed by further resamplings. This technical point will be solved by carefully analysing the Markovian dynamic induced by the successive resamplings in Section 3.4.2. The bound on the maximal facet length will be a simple byproduct of the bound on the maximal local roughness.

3.4.1 Upper bounds

We start by proving the upper bound on MaxFL. The correct control is given by the following proposition.

Proposition 3.4.1. *There exist constants $\tilde{c}, c, C > 0$ such that, for any $\tilde{c} \leq t \leq N^{2/3}(\log N)^{-2/3}$,*

$$\mathbb{P}_{\lambda}^{N^2}\left[\mathsf{MaxFL}(\Gamma) \geq tN^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\right] \leq Ce^{-ct^{3/2}\log N}.$$

The proof of this proposition follows the same lines as the proof of Proposition 3.2.10 with the exception that we modify the event GAC (x, y, η) of Subsection 3.3.1 to require a poly-logarithmic deviation of the area from its typical value.

Definition 3.4.2 (Logarithmic good area capture)**.** Let $\gamma \in \Lambda$ be a path, $\eta > 0$ and $x, y \in \mathbb{N}^2$. We say that γ realizes the event $\mathsf{LogGAC}(x, y, \eta)$ (meaning "logarithmic good area capture") if

(*i*) $\gamma \cap \mathbf{A}_{x,y} \in \Lambda^{x \to y}$, or in words, γ connects x and y by a oriented path,

(*ii*) $|\text{Enclose}(\gamma \cap \mathbf{A}_{x,y})| - |\mathbf{T}_{0,x,y}| \geq \eta \parallel x - y \parallel^{\frac{3}{2}} (\log \parallel x - y \parallel)^{\frac{1}{2}}$.

As previously, we need to estimate the probability of $\mathsf{LogGAC}(x, y, \eta)$ under $\mathbb{P}_{x,y}$, the uniform measure on $\Lambda^{x \to y}$. The following result, whose proof is postponed to the Appendix 3.5.3, gives us a lower bound on $\mathbb{P}_{x,y}[\mathsf{LogGAC}(x, y, \eta)]$. Recall that $\theta(x, y) \in [0, \pi/2]$ the angle formed by the horizontal axis and the segment joining x and y.

Lemma 3.4.3. *Let $\varepsilon, \eta > 0$. There exist $C = C(\varepsilon) > 0$ and $N_0 = N_0(\varepsilon, \eta) > 0$ such that for any $x, y \in \mathbb{N}^2$ satisfying $\Lambda^{x \to y} \neq \emptyset$, $\varepsilon \leq \theta(x, y) \leq \pi/2 - \varepsilon$ and $\|x - y\| \geq N_0$,*

$$\mathbb{P}_{x,y} [\mathsf{LogGAC}(x, y, \eta)] \geq \|x - y\|^{-C\eta^2}.$$

Proposition 3.2.5 will ensure that the condition on $\theta(x, y)$ holds with high probability.

Proof of Proposition 3.4.1. We repeat the same strategy as in the proof of Proposition 3.2.10 considering this time the event

$$\mathsf{BigFacet}(t) := \left\{ \mathsf{MaxFL}(\Gamma) \geq t N^{\frac{2}{3}} (\log N)^{\frac{1}{3}} \right\}$$

instead of $\mathsf{BigMeanFL}(t)$, LogGAC instead of GAC, and replacing $\mathsf{GoodHit}$ by the event that the random pair $(\mathbf{x}, \mathbf{y})$ hits the extremities of the largest facet. The control on the probability of LogGAC is given by Lemma 3.4.3 above. ☐

With this result, it is now easy to estimate the correct scale for MaxLR. The statement is given in the following proposition.

Proposition 3.4.4. *There exist constants $\tilde{c}, c, C > 0$ such that, for any $\tilde{c} \leq t \leq N^{5/6} (\log N)^{-5/6}$, then,*

$$\mathbb{P}_{\lambda}^{N^2} \left[\mathsf{MaxLR}(\Gamma) \geq t N^{\frac{1}{3}} (\log N)^{\frac{2}{3}} \right] \leq C e^{-ct^{6/5} \log N}.$$

Proof. Denote by MaxFac the facet where $\mathsf{MaxLR}(\Gamma)$ is attained, and by $\mathsf{MLRF}(\Gamma)$ its length. It is clear that

$$\mathsf{MLRF}(\Gamma) \leq \mathsf{MaxFL}(\Gamma).$$

With this remark and Proposition 3.4.1, we can conclude using the exact same method as in the proof of Proposition 3.3.2, the only difference being that we now use the coupling along the facet which has the largest local roughness.

More precisely, let $\delta > 0$ to be fixed later. Let $t > 0$, then

$$\left\{\mathsf{MaxLR}(\Gamma) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}}\right\} \subset$$
$$\left\{\mathsf{MLRF}(\Gamma) \geq t^{2-\delta}N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\right\} \cup A \cup \left\{\Gamma \subset (B_{K_1 N})^c\right\} \cup \bigcup_{j=N^{\frac{1}{3}}}^{t^{2-\delta}N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}} A_j, \quad (3.17)$$

where

$$A_j = \left\{\mathsf{MaxLR}(\Gamma) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}},\ \mathsf{MLRF}(\Gamma) = j,\ \Gamma \subset B_{K_1 N}\right\},$$
$$A = \left\{\mathsf{MaxLR}(\Gamma) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}},\ \mathsf{MLRF}(\Gamma) < N^{\frac{1}{3}}\right\},$$

and K_1 is defined as in Lemma 3.2.2. It is easy to check that $A = \emptyset$ for $t \geq 1$, which is now our assumption. In the right-hand side of (3.4.1), we already know how to control the probability of the first three terms. The proof will follow from obtaining an upper bound on $\mathbb{P}_\lambda^{N^2}[A_j]$. This bound is (again) given by the Lemma 3.2.7 (see Remark 3.2.8). Let us denote by $\{\|\mathsf{Fac}(x)\| = j\}$ the event that the facet of the vertex x has length j. Then, for $N^{1/3} \leq j \leq t^{2-\delta}N^{2/3}(\log N)^{1/3}$,

$$\begin{aligned}
\mathbb{P}_\lambda^{N^2}[A_j] &\leq \sum_{x \in B_{K_1 N}} \mathbb{P}_\lambda^{N^2}[\mathsf{LR}(x) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}},\ \|\mathsf{Fac}(x)\| = j] \\
&\leq \sum_{x \in B_{K_1 N}} \sum_{\substack{a,b \in \mathbb{N}^2 \\ \|b-a\|=j \\ \arg(a)<\arg(x)<\arg(b)}} \mathbb{P}_\lambda^{N^2}[\mathsf{LR}(x) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}} \bigm| \mathsf{Fac}(a,b)]\mathbb{P}_\lambda^{N^2}[\mathsf{Fac}(a,b)] \\
&\leq O(N^2 j^4)\exp(-c\frac{t^2 N^{2/3}(\log N)^{4/3}}{2j}) \\
&\leq \mathrm{e}^{-ct^\delta \log N}
\end{aligned}$$

when the third inequality is a consequence of Lemma 3.2.7 (and more particularly Remark 3.2.8). Set $\delta = \frac{6}{5}$ and take $t_0 \leq t \leq N^{5/6}(\log N)^{-5/6}$, where t_0 is large enough. Applying Lemma 3.2.2 and Proposition 3.4.1 yields the desired bound. □

3.4.2 Lower bounds

We start with the proof of the lower bound for MaxLR and then see how we can deduce the lower bound for MaxFL.

The strategy

We rely again on the strategy of resampling. We will divide the upper-right quarter plane in angular sectors of angle

$$\theta_N = \theta_N(\chi) := \chi N^{-\frac{1}{3}}(\log N)^{\frac{1}{3}},$$

where $\chi > 0$ is a small constant that we will specify later. Then, we will randomly process a resampling in some of these sectors. We will argue that each one of these sectors has a probability of producing a "favourable" output (i.e. with a large local roughness) which is decaying slowly (see Lemma 3.4.19) in N, i.e. roughly like $N^{-\delta}$ for some (small) $\delta > 0$. By iterating these resamplings on different (sufficiently many) sectors we ensure to recover a new path which presents at least one sector with a large local roughness. There is a crucial point we will have to deal with: if a given sector produces a favorable output, this could be undone by a further resample. That is why we will resample on sufficiently distant sectors (see Proposition 3.4.20).

Let us first fix some notations. Define

$$m_N := \left\lfloor \frac{\pi}{2\theta_N} \right\rfloor$$

to be the number of sectors subject to resampling. Note that

$$m_N \underset{N\to+\infty}{\sim} \frac{\pi}{2\chi} N^{\frac{1}{3}} (\log N)^{-\frac{1}{3}}. \tag{3.18}$$

Denote by $\mathbf{A}_j(N)$ the sector of the first quadrant defined by the angles $(j-1)\theta_N$ and $j\theta_N$ for $j \in \{1, \ldots, m_N\}$. Note that the first quadrant may contain a narrower sector which we will not make use of. Finally, we introduce the following notation: given two points a and b in the first quadrant, we denote by $\angle(a, b)$ the (non negative[9]) angle between the associated vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ [10]. For a closed shape S, recall that Enclose(S) is the region of the first quadrant enclosed by S (and if needed the coordinate axes).

Defining the resampling procedure

We want to resample in the sectors $\mathbf{A}_j(N)$. However, choosing the beginning and ending points of the pieces subject to resampling in each sector must be done precociously. Recall that for $a, b \in \mathbb{N}^2$, $\Lambda^{a\to b}$ is the set of oriented paths from a to b.

Definition 3.4.5. Let $x, y \in \mathbb{N}^2$ be such that $\Lambda^{x\to y} \neq \emptyset$. Denote by $\mathbf{A}_{x,y}$ the cone of apex 0 bounded by x and y. For all $\gamma \in \Lambda^{N^2}$, $\Psi_{x,y}(\gamma)$ is the random element of Λ that is equal to γ outside $\mathbf{A}_{x,y}$, and that is random on $\mathbf{A}_{x,y}$ having the marginal law of $\mathbb{P}_\lambda^{N^2}$ on the considered sector, given the area condition and the path outside the sector.

Remark 3.4.6. In words, the formation of $\Psi_{x,y}(\gamma)$ can be summed up in two steps:

- Step A: we condition on the marginal of γ on $\mathbf{A}_{x,y}$ by the information of $\gamma \cap \mathbf{A}^c_{x,y}$.
- Step B: we condition on the area: $|\text{Enclose}((\gamma \cap \mathbf{A}^c_{x,y}) \cup \gamma_{x,y})| \geq N^2$.

[9] We will always choose $\angle(a, b) \in [0, \pi/2]$.

[10] Sometimes we will also write $\angle(\overrightarrow{a}, \overrightarrow{b})$, depending on the context.

The strategy is now the following: for each sector $\mathbf{A}_j(N)$, we pick some points x_j, y_j that belong to this sector, and to the random path we are considering, and we apply Ψ_{x_j,y_j}.

Definition 3.4.7 (Definition of RES_j)**.** Let $j \in \{1,\dots,m_N\}$. We define a procedure $\mathsf{RES}_j : \Lambda^{N^2} \to \Lambda$ that only acts on $\mathbf{A}_j(N)$. Let $\mathbf{B}_j(N)$ be the cone of apex 0 defined by the angles $(j-1)\theta_N + \theta_N/4$ and $j\theta_N - \theta_N/4$. Notice that $\mathbf{B}_j(N) \subset \mathbf{A}_j(N)$. Let $\gamma \in \Lambda^{N^2}$. Let $x_j = x_j(\gamma)$ (resp. $y_j = y_j(\gamma)$) be the left-most (resp. right-most) point of $\gamma \cap \mathbf{B}_j(N)$. The procedure RES_j consists in resampling γ between x_j and y_j. More precisely, we set $\mathsf{RES}_j(\gamma) := \Psi_{x_j,y_j}(\gamma)$.

We will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space in which RES_j acts on an input Γ having the distribution $\mathbb{P}_\lambda^{N^2}$.

The following proposition is a straightforward consequence of the definition of $\Psi_{x,y}$,

Proposition 3.4.8 ($\mathbb{P}_\lambda^{N^2}$ is invariant under the resampling procedure)**.** *Let $j \in \{1,\dots,m_N\}$. Then, if $\Gamma \sim \mathbb{P}_\lambda^{N^2}$, $\mathsf{RES}_j(\Gamma) \sim \mathbb{P}_\lambda^{N^2}$. In words, the law $\mathbb{P}_\lambda^{N^2}$ is invariant under the map RES_j.*

We have now properly defined a resampling procedure on each sector $\mathbf{A}_j(N)$. As we saw, this procedure can be realized in two steps. Our next objective is to find a sufficient condition for step B to be realized: how can we ensure that the resampled path captures enough area.

A sufficient condition to capture enough area

Let a and b be two points in $\mathbb{N}^2$ (with arg$(a) >$ arg(b)). Assume we are given $\gamma \in \Lambda^{N^2}$ that passes through a and b. Modify γ between these two points replacing the original piece of path $\gamma_{a,b}$ by a new element $\tilde{\gamma}_{a,b} \in \Lambda^{a \to b}$. We are looking for a condition on $\tilde{\gamma}_{a,b}$ that is sufficient for the modified path $\tilde{\gamma}$ to belong to Λ^{N^2}. It is clear that a sufficient condition for $\tilde{\gamma}$ to enclose a sufficiently large area is

$$|\text{Enclose}\left(\tilde{\gamma}_{a,b} \cup [0,a] \cup [0,b]\right)| \geq |\text{Enclose}(\gamma) \cap \mathbf{A}_{a,b}| \,. \tag{3.19}$$

As in [75], we obtain a quantitative sufficient condition (see Corollary 3.4.13) for (3.4.2) to be satisfied. We directly import without a proof the result from there as they trivially extend to our setting.

We will need a notation: for $x, y \in \mathbb{N}^2$ distinct, we denote by $\ell_{x,y}$ the unique line that passes through x and y. The following result is illustrated in Figure 3.7.

Lemma 3.4.9. *Let $\gamma \in \Lambda^{N^2}$. Let $j \in \{1,\dots,m_N\}$. Denote by z_j the point of $\mathcal{C}(\gamma)$ of argument $j\theta_N$. Let ℓ_j be the tangent line of $\mathcal{C}(\gamma)$ at z_j. Let $x_j, y_j \in \gamma \cap \mathbf{A}_{z_{j-1},z_j}$ and $\mathbf{E}_{x_j,y_j} := \mathbf{E}_j$ be the pentagon delimited by the lines $\ell_{j-1}, \ell_j, \ell_{x_j,y_j}, \ell_{0,x_j}$ and ℓ_{0,y_j}. Then,*

$$(\text{Enclose}(\gamma) \cap \mathbf{A}_{x_j,y_j}) \subset \mathbf{T}_{0,x_j,y_j} \cup \mathbf{E}_j,$$

where $\mathbf{T}_{0,x_j,y_j}$ is the triangle of apexes $0, x_j, y_j$.

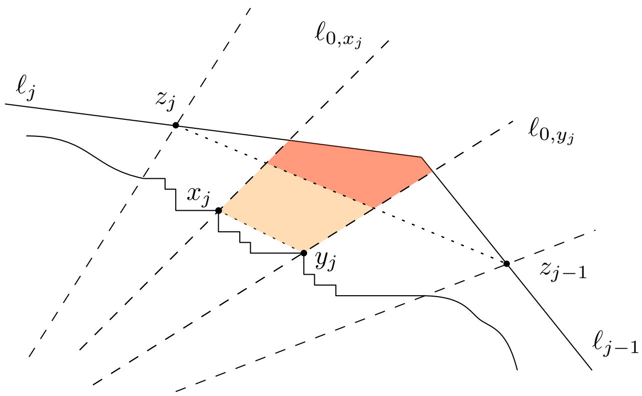

Figure 3.7: Illustration of Lemma 3.4.9. The light orange shaded region corresponds to $\mathbf{E}_j^0$, and the dark orange shaded region corresponds to $\mathbf{E}_j^1$.

An immediate consequence of the lemma is the fact that (3.4.2) is satisfied if

$$|\text{Enclose}\,(\tilde{\gamma}_{a,b} \cup [0,a] \cup [0,b])| \geq |\mathbf{T}_{0,a,b}| + |\mathbf{E}_{a,b}|\,.$$

We now write $\mathbf{E}_j = \mathbf{E}_j^0 \cup \mathbf{E}_j^1$ (where $\mathbf{E}_j^0$ denotes the part of $\mathbf{E}_j$ that lies below ℓ_{z_{j-1},z_j}), see Figure 3.7. It seems natural to seek an upper bound for $|\mathbf{E}_j|$. We start by determining an upper bound on $|\mathbf{E}_j^1|$. This is what we perform in the next lemma.

Lemma 3.4.10 ([75, Lemma 3.9])**.** *Let $\gamma \in \Lambda^{N^2}$. Assume $\gamma \subset B_{K_1 N}$. We keep the notations of the preceding lemma. Let $\overrightarrow{w_j}$ be the unit vector tangent to $\mathcal{C}(\gamma)$ at z_j oriented in the counterclockwise sense. We define the set of sectors with* moderate boundary turning *to be*

$$\text{MBT} = \{j \in \{1,\dots,m_N\},\ \|z_j - z_{j-1}\| \leq \frac{10\pi K_1 N}{m_N} \text{ and } \angle(\overrightarrow{w_j}, \overrightarrow{w_{j-1}}) \leq \frac{10\pi}{m_N}\}.$$

Then,

$$|\text{MBT}| \geq \frac{9m_N}{10},$$

and if $j \in$ MBT,

$$|\mathbf{E}_j^1| \leq \frac{1}{2}(20)^3 K_1^2 \chi^3 N \log N.$$

We now need to find an upper bound on $|\mathbf{E}_j^0|$. It is clear that such bound can only be obtained under the assumption that the points x_j and y_j defined above are not to far from $\mathcal{C}(\gamma)$. This remark motivates the following definition. Recall that if $\gamma \in \Lambda$ and $v \in \gamma$, $\mathsf{LR}(v,\gamma) = \mathrm{d}(v, \mathcal{C}(\gamma))$ denotes the local roughness of v in γ.

Definition 3.4.11 (Favourable sectors)**.** Let $\gamma \in \Lambda^{N^2}$. Let φ be a function that maps $(0,\infty)$ to itself and that satisfies, as $t \to 0$,

$$\varphi(t) = o(\sqrt{t}).$$

For $\chi > 0$, we say that the sector $\mathbf{A}_j(N)$ is *favourable under* γ if there exists $v \in \gamma \cap \mathbf{A}_j(N)$ such that

$$\mathsf{LR}(v,\gamma) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}}.$$

We define UNFAV$(\gamma,\chi) =$ UNFAV to be the set of $j \in \{1,\dots,m_N\}$ such that $\mathbf{A}_j(N)$ is not favourable under γ.

Lemma 3.4.12 ([75, Lemma 3.10])**.** *Let $\gamma \in \Lambda^{N^2}$. Assume that $\gamma \cap B_{K_2 N} = \emptyset$. Let $j \in$ MBT $\cap$ UNFAV. Then,*

$$|\mathbf{E}_j^0| \leq 20 K_1 \chi \varphi(\chi) N \log N.$$

From the preceding results, we can deduce the sufficient condition to capture enough area we were looking for,

Corollary 3.4.13 (Sufficient condition to capture a large area)**.** *Let $\gamma \in \Lambda^{N^2}$ be such that $\gamma \subset B_{K_1 N} \setminus B_{K_2 N}$. Let $j \in \{1,\dots,m_N\}$. Assume $j \in$ MBT $\cap$ UNFAV. Let $x_j, y_j \in \gamma \cap \mathbf{A}_j(N)$ be chosen accordingly to the procedure described in Definition 3.4.7. Then if we resample γ between x_j and y_j, replacing the path γ_{x_j,y_j} by a new path $\tilde{\gamma}_{x_j,y_j}$, the new path $\tilde{\gamma}$ will be an element of Λ^{N^2} if the following condition is satisfied,*

$$\left|\mathrm{Enclose}(\tilde{\gamma}_{x_j,y_j} \cup [0,x_j] \cup [0,y_j])\right| - \left|\mathbf{T}_{0,x_j,y_j}\right| \geq \left(\tfrac{1}{2}(20)^3 K_1^2 \chi^3 + 20 K_1 \varphi(\chi)\chi\right) N \log N.$$

The right shape

Now that we have a sufficient condition for a resampled path to be accepted, we need to exhibit an event that will ensure this sufficient condition. We make good use of the event $\mathsf{LogGAC}(x,y,\eta)$ introduced in Definition 3.4.2. As we are about to see, the realization of this event will ensure that the conditon of Corollary 3.4.13 will be satisfied. Nevertheless, we also need our resamplings to have a large local roughness and that is what motivates Definition 3.4.14.

Definition 3.4.14 (Significant inward deviation)**.** Let $\eta > 0$ and $x, y \in \mathbb{N}^2$ with $\arg(x) > \arg(y)$. Let $\gamma \in \Lambda$ be a path which passes by x and y. We say that γ realizes the event $\mathsf{LogSID}(x,y,\eta)$ (meaning "logarithmic significant inward deviation") if there exists $z \in \gamma_{x,y}$ such that

$$\mathrm{d}\left(z, \partial \mathrm{conv}([0,x] \cup [0,y] \cup \gamma_{x,y})\right) \geq \eta \parallel x-y \parallel^{\frac{1}{2}} (\log \parallel x-y \parallel)^{\frac{1}{2}},$$

where ∂conv is the border of the convex hull.

We are going to resample in the sectors $\mathbf{A}_j(N)$ introduced above. For a path $\gamma \in \Lambda^{N^2}$, call $x_j = x_j(\gamma), y_j = y_j(\gamma)$ the extremeties of the portion of γ subject to resampling in $\mathbf{A}_j(N)$ (as described in Definition 3.4.7). We first check that provided $\chi > 0$ is small enough, for a path γ satisfying "nice" properties, if we replace the portion of γ between x_j and y_j by a path satisfying $\mathsf{LogGAC}(x_j, y_j, \eta)$ and $\mathsf{LogSID}(x_j, y_j, \eta)$, we obtain an element of Λ^{N^2} with a large local roughness. This motivates the next definition.

Definition 3.4.15 (Successful action of RES_j)**.** Let $\eta > 0$ and $1 \leq j \leq m_N$. Let $\gamma \in \Lambda^{N^2}$. We say that our resampling operation in the j-th sector RES_j **acts** η**-successfully** on γ if the new path γ' obtained after the operation realizes the event $\mathsf{LogGAC}(x_j, y_j, \eta) \cap \mathsf{LogSID}(x_j, y_j, \eta)$.

We start by proving that the occurence of $\mathsf{LogGAC}(x_j, y_j, \eta)$ forces the area condition to be satisfied.

Lemma 3.4.16. *Let $\eta > 0$. There exists $\chi_1 = \chi_1(\eta) > 0$ such that for $\chi \in (0, \chi_1)$ the following assertion holds. Let $\gamma \in \Lambda^{N^2}$ such that $\gamma \subset B_{K_1 N} \setminus B_{K_2 N}$. Let $j \in \{1, \ldots, m_N\}$. Assume that $j \in$ MBT $\cap$ UNFAV. Assume that we modify γ between x_j and y_j and that the modified path γ' satisfies $\gamma' \in \mathsf{LogGAC}(x_j, y_j, \eta)$. Then, $\gamma' \in \Lambda^{N^2}$.*

Proof. It is easy to see from Definition 3.4.7 that

$$\angle(x_j, y_j) \geq \frac{\theta_N}{4}.$$

Using this and the fact that $x_j, y_j \notin B_{K_2 N}$, we get

$$\|x_j - y_j\| \geq \frac{K_2 \chi}{2\pi} N^{\frac{2}{3}} (\log N)^{\frac{1}{3}}.$$

The occurence of $\mathsf{LogGAC}(x_j, y_j, \eta)$ implies that

$$\left|\text{Enclose}(\gamma'_{x_j, y_j} \cup [0, x_j] \cup [0, y_j])\right| - |\mathbf{T}_{0, x_j, y_j}| \geq \frac{\eta}{2} \left(\frac{2}{3}\right)^{\frac{1}{2}} (K_2/2\pi)^{\frac{3}{2}} \chi^{\frac{3}{2}} N \log N.$$

Since $\varphi(t) = o(\sqrt{t})$ as $t \to 0$, we see that condition in Corollary 3.4.13 is ensured by fixing $\chi > 0$ small enough. ☐

A direct consequence of the proof is the following identity: if $\gamma \in \Lambda^{N^2}$ and $j \in \{1, \ldots, m_N\}$ satisfy the properties above, and $\chi > 0$ is small enough,

$$\mathbb{P}\left[\mathsf{RES}_j \text{ acts } \eta\text{-successfully on } \gamma\right] = \frac{|\{\gamma'_{x_j, y_j} \in \Lambda^{x_j \to y_j},\ (\gamma \setminus \gamma_{x_j, y_j}) \cup \gamma'_{x_j, y_j} \in \mathsf{LogGAC}(x_j, y_j, \eta) \cap \mathsf{LogSID}(x_j, y_j, \eta)\}|}{|\{\gamma'_{x_j, y_j} \in \Lambda^{x \to y},\ (\gamma \setminus \gamma_{x_j, y_j}) \cup \gamma'_{x_j, y_j} \in \Lambda^{N^2}\}|}.$$

Lemma 3.4.17. *Let $\eta > 0$. There exists $\chi_2 = \chi_2(\eta) > 0$ such that for $\chi \in (0, \chi_2)$ the following assertion holds. Let $\gamma \in \Lambda^{N^2}$ such that $\gamma \subset B_{K_1 N} \setminus B_{K_2 N}$. Let $j \in \{1, \ldots, m_N\}$. Assume that $j \in$ MBT $\cap$ UNFAV and that RES_j acts η-successfully on γ. Then, $\mathbf{A}_j(N)$ is favourable under $\gamma' = \mathsf{RES}_j(\gamma)$.*

Proof. Since $\gamma' \in \mathsf{LogSID}(x_j, y_j, \eta)$, there exists $z \in \gamma'_{x_j,y_j}$ such that

$$\mathrm{d}\left(z, \partial\mathrm{conv}\left([0,x_j] \cup [0,y_j] \cup \gamma'_{x_j,y_j}\right)\right) \geq \eta \|x_j - y_j\|^{\frac{1}{2}} (\log \|x_j - y_j\|)^{\frac{1}{2}}.$$

One may write $\gamma' = \left(\gamma \cap \mathbf{A}^c_{x_j,y_j}\right) \cup \gamma'_{x_j,y_j}$, so that we have

$$\begin{aligned} \mathrm{d}\left(z, \mathcal{C}\left(\gamma'\right)\right) &= \mathrm{d}\left(z, \mathcal{C}\left(\left(\gamma \cap \mathbf{A}^c_{x_j,y_j}\right) \cup \gamma'_{x_j,y_j}\right)\right) \\ &\geq \mathrm{d}\left(z, \partial\mathrm{conv}\left([0,x_j] \cup [0,y_j] \cup \gamma'_{x_j,y_j}\right)\right) \\ &\geq \eta \|x_j - y_j\|^{\frac{1}{2}} (\log \|x_j - y_j\|)^{\frac{1}{2}} \\ &\geq \frac{1}{2}\sqrt{K_2}\left(\frac{2}{3}\right)^{\frac{1}{2}} \eta \chi^{\frac{1}{2}} N^{\frac{1}{3}} (\log N)^{\frac{2}{3}}. \end{aligned}$$

Since $\varphi(t) = o(\sqrt{t})$ as $t \to 0$, we get the result choosing $\chi > 0$ small enough (in terms of η). ☐

We now fix $\eta > 0$ and $\chi = \chi(\eta) > 0$ sufficiently small so that Lemmas 3.4.16 and 3.4.17 are true. Combining Lemma 3.4.17 and Equation (3.4.2) we get, if $\gamma \in \Lambda^{N^2}$ satisfies the conditions above and if $j \in \mathrm{MBT} \cap \mathrm{UNFAV}$,

$$\mathbb{P}[\mathbf{A}_j(N) \text{ is favourable under } \mathsf{RES}_j(\gamma)] \geq \mathbb{P}_{x_j,y_j}[\mathsf{LogGAC}(x_j, y_j, \eta) \cap \mathsf{LogSID}(x_j, y_j, \eta)],$$

where we recall that $\mathbb{P}_{x_j,y_j}$ is the uniform law on $\Lambda^{x_j \to y_j}$. The last step is then to estimate the probability on the right-hand side of the above inequality. This step is actually done using the following result, whose proof is postponed to the appendix.

Lemma 3.4.18. *Let $\eta, \varepsilon > 0$. There exist $C = C(\varepsilon), N_0 = N_0(\varepsilon, \eta) > 0$ such that for any $N \geq N_0$, any $x, y \in \mathbb{N}^2$ such that $\arg(x) > \arg(y)$, $\theta(x,y) \in [\varepsilon, \pi/2 - \varepsilon]$ and $\|y - x\| \geq N_0$,*

$$\mathbb{P}_{x,y}[\mathsf{LogGAC}(x,y,\eta) \cap \mathsf{LogSID}(x,y,\eta)] \geq N^{-C\eta^2}.$$

We can now state the main result of this section.

Lemma 3.4.19 (Successful resampling)**.** *Let $\eta > 0$. Let $\chi = \chi(\eta) > 0$ be sufficiently small so that Lemmas 3.4.16 and 3.4.17 hold true. Let $\gamma \in \Lambda^{N^2}$ such that $\gamma \subset B_{K_1 N} \setminus B_{K_2 N}$. Let $j \in \{1, \dots, m_N\}$. Assume that $j \in \mathrm{MBT} \cap \mathrm{UNFAV}$ and that $\gamma \notin \mathsf{Bad}^-_{\varepsilon,\mathbf{A}} \cup \mathsf{Bad}^+_{\varepsilon,\mathbf{A}}$ for $\mathbf{A} = \mathbf{B}_j(N)$ where $\varepsilon > 0$ is given by Proposition 3.2.5. Then, there exist $C = C(\varepsilon) > 0$, and $N_0 = N_0(\varepsilon, \eta) \in \mathbb{N}$ such that for $N \geq N_0$,*

$$\mathbb{P}[\mathbf{A}_j(N) \textit{ is favourable under } \mathsf{RES}_j(\gamma)] \geq \mathbb{P}[\mathsf{RES}_j \text{ acts } \eta\text{-successfully on } \gamma] \geq N^{-C\eta^2}.$$

How not to undo a large local roughness

As described earlier, we want to perform our random surgery successively on different sectors $\mathbf{A}_j(N)$ in order to maximize the probability to obtain a favourable output. However, if we do it without caution, might it be that we will lose in the process favourable sectors. In order to avoid such problem we must only resample sufficiently distant sectors. This problem was handled by Hammond in [75]. Once again, the proof immediately extends to our setup.

Proposition 3.4.20 ([75, Lemmas 3.14 and 3.16])**.** *Let* $s_1(N) \in \{1, \ldots, m_N\}$*. Let* $s_2(N) \in (0, \infty)$ *such that*

$$s_2(N) < \frac{\chi}{2} s_1(N).$$

Set $s_3(N) = s_1(N) + 1$*. Let* $(k, j) \in \{1, \ldots, m_N\}^2$ *be such that* $|k - j| \geq s_3(N)$*. Let* $\gamma \in \Lambda^{N^2}$ *be such that* $\gamma \cap B_{K_2 N} = \emptyset$*. Define* $\gamma' = \mathsf{RES}_j(\gamma)$*. Assume that*

$$\max\left\{\mathsf{MaxFL}(\gamma), \mathsf{MaxFL}(\gamma')\right\} \leq s_2(N) N^{\frac{2}{3}} (\log N)^{\frac{1}{3}}.$$

Then, the sector $\mathbf{A}_k(N)$ *is favourable under* γ *if and only if it is favourable under* γ'*. Moreover,*

$$\mathcal{C}(\gamma) \cap \mathbf{A}_k(N) = \mathcal{C}(\gamma') \cap \mathbf{A}_k(N).$$

The proof of the lower bound for MaxLR

A naive approach to the proof of the lower bound would be to apply the resampling successively on all the sectors $\mathbf{A}_j(N)$. As we saw in the preceding section, this raises a major problem. Applying the random surgery consecutively on two successive sectors might have for consequence the disappearance of a favourable sector, which is something we clearly seek to avoid. This problem can be solved working on distant sectors as seen in the last section. This is why we decide to resample roughly one in $s_3(N)$ sectors. We decide to randomly choose the sectors we resample: the reason is obvious, we do not want to resample a sector where there is already a favourable sector. We now explain in detail our strategy.

We keep the notations of the above sections. We want to define a complete resampling RES that will involve all the procedures RES_j we constructed before and the quantities $\{s_i(N)\}_{i=1,2,3}$ introduced in Proposition 3.4.20, with their form being specified later.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in which are defined: a random path Γ of law $\mathbb{P}_\lambda^{N^2}$, and the operations of resampling RES_j for $1 \leq j \leq m_N$, which act independently. We also generate a sequence of i.i.d random variables $\{X_k\}_{1 \leq k \leq m_N}$ that have the law of a Bernoulli of parameter $\frac{1}{s_3(N)}$. We now properly define our entire resampling procedure RES. This operation will be defined under $(\Omega, \mathcal{F}, \mathbb{P})$. We build it by induction. Set $\Gamma_0 = \Gamma$ (the input). For $1 \leq j \leq m_N$, if $X_j = 1$ then set $\Gamma_j = \mathsf{RES}_j(\Gamma_{j-1})$, else set $\Gamma_j = \Gamma_{j-1}$ (no action is taken and the sector $\mathbf{A}_j(N)$ remains unchanged). We also set $\mathrm{UNFAV}_j = \mathrm{UNFAV}(\Gamma_j, \chi)$ and $\mathrm{MBT}_j = \mathrm{MBT}(\Gamma_j)$. As we previously saw, the law $\mathbb{P}_\lambda^{N^2}$ is invariant under RES. We will analyse $\mathbb{P}_\lambda^{N^2}$ by identifying it with the law of the output Γ_{m_N} of RES under the measure $\mathbb{P}$. However, the only way to analyse the output of the procedure is if it the input lies in a space $\mathcal{G}$ of "good" paths. Introduce for $0 \le i \le m_N$,

$$\mathcal{G}_{1,i} := \{\mathsf{MaxFL}(\Gamma_i) \le s_2(N) N^{\frac{2}{3}} (\log N)^{\frac{1}{3}}\},$$

$$\mathcal{G}_{2,i} := \{\Gamma_i \subset B_{K_1 N} \setminus B_{K_2 N}\},$$

$$\mathcal{G}_{3,i} := \{\Gamma_i \notin \mathsf{Bad}^-_{\varepsilon, \mathbf{B}_i(N)} \cup \mathsf{Bad}^+_{\varepsilon, \mathbf{B}_i(N)}\},$$

where $\varepsilon > 0$ is given by Proposition 3.2.5. We also define for $0 \le i \le m_N$,

$$\mathcal{G}_i := \bigcap_{1 \le j \le 3} \mathcal{G}_{j,i}, \text{ and } \qquad \mathcal{G}_{(i)} := \bigcap_{0 \le j \le i} \mathcal{G}_j.$$

Finally, the space of good outcomes that is interesting is nothing but

$$\mathcal{G} := \mathcal{G}_{(m_N)}.$$

For $\varepsilon_1 \in (0, \frac{2}{3})$ to be fixed later, we define

$$s_3(N) := N^{\varepsilon_1}, \qquad \text{and} \qquad s_2(N) := \frac{\chi}{4} N^{\varepsilon_1},$$

so that the conditions of Proposition 3.4.20 are satisfied. Let us first prove that the set of good outcomes $\mathcal{G}$ happens with sufficiently large probability.

Lemma 3.4.21. *There exist two constants $c, C > 0$ such that*

$$\mathbb{P}\left[\mathcal{G}^c\right] \le C \exp(-c\chi^{\frac{3}{2}} N^{\frac{3\varepsilon_1}{2}} \log N) + C \exp(-cN) + C \exp(-cN\theta_N).$$

Proof. By definition, we have that

$$\mathbb{P}[\mathcal{G}^c] \le m_N \mathbb{P}_\lambda^{N^2}\left[\mathsf{MaxFL}(\Gamma) > s_2(N) N^{\frac{2}{3}} (\log N)^{\frac{1}{3}}\right] + m_N \mathbb{P}_\lambda^{N^2}[\Gamma \subset (B_{K_1 N} \setminus B_{K_2 N})^c] + m_N \mathbb{P}_\lambda^{N^2}[\mathsf{Bad}^-_{\varepsilon, \mathbf{B}_1(N)} \cup \mathsf{Bad}^+_{\varepsilon, \mathbf{B}_1(N)}].$$

We then conclude by applying Proposition 3.4.1, Lemma 3.2.2 and Proposition 3.2.5. □

Our second objective is to control the evolution of the sets MBT and UNFAV during the procedure (under the assumption of $\mathcal{G}$). This control is obtained in the following lemma.

Lemma 3.4.22 ([75, Lemma 3.17])**.** *For $1 \le j \le m_N - s_3(N)$, if $\mathcal{G}$ occurs then*

$$\text{UNFAV}_j \cap \{j + s_3(N), \dots, m_N\} = \text{UNFAV}_0 \cap \{j + s_3(N), \dots, m_N\},$$

and

$$\text{MBT}_j \cap \{j + s_3(N), \dots, m_N\} = \text{MBT}_0 \cap \{j + s_3(N), \dots, m_N\}.$$

Proof. Let $(j,k) \in \{1,\dots,m_N\}^2$ satisfy $j + s_3(N) \le k \le m_N$. If $\mathcal{G}$ occurs, then we may apply Proposition 3.4.20 to each of the first j stages of the procedure RES. In particular, this tells us that $\mathbf{A}_k(N)$ is favourable under Γ_j if and only if it is favourable under Γ_0: this is exactly (3.4.22).

Then, the condition $k \in \mathrm{MBT}_j$ is determined by the data $\mathcal{C}(\Gamma_j) \cap \mathbf{A}_k(N)$. However, one may also successively apply Proposition 3.4.20 to the first j stages of the procedure RES to obtain that

$$\mathcal{C}(\Gamma_j) \cap \mathbf{A}_k(N) = \mathcal{C}(\Gamma_0) \cap \mathbf{A}_k(N),$$

which yields (3.4.22). □

Now, recall from Lemma 3.4.10 that if $\Gamma_0 \subset B_{K_1 N}$ then

$$|\mathrm{MBT}_0| \ge \frac{9m_N}{10} \ge \frac{m_N}{2}.$$

By the occurence of $\mathcal{G}_{2,0}$, we may then define a set $\overline{\mathrm{MBT}_0} \subset \mathrm{MBT}_0$ that satisfies the following properties:

- $|\overline{\mathrm{MBT}_0}| \ge \frac{m_N}{4s_3(N)}$,
- each pair of consecutive elements of $\overline{\mathrm{MBT}_0}$ differ by at least $2s_3(N)+1$.

We also write $\overline{\mathrm{MBT}_0} \cap \mathrm{UNFAV}_0 = \{p_1,\dots,p_{r_1}\}$ and $\overline{\mathrm{MBT}_0} \cap \mathrm{UNFAV}_0^c = \{q_1,\dots,q_{r_2}\}$ with $r_1 + r_2 = |\overline{\mathrm{MBT}_0}|$. For $1 \le r \le r_1$, let P_r denote the event that in the action of RES, at stage p_r, RES_{p_r} is chosen to act and acts successfully while no action is taken at stages j for $j \in \{p_r - s_3(N),\dots,p_r - 1\} \cup \{p_r+1,\dots,p_r+s_3(N)\}$. Similarly, for $1 \le r \le r_2$, let Q_r denote the event that in the action of RES, at stage q_r no action is taken, and same for the stages j with $j \in \{q_r - s_3(N),\dots,q_r\} \cup \{q_r+1,\dots,q_r+s_3(N)\}$. We make good use of these events to make favourable sectors appear.

Lemma 3.4.23. *We keep the notations introduced above. Then,*

(i) for each $r \in \{1,\dots,r_1\}$, $\mathcal{G} \cap P_r$ implies that the sector $\mathbf{A}_{p_r}(N)$ is favourable under the output Γ_{m_N},

(ii) for each $r \in \{1,\dots,r_2\}$, $\mathcal{G} \cap Q_r$ implies that the sector $\mathbf{A}_{q_r}(N)$ is favourable under the output Γ_{m_N}.

Proof. (i) By Lemma 3.4.22, we have $p_r \in \mathrm{UNFAV}_{p_r - s_3(N)} \cap \mathrm{MBT}_{p_r - s_3(N)}$, since $p_r \in \mathrm{UNFAV}_0 \cap \mathrm{MBT}_0$. Given P_r, we then have $p_r \in \mathrm{UNFAV}_{p_r-1} \cap \mathrm{MBT}_{p_r-1}$ because $\Gamma_{p_r - s_3(N)} = \Gamma_{p_r-1}$. Applying Lemma 3.4.17 to the η-successful action of RES_{p_r} on Γ_{p_r-1}, we find that $\mathbf{A}_{p_r}(N)$ is favourable under Γ_{p_r}. This remains the case for $\Gamma_{p_r+s_3(N)}$ because $\Gamma_{p_r} = \Gamma_{p_r+s_3(N)}$. Now, thanks to Proposition 3.4.20, $\mathbf{A}_{p_r}(N)$ stays a favourable sector during the remaining stages of the procedure RES (recall that we also work under $\mathcal{G}$ in which $\mathcal{G}_{1,i}$ occurs for $1 \le i \le m_N$).

(ii) This is essentially the same argument as the one depicted above. We have that $q_r \notin \mathrm{UNFAV}_{q_r - s_3(N)}$ so that the inaction of RES_j for $j \in \{q_r - s_3(N), \dots, q_r + s_3(N)\}$ entails $q_r \notin \mathrm{UNFAV}_{q_r + s_3(N)}$. We can then conclude as above using Proposition 3.4.20.

□

An immediate consequence of Lemma 3.4.23 is that

$$\left(\bigcup_{i=1}^{r_1} P_i \cup \bigcup_{i=1}^{r_2} Q_i\right) \cap \mathcal{G} \subset \left\{\mathsf{MaxLR}\left(\Gamma_{m_N}\right) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}}\right\}.$$

Also, note that by the discussion above, if $\mathcal{G}$ happens then $r_1 + r_2 \geq \frac{m_N}{4s_3(N)}$. Then,

$$\left(\bigcup_{i=1}^{r_1} P_i \cup \bigcup_{i=1}^{r_2} Q_i\right)^c \cap \mathcal{G} \subset \bigcap_{i=1}^{\frac{m_N}{8s_3(N)}} \left(\{r_1 \geq i\} \cap P_i^c \cap \mathcal{G}_{(p_i - s_3(N))}\right) \cup \bigcap_{i=1}^{\frac{m_N}{8s_3(N)}} \left(\{r_2 \geq i\} \cap Q_i^c \cap \mathcal{G}_{(q_i - s_3(N))}\right).$$

Using all the material above, we claim that, for any $K \in \mathbb{N}$, given $\{r_1 \geq K\} \cap \mathcal{G}_{(p_K - s_3(N))}$ and the values of $\mathbb{1}_{P_1}, \dots, \mathbb{1}_{P_{K-1}}$, the conditional probability that P_K occurs is at least

$$N^{-C\eta^2} \frac{1}{s_3(N)} \left(1 - \frac{1}{s_3(N)}\right)^{2s_3(N)},$$

where $\eta > 0$ and $C = C(\varepsilon)$ is given by Lemma 3.4.19. Indeed, the event on which we condition is measurable with respect to $\sigma(\Gamma_0, \dots, \Gamma_{p_K - s_3(N)})$, and, if it occurs, $\Gamma_{p_K - s_3(N)}$ satisfies the hypothesis of Lemma 3.4.19. The claim then follows by this lemma. As a result,

$$\mathbb{P}\left[\bigcap_{i=1}^{\frac{m_N}{8s_3(N)}} \left(\{r_1 \geq i\} \cap P_i^c \cap \mathcal{G}_{(p_i - s_3(N))}\right)\right] \leq \left(1 - N^{-C\eta^2} \frac{1}{s_3(N)} \left(1 - \frac{1}{s_3(N)}\right)^{2s_3(N)}\right)^{\frac{m_N}{8s_3(N)}}. \tag{3.20}$$

Similarly,

$$\mathbb{P}\left[\bigcap_{i=1}^{\frac{m_N}{8s_3(N)}} \left(\{r_2 \geq i\} \cap Q_i^c \cap \mathcal{G}_{(q_i - s_3(N))}\right)\right] \leq \left(1 - \left(1 - \frac{1}{s_3(N)}\right)^{2s_3(N)+1}\right)^{\frac{m_N}{8s_3(N)}}. \tag{3.21}$$

Now, Proposition 3.4.8 implies that

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MaxLR}(\Gamma) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}}\right] = \mathbb{P}\left[\mathsf{MaxLR}\left(\Gamma_{m_N}\right) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}}\right].$$

Using (3.4.2) we then obtain,

$$\mathbb{P}_{\lambda}^{N^2}[\mathsf{MaxLR}(\Gamma) \geq \varphi(\chi)N^{\frac{1}{3}}(\log N)^{\frac{2}{3}}] \geq 1 - \mathbb{P}[\mathcal{G}^c] - \mathbb{P}\Big[\left(\bigcup_{i=1}^{r_1} P_i \cup \bigcup_{i=1}^{r_2} Q_i\right)^c \cap \mathcal{G}\Big].$$

Using Lemma 3.4.21, Equations (3.4.2), (3.4.2), (3.4.2) and Definition 3.4.2, we obtain the vanishing of $\mathbb{P}_{\lambda}^{N^2}\big[\mathsf{MaxLR}(\Gamma) < \varphi(\chi)N^{\frac{1}{3}}(\log N)^{\frac{2}{3}}\big]$ to 0 as N goes to infinity. More precisely, we have a quantitative result. Recall that one may take $\eta > 0$ sufficiently small so that it always satisfies $C\eta^2 < \varepsilon_1$. For some constant $c(\chi) > 0$,

$$\mathbb{P}[\mathcal{G}^c] \leq \exp(-c(\chi)N^{\frac{3\varepsilon_1}{2}} \log N),$$

$$\left(1 - N^{-C\eta^2}\frac{1}{s_3(N)}\left(1 - \frac{1}{s_3(N)}\right)^{2s_3(N)}\right)^{\frac{m_N}{8s_3(N)}} \leq \exp(-c(\chi)N^{\frac{1}{3}-3\varepsilon_1}(\log N)^{\frac{1}{3}}),$$

and

$$\left(1 - \left(1 - \frac{1}{s_3(N)}\right)^{2s_3(N)+1}\right)^{\frac{m_N}{8s_3(N)}} \leq \exp(-c(\chi)N^{\frac{1}{3}-\varepsilon_1}(\log N)^{\frac{1}{3}}).$$

Taking $\varepsilon_1 = \frac{2}{27}$, we obtain the following result,

Proposition 3.4.24. *For any $\chi > 0$ sufficiently small, there exist $c(\chi), N_0 > 0$ such that for $N \geq N_0$,*

$$\mathbb{P}_{\lambda}^{N^2}\big[\mathsf{MaxLR}(\Gamma) < \varphi(\chi)N^{\frac{1}{3}}(\log N)^{\frac{2}{3}}\big] \leq \exp(-c(\chi)N^{\frac{1}{9}} \log N).$$

Remark 3.4.25. Recall that above we first consider values of ε_1 sufficiently small (typically $\varepsilon_1 < 1/9$), and then fix a value of $\eta > 0$ such that $C\eta^2 \ll \varepsilon_1$ (where C is the constant given by Lemma 3.4.19. Finally, we choose $\chi = \chi(\eta)$ sufficiently small such that Lemmas 3.4.16 and 3.4.17 hold. We did not try to obtain an optimal bound in Proposition 3.4.24.

The proof of the lower bound for MaxFL

Once we have proved (3.1.5), obtaining a lower bound on MaxFL is immediate with the use of the coupling described in Section 3.2. Indeed, assume that a path Γ satisfies the event $\{\mathsf{MaxLR}(\Gamma) \geq tN^{1/3}\log N^{2/3}\}$, but with a maximum facet length of order $o(N^{2/3}\log N^{1/3})$, this would imply the existence of a facet of length $o(N^{2/3}\log N^{1/3})$ and whose local roughness is greater than $tN^{1/3}\log N^{2/3}$. If the piece of path along this facet had true Gaussian fluctuations it would be impossible, but as we said earlier, thanks to Proposition 3.2.6, it is possible to compare the fluctuations of the path along a given facet to the ones of a random walk.

Proposition 3.4.26. *Let $\chi > 0$ be sufficiently small. There exist a function ϕ such that $\phi(t) \xrightarrow[t\to 0^+]{} +\infty$ such that for N large enough,*

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}] \le N^{-\phi(\chi)}.$$

Proof. We split the event $\{\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\}$ into the union of two events, according to the value of the maximal local roughness,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}] \le \mathbb{P}_\lambda^{N^2}[\mathsf{MaxLR} < \varphi(\chi)(\log N)^{\frac{2}{3}} N^{\frac{1}{3}}] \\ + \mathbb{P}_\lambda^{N^2}[\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}},\ \mathsf{MaxLR} \ge \varphi(\chi)(\log N)^{\frac{2}{3}} N^{\frac{1}{3}}].$$

The first term of this sum has been shown to be smaller than $\exp(-c(\chi)N^{1/9}\log N)$ for χ sufficiently small and N sufficiently large. To bound the second term, we use the coupling described in Proposition 3.2.6. Indeed, we invoke the proof of Proposition 3.4.4 to get that for some $c > 0$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}},\ \mathsf{MaxLR} \ge \varphi(\chi)(\log N)^{\frac{2}{3}} N^{\frac{1}{3}}] \le \\ O(N^6)\exp(-\tfrac{c}{\varphi(\chi)}\log N) + \exp(-cN),$$

where the second term comes from the bound on $\mathbb{P}_\lambda^{N^2}[\Gamma \cap (B_{K_1 N})^c \neq \emptyset]$. This gives the result provided $\chi > 0$ is sufficiently small in which case one can set $\phi(\chi) := -(6 - c\varphi(\chi)^{-1}) > 0$. □

3.5 Discussion: extending the results to the Wulff setting

In that section, we briefly explain the fact that this model — even though it might look very simple — is an appropriate model for modelling the phase separation interface in the Wulff setting. In particular, we argue that it shares all the crucial properties of the latter object. We chose to illustrate this fact through the example of the outermost circuit in a constrained subcritical random-cluster model, a setting that was introduced by Hammond in [73, 74, 75].

3.5.1 Definition of the random-cluster model

We quickly recall the definition of the random-cluster model, also known as FK percolation and give a few of its basic properties (we refer to [49] for a complete exposition). The random-cluster model on $\mathbb{Z}^2$ is a model of random subgraphs of $\mathbb{Z}^2$. Its law is described by two parameters, $p \in [0,1]$ and $q > 0$. Let $G = (V(G), E(G))$ be a finite subgraph of $\mathbb{Z}^2$. We denote its *boundary* by

$$\partial G = \left\{x \in V(G), \exists y \notin V(G), \{x,y\} \in E(\mathbb{Z}^2)\right\}.$$

A *percolation configuration* on G is an element $\omega \in \{0,1\}^{E(G)}$. Hence, each edge of G is assigned a value 0 or 1. We say that an edge $e \in G$ is *open* if $\omega(e) = 1$ and is *closed* otherwise. Two vertices $x, y \in \mathbb{Z}^2$ are said to be connected if there exists a path of nearest neighbour vertices $x = x_0, x_1, \dots, x_n = y$ such that the edges $\{x_i, x_{i+1}\}$ are open for every $0 \leq i \leq n-1$. In this case we say that the event $\{x \leftrightarrow y\}$ occurs. A *vertex cluster* of ω is a maximal connected component of the set of vertices (it can be an isolated vertex). Given a percolation configuration ω, we denote by $o(\omega)$ its number of open edges, and $k(\omega)$ its number of vertex clusters.

A *boundary condition* on G is a partition $\eta = P_1 \cup \dots \cup P_k$ of ∂G. From a configuration $\omega \in \{0,1\}^{E(G)}$, we create a configuration ω^η by identifying together the vertices that belong to the same P_i of η. Two particular boundary conditions, that we shall call the *free boundary* condition (resp. wired boundary condition), consist in the partition made of singletons (resp. of the whole set ∂G). We shall write $\eta = 0$ (resp. $\eta = 1$) for this specific boundary condition.

Definition 3.5.1. Let $G = (V(G), E(G))$ be a finite subgraph of $\mathbb{Z}^2$, and η be a boundary condition on G. Let $p \in [0,1]$ and $q > 0$. The random-cluster measure on G with boundary condition η is the following measure on percolation configurations on G:

$$\phi^\eta_{p,q,G}(\omega) = \frac{1}{Z^\eta_{p,q,G}} \left(\frac{p}{1-p}\right)^{o(\omega)} q^{k(\omega^\eta)},$$

where $Z^\eta_{p,q,G} > 0$ is the normalization constant ensuring that $\phi^\eta_{p,q,G}$ is indeed a probability measure. We shall refer to $Z^\eta_{p,q,G}$ as the *partition function* of the model.

It is classical that for $\eta = 0$ and $\eta = 1$, this measure can be extended to the whole $\mathbb{Z}^2$, by taking the weak limit of the measures ϕ^η_{p,q,G_n} over any exhaustion $(G_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}^2$, and that the limit measure does not depend of the choice of the exhaustion. Below, we will simply write $\phi^\eta_{p,q}$ instead of $\phi^\eta_{p,q,\mathbb{Z}^2}$. A very fundamental feature of this model is that it undergoes a *phase transition*: namely for any $q \geq 1$, there exists a critical parameter $p_c = p_c(q) \in (0,1)$ such that:

- $\forall p < p_c(q), \phi^1_{p,q}(0 \leftrightarrow \infty) = 0.$
- $\forall p > p_c(q), \phi^0_{p,q}(0 \leftrightarrow \infty) > 0.$

We are going to be interested in the first case - called *the subcritical regime*. In this case it is well known that the choice of boundary conditions does not affect the infinite volume measure: thus we drop η from the notation and simply write $\phi_{p,q}$ for the unique infinite volume measure when $p < p_c(q)$.

Another well-known feature of the subcritical regime is the existence and the positivity of the following limit for any $x \in \mathbb{S}^1$, called the correlation length:

$$\xi_{p,q}(x) := -\lim_{n\to\infty} \frac{n}{\log \phi_{p,q}\left[0 \leftrightarrow \lfloor nx \rfloor\right]} > 0.$$

3.5.2 Extension of the results to the random-cluster model

For the rest of this section, let us fix $q \geq 1$ and $0 < p < p_c(q)$. Because of the exponential decay of the size of the cluster of 0, it is easy to see that almost surely, the outermost open circuit surrounding 0 is well-defined. Following [74], we call it Γ_0. Γ_0 sampled according to $\phi_{p,q}$ has to be seen as the analog of Γ sampled according to $\mathbb{P}_\lambda$ in our model. It remains to introduce the analog of the event $\{\mathcal{A}(\Gamma) \geq N^2\}$. Indeed introduce the area trapped by the circuit $\mathcal{A}(\Gamma_0)$, and define the conditioned measure $\phi_{p,q}^{N^2} := \phi_{p,q}\left[\cdot \mid \mathcal{A}(\Gamma_0) \geq N^2\right]$. This measure is the analog of $\mathbb{P}_\lambda^{N^2}$. We now compare the properties of Γ sampled according to $\mathbb{P}_\lambda^{N^2}$ with the properties of Γ_0 sampled according to $\phi_{p,q}^{N^2}$.

Global curvature: the Wulff shape. The first essential feature exhibited by $\mathbb{P}_\lambda^{N^2}$ is the convergence of its sample paths at the macroscopic scale towards a deterministic curve given by the solution of a variationnal problem. This is a very classical result in statistical mechanics known as the *Wulff phenomenon* (see [47, 29] for detailed monographs about this theory). Define the following compact set:

$$\mathcal{W} = \nu \bigcap_{u \in \mathbb{S}^1} \{t \in \mathbb{R}^2, \langle t, u\rangle \leq \xi_{p,q}^{-1}(u)\},$$

with the constant $\nu > 0$ being chosen so that the set $\mathcal{W}$ is of volume 1 (and $\langle \cdot, \cdot \rangle$ denoting the usual scalar product on $\mathbb{R}^2$). The boundary $\partial\mathcal{W}$ of the Wulff shape plays the role of the limit shape f_λ in our model. Indeed, one has the following result, which is the exact analog of Theorem 3.1.6, and appears in [74, Proposition 1]: for any $\varepsilon > 0$,

$$\phi_{p,q}^{N^2}[d_{\mathcal{H}}\left(\partial\mathcal{W}, N^{-1}\Gamma_0\right) > \varepsilon] \xrightarrow[n\to\infty]{} 0.$$

Moreover we used at several occasions the concavity and differentiability of f_λ: it is well known (see [25, Theorem B]), that $\mathcal{W}$ is indeed a strictly convex set and that its boundary $\partial\mathcal{W}$ is analytic.

Local Brownian nature of the interface. A second dramatic feature of the model is the Brownian nature of the interface under a scaling of the type $N^{-1/3}\Gamma(tN^{2/3})$. Indeed, Donsker's invariance principle is heavily used in Subsection 3.3.2 to argue that the area captured by Γ in a cone of angular opening of order $N^{-1/3}$ is of order at most βN, with some Gaussian tails on β. This is also the case in the random-cluster model, and is an instance of the celebrated *Ornstein–Zernike theory*. Indeed it is known since the breakthrough work of [25] that a subcritical cluster of FK percolation conditionned to link two distant points is asymptotically Brownian whenever the distance between the points goes to infinity. Hence, an analog of Donsker's invariance principle holds for a percolation interface at the scale $(N^{1/3}, N^{2/3})$, and the arguments of Subsection 3.3.2 can be reproduced *mutatis mutandi*. For precise statements, see [25, Theorem A, Theorem C] for the local Gaussian asymptotic for a subcritical cluster and the invariance principle towards a Brownian bridge for a rescaled subcritical cluster.

Brownian Gibbs property in the random-cluster model. Perhaps the most essential tool used through the work is what was referred to - according to [37] - as the *Brownian Gibbs property* of the model, stating that if one forgets some portion of the walk Γ, then conditionally on the remaining portion of Γ, the distribution of the erased part is simply the distribution of a random walk conditioned to link both parts of the remaining non-erased walk Γ, conditioned on the event that the total enclosed area is at least N^2.

For the random-cluster model, the existence of this Brownian Gibbs property is less clear, for two distinct reasons. The first reason is that Γ_0 lacks - at least locally - the oriented structure of our random walk model. Thus, due to possible backtracks of Γ_0 it is not *a priori* clear that one is able to perform the resampling operation between any two points, which could be a possible obstruction to the strategy described in this work. The second possible obstacle to the Brownian Gibbs property is the lack of independence: indeed the Spatial Markov property of the random-cluster model (see [49]) is the exact analog of the Brownian Gibbs property. However, opposed to our random walk model, one has to take into consideration the *boundary conditions* enforced by the conditionning on the circuit outside of some fixed region. This could be a problem, as our strategy strongly relies on independence of the resampled piece - conditionnally on the area constraint.

These two obstructions have successfully been overcome by Alan Hammond in [73]. We refer to this work to observe that one can indeed implement the resampling strategy in the context of the random-cluster model. However, in the remainder of this section, we quickly explain the strategy to overcome the two difficulties pointed out previously.

(i) **Renewal structure and regularity of the outermost circuit.** It was observed in [73] that the first difficulty can be overcome by implementing the resampling strategy only between *renewal points* of the outermost circuit. For the precise definition of these points, see [73, Definition 1.9]: these are precisely the sites where the circuit can be resampled without any backtrack problem. Then, the main result of [73], stated in Theorem 1.1, is that the probability of not seeing such a point in a cone of angular opening uN^{-1} decays faster than $\exp(-u^2)$. Since all our resamplings take place in sectors of angular opening of order $N^{-1/3}$, one sees that imposing that the resampling occurs between renewals does not change the scaling of the quantities that we are computing.

(ii) **Exponential mixing versus independence.** The second obstruction is ruled out by the exponential mixing property of subcritical random-cluster models (see [49]). Again, since our reasoning takes place at scales of order $N^{1/3}$ while the latter property ensures that random-cluster configurations decorrelate at a polynomial rate between spatial scales of logarithmic order, the lack of independence is not a problem in our setting (see the discussion in [74, Section 2.2]).

We hope to have convinced the careful reader that our techniques, even though written in a simple context, are robust and suitable to the analysis of any subcritical statistical mechanics models exhibiting the features discussed above (which in turn should be the case at least in a wide range of models). Finally let us conclude this section by mentioning that thanks to the well-known Edwards–Sokal coupling, this analysis might possibly allow the identification of the fluctuations scale of the facets of a droplet in a supercritical Potts model with q colours, which is maybe a more physically appealing conclusion.

Appendix

3.5.3 Computations on the simple random walk

This subsection is devoted to the proofs of Lemmas 3.3.4, 3.4.3 and 3.4.18 . The proofs are adapted from [74].

Proof of Lemma 3.3.4. We assume that there exists some $\varepsilon > 0$ such that $\varepsilon \leq \theta(x, y) \leq \pi/2 - \varepsilon$. Let $h = \|x - y\|$. Let $\mathbf{R}$ be the rectangle whose up-left corner is x, down-right corner is y and which has horizontal and vertical sides. Since $\theta(x, y) \in [\varepsilon, \pi/2 - \varepsilon]$, $\mathbf{R}$ is not degenerate. We start by changing the coordinates. We set $z_0 = (0, 0)$, $z_1 = (h/4, 10\eta\sqrt{h})$, $z_2 = (h/2, 5\eta\sqrt{h})$, $z_3 = (3h/4, 10\eta\sqrt{h})$ and $z_4 = (h, 0)$. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plane which maps $[x, y]$ to an horizontal segment and T the translation which maps x to 0. Let $\tilde{\mathbf{R}} = (T \circ \mathbb{R}_\theta)(\mathbf{R})$. To ensure that $z_1, z_3 \in \tilde{\mathbf{R}}$, we set $N_0 = N_0(\varepsilon, \eta)$ large enough so that for $h \geq N_0$,

$$\arctan(40\eta h^{-\frac{1}{2}}) \leq \varepsilon.$$

We now assume that $h \geq N_0$. We will need the following definitions.

A θ-*path* is an element of $(T \circ \mathbb{R}_\theta)(\Lambda^{x \to y})$. Given $u, v \in \mathbb{R}^2$ and $\gamma \in \Lambda^{x \to y}$, we say that u and v are θ-*connected* in γ if u and v belong to $(T \circ \mathbb{R}_\theta)(\gamma)$. For $i \in \{0, \dots, 3\}$, let H_i be the event that z_i and z_{i+1} are θ-connected by a θ-path which fluctuates less than $10 \parallel z_{i+1} - z_i \parallel^{\frac{1}{2}}$ around the segment $[z_i, z_{i+1}]$. Finally, define the event $\mathsf{Shape} = H_0 \cap H_1 \cap H_2 \cap H_3$, see Figure 3.8.

Notice that

$$\mathsf{Shape} \subset \mathsf{GAC}\,(x, y, \eta)\,.$$

Indeed, let Q be the rectangle $[h/4, 3h/4] \times [0, 5\eta\sqrt{h}]$ (see Figure 3.8). Then, if γ realises Shape, $(T \circ \mathbb{R}_\theta)^{-1}(Q)$ and γ do not cross, except maybe on a small region of area $O(h)$

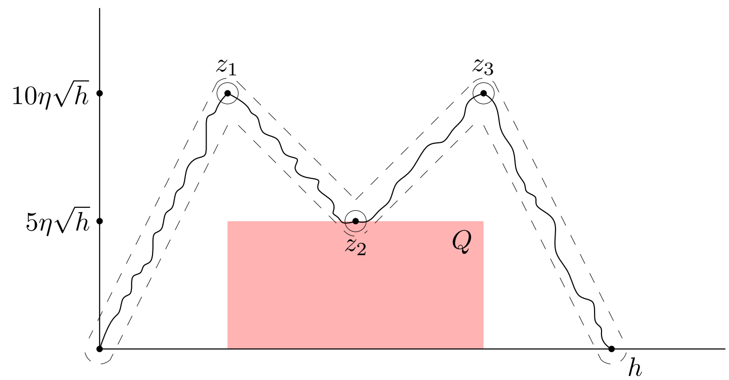

Figure 3.8: Illustration of the event Shape introduced in the proof of Lemma 3.3.4. The θ-path is allowed to fluctuate in the region surrounded by the dashed line.

around $(T \circ \mathbb{R}_\theta)^{-1}(z_2)$. Hence, provided that N is large enough,

$$\begin{aligned} |\text{Enclose}(\gamma \cap \mathbf{A}_{x,y})| &\geq |\mathbf{T}_{0,x,y}| + \frac{5}{2}\eta h^{\frac{3}{2}} - O(h) \\ &\geq |\mathbf{T}_{0,x,y}| + \eta h^{\frac{3}{2}}. \end{aligned}$$

Now, it remains to get a good estimate on $\mathbb{P}_{x,y}[\mathsf{Shape}]$. A simple computation coming from Gaussian fluctuations gives that there exists a constant $c > 0$ such that for $i \in \{0, \ldots, 3\}$,

$$\mathbb{P}_{x,y}[H_i] \geq c\mathbb{P}_{x,y}[z_i \leftrightarrow z_{i+1}]^{11}.$$

We then have,

$$\begin{aligned} \mathbb{P}_{x,y}[\mathsf{Shape}] &= \mathbb{P}_{x,y}\left[\bigcap_{i=0}^{3} H_i\right] \\ &\geq c^4 \frac{\prod_{i=0}^{3} |\{\theta\text{-paths from } z_i \text{ to } z_{i+1}\}|}{|\{\theta\text{-paths from } z_0 \text{ to } z_3\}|}. \end{aligned}$$

Let $a = \cos\theta$ and $b = \sin\theta$. Define

$$\begin{cases} \alpha_0 = \frac{h}{4}a + 10\eta h^{\frac{1}{2}} b \\ \beta_0 = \frac{h}{4}b - 10\eta h^{\frac{1}{2}} a \end{cases}, \qquad \begin{cases} \alpha_1 = \frac{h}{4}a - 5\eta h^{\frac{1}{2}} b \\ \beta_1 = \frac{h}{4}b + 5\eta h^{\frac{1}{2}} a \end{cases},$$

$$\begin{cases} \alpha_2 = \frac{h}{4}a + 5\eta h^{\frac{1}{2}} b \\ \beta_2 = \frac{h}{4}b - 5\eta h^{\frac{1}{2}} a \end{cases}, \qquad \begin{cases} \alpha_3 = \frac{h}{4}a - 10\eta h^{\frac{1}{2}} b \\ \beta_3 = \frac{h}{4}b + 10\eta h^{\frac{1}{2}} a \end{cases}, \qquad \begin{cases} \alpha_4 = ha \\ \beta_4 = hb \end{cases}.$$

[11] $\{z_i \leftrightarrow z_{i+1}\}$ is the event that z_i and z_{i+1} are θ-connected.

It is nothing but a little combinatorial fact that the cardinal of the set of θ-paths linking z_i and z_{i+1} is $\binom{\alpha_i+\beta_i}{\alpha_i}$. So that[12],

$$\frac{\prod_{i=0}^{3} |\{\theta\text{-paths from } z_i \text{ to } z_{i+1}\}|}{|\{\theta\text{-paths from } z_0 \text{ to } z_3\}|} = \frac{\prod_{i=0}^{3} \binom{\alpha_i+\beta_i}{\alpha_i}}{\binom{\alpha_4+\beta_4}{\alpha_4}}.$$

A quite tedious computation using Stirling's estimate yields the existence of some constant $C = C(\varepsilon, \eta) > 0$ such that

$$\mathbb{P}_{x,y}[\mathsf{Shape}] \geq Ch^{-\frac{3}{2}}.$$

We are a factor $h^{-3/2}$ away from the desired result. This factor can be removed by considering variants of the event Shape where the vertical coordinates of z_1, z_2, z_3 may differ from the original ones by at most $2\eta h^{1/2}$. Since these variants of Shape are still included in $\mathsf{GAC}\,(x, y, \eta)$ and there being $(2\eta)^3 h^{3/2}$ such events, we get the desired conclusion. We obtained ,

$$\mathbb{P}_{x,y}\left[\mathsf{GAC}\,(x, y, \eta)\right] \geq C(\eta, \varepsilon).$$

□

Remark 3.5.2. It is clear that the constant $C(\eta, \varepsilon)$ degenerates when ε goes to 0, which explains the need for the *a priori* estimate given by Proposition 3.2.5.

Proof of Lemmas 3.4.3 *and* 3.4.18. Note that the statement of Lemma 3.4.18 implies the one of Lemma 3.4.3. We then focus on the first one. The proof of Lemma 3.3.4 can be reproduced with a minor change of parameters. Indeed, define this time $z_0 = (0, 0)$, $z_1 = (\frac{h}{4}, 10\eta(h \log h)^{\frac{1}{2}})$, $z_2 = (\frac{h}{2}, 5\eta(h \log h)^{\frac{1}{2}})$, $z_3 = (\frac{3h}{4}, 10\eta(h \log h)^{\frac{1}{2}})$ and $z_4 = (h, 0)$.

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plane which maps $[x, y]$ on an horizontal segment and T the translation which maps x to 0. Let $\tilde{\mathbf{R}} = T \circ \mathbb{R}_\theta(\mathbf{R})$. To ensure that $z_1, z_3 \in \tilde{\mathbf{R}}$, we set $N_0 = N_0(\varepsilon, \eta)$ large enough so that for $h \geq N_0$,

$$\arctan(40\eta(h \log h)^{-\frac{1}{2}}) \leq \varepsilon.$$

Assume that $h \geq N_0$. As previously, let, for $i \in \{0, \ldots, 3\}$, H_i be the event that z_i and z_{i+1} are θ-connected by a θ-path which fluctuates less than $10 \parallel z_{i+1} - z_i \parallel^{\frac{1}{2}}$ around the segment $[z_i, z_{i+1}]$, and define the event $\mathsf{LogShape} = H_0 \cap H_1 \cap H_2 \cap H_3$. As previously, we notice that

$$\mathsf{LogShape} \subset \mathsf{LogGAC}(x, y, \eta).$$

Moreover,

$$\mathsf{LogShape} \subset \mathsf{LogSID}(x, y, \eta).$$

[12]We omit integer rounding and the fact that there might not be a θ-path from z_i to z_{i+1} (but rather a θ-path from $z_i + B_1$ to $z_{i+1} + B_1$ where B_1 is the unit ball).

Indeed, it is straightforward to check that $(T \circ \mathbb{R}_\theta)^{-1}(z_2)$ accomplishes the born on the local roughness:

$$\begin{aligned} \mathrm{d}((T \circ \mathbb{R}_\theta)^{-1}(z_2), \mathcal{C}(\gamma)) &\geq \mathrm{d}(z_2, [z_1, z_3]) \\ &= 5\eta(h \log h)^{\frac{1}{2}} + O(1). \end{aligned}$$

Finally, the estimate of $\mathbb{P}_{x,y}\,[\mathsf{LogShape}]$ follows the one of $\mathbb{P}_{x,y}[\mathsf{Shape}]$ conducted in the preceding lemma. Defining $a = \cos\theta, b = \sin\theta$ and

$$\begin{cases} \alpha_0 = \frac{h}{4}a + 10\eta(h\log h)^{\frac{1}{2}}b \\ \beta_0 = \frac{h}{4}b - 10\eta(h\log h)^{\frac{1}{2}}a \end{cases}, \qquad \begin{cases} \alpha_1 = \frac{h}{4}a - 5\eta(h\log h)^{\frac{1}{2}}b \\ \beta_1 = \frac{h}{4}b + 5\eta(h\log h)^{\frac{1}{2}}a \end{cases},$$

$$\begin{cases} \alpha_2 = \frac{h}{4}a + 5\eta(h\log h)^{\frac{1}{2}}b \\ \beta_2 = \frac{h}{4}b - 5\eta(h\log h)^{\frac{1}{2}}a \end{cases}, \qquad \begin{cases} \alpha_3 = \frac{h}{4}a - 10\eta(h\log h)^{\frac{1}{2}}b \\ \beta_3 = \frac{h}{4}b + 10\eta(h\log h)^{\frac{1}{2}}a \end{cases}, \qquad \begin{cases} \alpha_4 = ha \\ \beta_4 = hb \end{cases}.$$

As previously, we can use Stirling's estimate to infer that there exists $C = C(\varepsilon) > 0$ such that

$$\mathbb{P}_{x,y}[\mathsf{LogShape}] \geq h^{-\frac{3}{2}} h^{-C\eta^2}.$$

We recover the result by considering $h^{3/2}$ variants of the event $\mathsf{LogShape}$, as above. □

Multivalued map principle

If A and B are two finite sets, we say that any function $T : A \to \mathcal{P}(B)$ is a *multivalued map* and for any $b \in B$, we write $T^{-1}(b) = \{a \in A, b \in T(a)\}$.

Lemma 3.5.3 (Probabilistic multivalued map principle)**.** *Let* $(\Omega_1, \mathbb{P}_1), (\Omega_2, \mathbb{P}_2)$ *be two discrete probability spaces, let* A *(resp.* B*) be a measurable subset of* Ω_1 *(resp.* Ω_2*) and let* $T : A \to \mathcal{P}(B)$ *be a multivalued map. We assume that the following quantities are finite:*

$$\varphi(T) := \max_{a\in A} \max_{b\in T(a)} \frac{\mathbb{P}_1(a)}{\mathbb{P}_2(b)} < \infty,$$

$$\psi(T) := \frac{\max_{b\in B} \left|T^{-1}(b)\right|}{\min_{a\in A} |T(a)|} < \infty.$$

Then,

$$\mathbb{P}_1[A] \leq \varphi(T)\psi(T)\mathbb{P}_2[B].$$

Proof. The lemma follows by the following simple computation:

$$\begin{aligned} \mathbb{P}_1\,[A] &= \sum_{a\in A} \sum_{b\in T(a)} \frac{\mathbb{P}_1\,[a]}{\mathbb{P}_2\,[b]} \mathbb{P}_2[b]\,|T(a)|^{-1} \\ &\leq \varphi(T) \sum_{b\in B} \mathbb{P}_2[b] \sum_{a\in T^{-1}(b)} |T(a)|^{-1} \\ &\leq \varphi(T)\psi(T)\mathbb{P}_2[B]. \end{aligned}$$

□

Chapter 4

Multiple interfaces and entropic repulsion

4.1 Introduction

Rigorous understanding of the behaviour of *interfaces* in statistical mechanics models has been the focus of intensive study for more than 50 years, especially in the case of the Ising model. The first rigorous results were perturbative and made use of the Pirogov–Sinaï theory to show that a low temperature two-dimensional Ising interface converges, after an appropriate diffusive scaling, towards a Brownian bridge [63, 79]. However, these works are restricted to the very low temperature regime, even if the belief was that the result should hold for any subcritical temperature.

In the beginning of the XXIst century, the development and the understanding of the rigorous *Ornstein–Zernike theory*, first in Bernoulli percolation and later on in the context of more dependent models such as the Ising and Potts models [22, 24, 25], provided a new powerful tool for a detailed study of the subcritical phase of these percolation or spin models. The structural output of this theory is the probabilistic description of long clusters (or equivalently of long interfaces as we shall see below) in terms of one-dimensional "irreducible pieces" behaving almost independently (for a precise statement, we refer to Theorem 4.2.7). In particular, the diffusive scaling limit of interfaces at any subcritical temperature could be obtained in the case of the Ising model as a quite simple byproduct of this robust theory in the work of Greenberg and Ioffe [68] (see [92] for the simpler case of Bernoulli percolation).

Later on, this technique has been found to be efficient for studying interfaces interacting with their environment. Indeed, the above mentioned works deal with unconstrained (also called *free*) interfaces, but recent works have been extending the study of these interfaces to broader settings in which non-trivial interactions with the environment are added. Let us cite [102] for the case of a defect line in the Potts model, and — much more related to this work — [84] for the treatment of a Potts interface above a boundary wall. These examples of interfaces interacting with their close environment have turned out to be more delicate to handle and in certain conditions have been shown to exhibit highly non-trivial behaviours such as *wetting transitions*, which have been studied in [87].

Of the same nature is the study of a system of multiple interacting interfaces, which is the focus of the present work. Indeed, this paper determines the scaling behaviour of a finite number of long clusters of subcritical Fortuin–Kasteleyn (FK) percolation, conditioned not to intersect; subcritical percolation clusters mimic interfaces in the low temperature regime.

An interesting feature of this setting is that when conditioned not to intersect, the interaction between the clusters can turn out to be *attractive*, *a priori* allowing the existence of a *pinning* transition — a regime where this attraction is so strong that the clusters actually remain at a bounded distance from each other. We rule out the existence of such a transition. In the fashion of [84], we show that the behaviour of this system obeys an *entropic repulsion* phenomenon: the entropy caused by the large number of possible clusters wins over the energetic reward obtained by keeping them close together, all the way up to the critical point. Such a phenomenon has been previously identified in a variety of settings, for instance in the three-dimensional semi-infinite Ising model at low temperatures [62], the 2+1-dimensional SOS model above a hard wall [27], a 1+1-dimensional interface above an attractive field in presence of a magnetic field [109] or a supercritical Potts interface above a wall [84], to mention but a few works studying this phenomenon.

In this work, entropic repulsion of the FK clusters at any subcritical temperature is established in Proposition 4.4.8, which is probably the most important output of this work. As a byproduct, we derive two results regarding the global behaviour of such a system of conditioned clusters. The first one is the diffusive scaling limit of such a system, which is shown to be a system of Brownian bridges conditioned not to intersect: the so-called *Brownian watermelon*. Moreover, we observe that the entropic repulsion phenomenon also allows the computation — up to a multiplicative constant — of the probability of the existence of such a system of clusters. Finally, as a byproduct of the latter observation, we also obtain the asymptotics of the probability of the occurrence of a large finite connection in the supercritical random-cluster model.

The method is in spirit close to that of [84], but with considerable additional difficulties. These are essentially due to the fact that the interaction is not only between a random object and a deterministic one, but between several random objects, forcing one to control their *joint* behaviour. The proofs make heavy use of the Ornstein–Zernike theory for subcritical random-cluster models, developed in [25].

4.1.1 Definitions of the random-cluster model and the Brownian watermelon

The random-cluster model

The model of interest is the so-called *random-cluster model* (also known as *FK-percolation*). We first recall its definition and a few basic properties (we refer to [49] for a complete exposition). The random-cluster model on $\mathbb{Z}^2$ is a model of random subgraphs of $\mathbb{Z}^2$. Its law is described by two parameters, $p \in [0,1]$ and $q > 0$.

Let $G = (V(G), E(G))$ be a finite subgraph of $\mathbb{Z}^2$. We denote its *inner boundary* (resp. *outer boundary*) by

$$\partial G = \left\{x \in V(G), \exists y \notin V(G), \{x,y\} \in E(\mathbb{Z}^2)\right\} \text{ and}$$
$$\partial_{\mathsf{ext}} G = \left\{x \notin V(G), \exists y \in V(G), \{x,y\} \in E(\mathbb{Z}^2)\right\}, \text{ respectively.}$$

A *percolation configuration* on G is an element $\omega \in \{0,1\}^{E(G)}$. We say that an edge $e \in G$ is *open* if $\omega(e) = 1$ and *closed* otherwise. Two vertices $x, y \in \mathbb{Z}^2$ are said to be connected if there exists a path of nearest neighbour vertices $x = x_0, x_1, \ldots, x_n = y$ such that the edges $\{x_i, x_{i+1}\}$ are open for every $0 \leq i \leq n-1$. In this case, we say that the event $\{x \leftrightarrow y\}$ occurs. A *vertex cluster* of ω is a maximal connected component of the set of vertices (it can be an isolated vertex). Given a percolation configuration ω, we denote by $o(\omega)$ its number of open edges, and by $k(\omega)$ its number of vertex clusters.

A *boundary condition* on G is a partition $\eta = P_1 \cup \cdots \cup P_k$ of ∂G. From a configuration $\omega \in \{0,1\}^{E(G)}$, we create a configuration ω^η by identifying the vertices that belong to the same P_i of η. Two particular boundary conditions, that we shall call the *free boundary* condition (resp. wired boundary condition), consist in the partition made of singletons (resp. of the whole set ∂G). We shall write $\eta = 0$ (resp. $\eta = 1$) for this specific boundary condition.

Definition 4.1.1. Let $G = (V(G), E(G))$ be a finite subgraph of $\mathbb{Z}^2$, and η be a boundary condition on G. Let $p \in [0,1]$ and $q > 0$. The random-cluster measure on G with boundary condition η is the following probability measure on percolation configurations on G:

$$\phi^{\eta}_{p,q,G}(\omega) = \frac{1}{Z^{\eta}_{p,q,G}} \left(\frac{p}{1-p}\right)^{o(\omega)} q^{k(\omega^\eta)},$$

where $Z^{\eta}_{p,q,G} > 0$ is the normalisation constant ensuring that $\phi^{\eta}_{p,q,G}$ is indeed a probability measure. We shall refer to $Z^{\eta}_{p,q,G}$ as the *partition function* of the model.

It is classical that for $\eta = 0$ and $\eta = 1$, this measure can be extended to the whole plane $\mathbb{Z}^2$, by taking the weak limit of the measures ϕ^{η}_{p,q,G_n} over any exhaustion $(G_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}^2$, and that the limit measure does not depend of the choice of the exhaustion. Below, we will simply write $\phi^{\eta}_{p,q}$ instead of $\phi^{\eta}_{p,q,\mathbb{Z}^2}$.

A very fundamental feature of this model is that it undergoes a *phase transition*. Namely for any $q \geq 1$, there exists a critical parameter $p_c = p_c(q) \in (0,1)$ such that:

- $\forall p < p_c(q), \phi^1_{p,q}\left(0 \leftrightarrow \infty\right) = 0$;
- $\forall p > p_c(q), \phi^0_{p,q}\left(0 \leftrightarrow \infty\right) > 0$,

where $\{0 \leftrightarrow \infty\}$ is the event that the cluster of 0 is infinite.

We are going to be interested in the first case — called *the subcritical regime*. In this case it is well known that the choice of boundary conditions does not affect the infinite volume measure. We thus drop η from the notation and simply write $\phi_{p,q}$ for the unique infinite volume measure when $p < p_c(q)$. Another important feature of the subcritical random-cluster model is the existence and the positivity of the following limit:

$$\tau_{p,q} := \lim_{n\to\infty} -\frac{1}{n} \log\left[\phi_{p,q}\left(0 \leftrightarrow (n,0)\right)\right]. \tag{4.1}$$

We call this quantity the *inverse correlation length* in the direction $\vec{e_1}$. Moreover, standard subadditivity arguments yield that

$$\forall x \in \mathbb{Z}^2, \phi_{p,q}\left[x \leftrightarrow x + (n,0)\right] \leq \mathrm{e}^{-\tau_{p,q} n}. \tag{4.2}$$

Since p, q will be fixed through this work, we shall simply write $\tau > 0$ instead of $\tau_{p,q}$.

The Brownian watermelon

The Brownian watermelon is a stochastic process that arises in various areas of probability theory, like random matrix theory [11], integrable probability [88], but also more recently in the study of the KPZ universality class [76].

We give a brief definition of this object, and we refer to [99], [67] and [43] for the full construction and details. Let $r \geq 1$ be an integer. We define the Weyl chamber of order r:

$$W = \left\{(x_1, \ldots, x_r) \in \mathbb{R}^r, x_1 < \cdots < x_r\right\}.$$

We shall also introduce the functional Weyl chamber in the interval $[s,t]$ for $0 \leq s < t$ (for any set $A \subset \mathbb{R}$, the set $\mathcal{C}(A, \mathbb{R}^r)$ denotes the space of continuous functions from A to $\mathbb{R}^r$):

$$\mathcal{W}_{[s,t]} = \{f \in \mathcal{C}([s,t], \mathbb{R}^r), \forall s \leq \ell \leq t, f(\ell) \in W\}.$$

Moreover let Δ denote the Vandermonde function, defined for any $(x_1, \ldots, x_r) \in \mathbb{R}^r$ by:

$$\Delta(x_1, \ldots, x_r) = \prod_{1 \leq i < j \leq r} (x_j - x_i).$$

Definition 4.1.2 (Brownian watermelon)**.** The Brownian watermelon with r bridges is the continuous process $\left(\mathsf{BW}_t^{(r)}\right)_{0\leq t\leq 1}$ obtained by conditioning r independent standard Brownian bridges not to intersect in $(0,1)$. It is a random object of $\mathcal{C}([0,1],\mathbb{R}^r)$.

Remark 4.1.3. Since the non-intersection event has null probability for r random bridges as soon as $r\geq 2$, the latter conditioning is rigorously done by means of a Doob h-transformation by the harmonic function Δ. We refer to [99] and [54] for the details of the construction (and the fact that Δ is harmonic for a system of r standard bridges). Moreover, it can be shown, by means of the Karlin–McGregor formula, that for any $0<t<1$

$$\mathbb{P}\left[\mathsf{BW}_t^{(r)}\in \mathrm{d}z\right]\propto \frac{1}{\left(t(1-t)\right)^{r^2/2}}\Delta^2(z)\mathrm{e}^{-\frac{|z|^2}{2t(1-t)}}\mathbb{1}_{z\in W}\mathrm{d}z. \tag{4.3}$$

Remark 4.1.4. Alternatively, the Brownian watermelon can be built *via* the following method: consider a system $(B_t^\varepsilon)_{0\leq t\leq 1}$ of r independent standard Brownian bridges started from 0, $\varepsilon,\dots,(r-1)\varepsilon$ respectively. Then under the conditioning on the event $\{B_t^\varepsilon\in\mathcal{W}_{0,1}\}$ (this happens with positive probability), the following weak limit exists in $\mathcal{C}([0,1],\mathbb{R}^r)$ when $\varepsilon\to 0$ and is called the Brownian watermelon:

$$(B_t^\varepsilon)_{0\leq t\leq 1}\xrightarrow[\varepsilon\to 0]{(d)}(\mathsf{BW}_t^{(r)})_{0\leq t\leq 1}.$$

For more information on this construction, see [99] and [76].

Notations and conventions. If a_n and b_n are two sequences of real numbers, we shall write $a_n\sim b_n$ when $\frac{a_n}{b_n}\xrightarrow[n\to\infty]{}1$. We shall also write $a_n=o(b_n)$ when $\frac{a_n}{b_n}\xrightarrow[n\to\infty]{}0$ and $a_n=O(b_n)$ when there exists a constant $C>0$ such that $|a_n|\leq C|b_n|$ for all $n\geq 0$. Moreover, we shall write $a_n\asymp b_n$ whenever $a_n=O(b_n)$ and $b_n=O(a_n)$. Finally, the generic notations $c,C>0$ will denote constants depending only on p and q, that may change from line to line during computations. We denote by $\|\cdot\|$ the Euclidian norm on $\mathbb{R}^d$.

4.1.2 Exposition of the results

In this paper, we study the scaling limit of a system of subcritical clusters conditioned on a connection and a non-intersection event. We first start by defining these percolation events.

Definition 4.1.5 (Connection event, Non-intersection event)**.** Let $x,y\in W\cap\mathbb{Z}^r$ and $n\geq 0$. Then we define the multiple connection event $\mathsf{Con}_{x,y}^n$ by

$$\mathsf{Con}_{x,y}^n=\left\{\forall 1\leq i\leq r,(0,x_i)\leftrightarrow(n,y_i)\right\}.$$

The non-intersection event will be defined by

$$\mathsf{NI}_x=\left\{\forall 1\leq i<j\leq r,\ \mathcal{C}_{(0,x_i)}\cap\mathcal{C}_{(0,x_j)}=\emptyset\right\},$$

where $\mathcal{C}_u$ denotes the cluster of the vertex $u \in \mathbb{Z}^2$. In the rest of this work, as n, x, y will be fixed, we shall abbreviate $\mathsf{Con}^n_{x,y}$ by Con and NI_x by NI. Moreover, we will also abbreviate $\mathcal{C}_{(0,x_i)}$ by $\mathcal{C}_i$.

Our main result consists in the estimation of the probability that $\{\mathsf{Con}, \mathsf{NI}\}$ occurs in a subcritical random-cluster measure.

Theorem 4.1.6. *Let $q \geq 1$, and $0 < p < p_c(q)$. Let $r \geq 1$ be a fixed integer. Then, there exist two constants $C_-, C_+ > 0$ such that for any sequences x_n, y_n of elements of W satisfying $\|x_n\|, \|y_n\| = o(\sqrt{n})$, when n is sufficiently large,*

$$C_- V(x_n)V(y_n)n^{-\frac{r^2}{2}}\mathrm{e}^{-\tau r n} \leq \phi\left[\mathsf{Con}, \mathsf{NI}\right] \leq C_+ V(x_n)V(y_n)n^{-\frac{r^2}{2}}\mathrm{e}^{-\tau r n},$$

where V is the function defined in Theorem 4.5.5.

Remark 4.1.7. The function V is not explicit. However, it is known that (see Theorem 4.5.5) :

$$\text{When } \min_{1 \leq i \leq r-1}\{|(x_n)_{i+1} - (x_n)_i|\} \xrightarrow[n\to\infty]{} +\infty, \text{ then } V(x_n) \sim \Delta(x_n) \text{ as } n \to \infty.$$

Moreover, when x, y are fixed elements of W, the statement simplifies as

$$\phi\left[\mathsf{Con}, \mathsf{NI}\right] \asymp n^{-\frac{r^2}{2}}\mathrm{e}^{-\tau r n}.$$

An interesting corollary, which is a direct consequence of Theorem 4.1.6 in the case $r = 2$, can be obtained using the methods of [23], where the same result is proved in the case of Bernoulli percolation (corresponding to $q = 1$ in the random-cluster model). Let us define the *truncated inverse correlation length* in the direction $\vec{e_1}$ by

$$\tau_p^{\mathsf{f}} = \lim_{n\to\infty} -\frac{1}{n}\log\phi\left[0 \leftrightarrow (n,0), |\mathcal{C}_0| < \infty\right].$$

It is well known that on $\mathbb{Z}^2$, whenever $p \neq p_c(q)$, one has that $\tau_p^{\mathsf{f}} > 0$. Moreover, it is clear that whenever $p < p_c(q)$, $\tau_p^{\mathsf{f}} = \tau_p$, where τ_p has been defined in (4.1). Then, Theorem 4.1.6 allows to compute the prefactor in the supercritical truncated correlation function.

Corollary 4.1.8. *Let $q \geq 1$ and $p \in (p_c, 1)$. Let ϕ be the unique infinite-volume random-cluster measure on $\mathbb{Z}^2$. Then,*

$$\phi\left[0 \leftrightarrow (n,0), |\mathcal{C}_0| < \infty\right] \asymp \frac{1}{n^2}\mathrm{e}^{-2\tau^{\mathsf{f}}_{p^*} n},$$

where p^* stands for the dual parameter of p (see (4.6) for the relation linking p and p^*).

Remark 4.1.9. In particular, we obtain the following equality, holding for any supercritical $p > p_c$

$$\tau_p^{\mathsf{f}} = 2\tau_{p^*}^{\mathsf{f}} (= 2\tau_{p^*}).$$

This is a very specific instance of duality, and such a relation is not expected to hold in higher dimensions. The result was already well known in the case of Bernoulli percolation, see for instance [35] or [69, Theorem 11.24].

Our second result consists in the study of the behaviour of the r clusters created by conditioning on $\{\mathsf{Con}, \mathsf{NI}\}$. It will be formulated in terms of the *envelopes* of a cluster.

Definition 4.1.10 (Upper and lower envelopes of a cluster)**.** Let $\omega \in \mathsf{Con}$. Then for any $0 \leq k \leq n$ and $1 \leq i \leq r$ we define (see Figure 4.1)

$$\Gamma_i^+(k) = \max\{\ell \in \mathbb{Z}, (k,\ell) \in \mathcal{C}_i\} \text{ and } \Gamma_i^-(k) = \min\{\ell \in \mathbb{Z}, (k,\ell) \in \mathcal{C}_i\}.$$

It is clear that $\Gamma_i^\pm$ are well defined, since all clusters are almost surely finite in the subcritical regime, and the sets above are not empty due to Con. We will see these quantities as functions from $[0,n]$ to $\mathbb{R}$ by considering the piecewise affine functions $\Gamma_i^\pm(t)$ that coincide with $\Gamma_i^\pm$ on the integers $t = k$.

Our second result is the following:

Theorem 4.1.11. *Fix $x, y \in W \cap \mathbb{Z}^r$ and $p \in (0, p_c(q))$. Then under the family of measures $\phi_{p,q}[\ \cdot\ |\mathsf{Con}, \mathsf{NI}]$ (we recall that* Con, NI *depend on n), there exists $\sigma > 0$ such that:*

$$\left(\frac{1}{\sqrt{n}}\left(\Gamma_1^+(nt), \dots, \Gamma_r^+(nt)\right)\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (\sigma \mathsf{BW}_t^{(r)})_{0\leq t\leq 1}, \tag{4.4}$$

where $\mathsf{BW}^{(r)}$ is the Brownian watermelon with r bridges, and where the convergence holds in the space $\mathcal{C}([0,1],\mathbb{R}^r)$ endowed with the topology of uniform convergence. Moreover, almost surely, for all $1 \leq i \leq r$,

$$\frac{1}{\sqrt{n}}\left\|\Gamma_i^+ - \Gamma_i^-\right\|_\infty \xrightarrow[n\to\infty]{} 0 \tag{4.5}$$

Remark 4.1.12. A consequence of (4.5) is that in the setting of Theorem 4.1.11, the clusters remain of width $o(\sqrt{n})$. Actually, we prove that almost surely, $\|\Gamma_i^+ - \Gamma_i^-\|_\infty = O(\log n)$. In particular, the choice of the upper interfaces Γ_i^+ in (4.4) is arbitrary and can be replaced by any assignment of $\pm$ for the choice of interfaces to converge.

Remark 4.1.13. The result is stated for fixed $x, y \in W \cap \mathbb{Z}^r$. However, the careful reader may check that our method allows to treat the case where x and y depend on n. Indeed, as soon as x_n, y_n are two sequences of $W \cap \mathbb{Z}^r$ satisfying

$$\|x_n\| = o(\sqrt{n}) \quad \text{and} \quad \|y_n\| = o(\sqrt{n}),$$

our methods may apply and yield the same scaling limit.

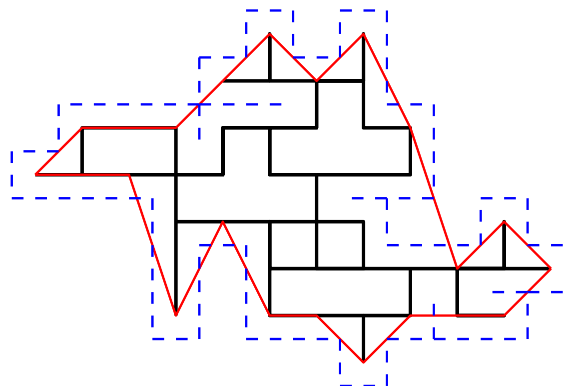

Figure 4.1: Depiction (in red) of the envelopes Γ^+ and Γ^- of a percolation cluster. The blue dashed path corresponds to the more natural notion of interface that could have been considered instead. However, as explained in Remark 4.1.14, this blue path converges in the space of continuous curves towards the Brownian watermelon as well.

Remark 4.1.14. For the reader familiar with statistical mechanics, it might seem strange that our result is formulated in terms of these envelopes and not in terms of the upper and lower interfaces running along the boundary of the clusters $\mathcal{C}_i$. However, it may be shown that the interfaces also converge to the paths of $\mathsf{BW}_t^{(r)}$ (as paths in $[0,1]\times\mathbb{R}^r$). We then chose to work with $\Gamma^\pm$ since we can use the space of continuous functions from $[0,n]$ to $\mathbb{R}^r$ for studying convergence questions, which is easier to treat than the space of continuous curves which would be needed when considering those interfaces.

4.1.3 Background on the random-cluster model

We first recall some basic properties of the random-cluster model (once again we refer to [49] for a complete exposition). These properties are valid for any choice of parameters p and q.

Positive association. The space $\{0,1\}^{E(\mathbb{Z}^2)}$ can be equipped with a partial order: we say that $\omega_1 \le \omega_2$ if for any $e\in E(\mathbb{Z}^2)$, $\omega_1(e)\le\omega_2(e)$. An event $\mathcal{A}$ will be called *increasing* if for any $\omega_1\le\omega_2$, $\omega_1\in\mathcal{A}\Rightarrow\omega_2\in\mathcal{A}$. The *FKG inequality* then states that for any increasing events $\mathcal{A},\mathcal{B}$, any graph G and any boundary conditions η,

$$\phi^{\eta}_{G,p,q}[\mathcal{A}\cap\mathcal{B}]\ge\phi^{\eta}_{G,p,q}[\mathcal{A}]\phi^{\eta}_{G,p,q}[\mathcal{B}]. \qquad \text{(FKG)}$$

This property implies in particular that for any boundary conditions $\eta_1\le\eta_2$ (meaning that the partition η_1 is finer than η_2), for any increasing event $\mathcal{A}$,

$$\phi^{\eta_1}_{G,p,q}\left[\mathcal{A}\right]\le\phi^{\eta_2}_{G,p,q}\left[\mathcal{A}\right]. \qquad \text{(CBC)}$$

This property is called the *comparison of boundary conditions* and may also be stated as "ϕ^{η_1} is stochastically dominated by ϕ^{η_2}".

Duality. Let $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ and consider the lattice $(\mathbb{Z}^2)^*$ with edges between nearest neighbours. This lattice is called the *dual lattice*. It has the property that for any $e \in E(\mathbb{Z}^2)$, there exists a unique edge $e^* \in E((\mathbb{Z}^2)^*)$ that crosses e. To a percolation configuration $\omega \in \{0,1\}^{E(\mathbb{Z}^2)}$ we can associate a dual configuration ω^* on the dual lattice by setting $\omega^*(e^*) = 1 - \omega(e)$. Then we remark that — as soon as the parameters guarantee that there exists a unique Gibbs measure — if ω is sampled according to $\phi_{p,q}$, then ω^* has the distribution of ϕ_{p^*,q^*}, where

$$q = q^* \text{ and } \frac{pp^*}{(1-p)(1-p^*)} = q. \tag{4.6}$$

It has been proved by V. Beffara and H. Duminil-Copin in [13] that $p_c(q) = p_c^*(q)$, meaning that the parameter $p_c(q)$ is *self-dual*. Also observe that if $\phi_{p,q}$ is subcritical, then ϕ_{p^*,q^*} is supercritical and vice-versa.

Spatial Markov property. Let G be a subgraph of $\mathbb{Z}^2$, and $G' \subset G$ a subgraph of G. Let ξ be a percolation configuration on $\mathbb{Z}^2$. Observe that it induces a boundary condition on G — that we name $\eta(\xi)$ — by identifying the vertices wired together by ξ outside G, and a boundary condition on G' - that we name $\eta'(\xi)$ by the same principle. Then,

$$\phi_{G,p,q}^{\eta(\xi)}\left[\ \cdot\ |\omega(e) = \xi(e),\ \forall e \notin G'\right] = \phi_{G',p,q}^{\eta'(\xi)}[\cdot]. \tag{SMP}$$

Finite energy property. When $p \notin \{0,1\}$, there exists a constant $\varepsilon > 0$ depending only on p and q such that for any finite graph G, any finite $F \subset E(G)$, any boundary condition η, and any percolation configuration ω_0,

$$\varepsilon^{|F|} \leq \phi_{G,p,q}^{\eta}\left[\omega(e) = \omega_0(e),\ \forall e \in F\right] \leq (1-\varepsilon)^{|F|}.$$

Weak ratio mixing. In the subcritical regime, the random-cluster measure also enjoys the following *weak ratio mixing property*. For two finite connected sets of edges E_1 and E_2, define their distance $d(E_1, E_2)$ as the Euclidean distance between the set of their respective endpoints. Then, for any graph G, any boundary condition η, any $q \geq 1$ and any $p < p_c(q)$, there exists a constant $c > 0$ such that for any events $\mathcal{A}$ and $\mathcal{B}$ depending on edges of E_1 and E_2 respectively,

$$\left|1 - \frac{\phi_{G,p,q}^{\eta}[\mathcal{A} \cap \mathcal{B}]}{\phi_{G,p,q}^{\eta}[\mathcal{A}]\phi_{G,p,q}^{\eta}[\mathcal{B}]}\right| < \mathrm{e}^{-cd(E_1,E_2)}. \tag{MIX}$$

4.1.4 Outline of the proof

The main idea of modern Ornstein–Zernike theory is to couple a subcritical percolation cluster conditioned on realizing a connection event $\{x \leftrightarrow y\}$ with a random walk started from x and conditioned to reach y. Such a cluster is essentially a one-dimensional object. As the knowledge on conditioned random walks is very broad, in particular in terms of Local Limit Theorems and invariance principles, such a coupling allows to derive properties of the original cluster. In our setting, we would like to couple a system of r percolation clusters conditioned on $\mathsf{Con} \cap \mathsf{NI}$ with a system of r random walks conditioned on a hitting event and on not intersecting each other. However, such a coupling is not immediately available in this setting and we have to rely on several comparison principles to show that the behaviours of these two types of systems are close. Once this task is accomplished, we use an invariance principle for a system of non-intersecting random walks to derive Theorem 4.1.11.

Let us be a bit more precise about the method. We first show that Ornstein–Zernike theory extends to r non-intersecting clusters sampled according to $\phi^{\otimes r}$ (the product of r random cluster measures on $\mathbb{Z}^2$) and thus interacting only through the conditioning. This allows us to derive an invariance principle for this product measure.

The next step is to transmit the results obtained for the product measure to the "true" FK-percolation measure. As crucially observed in [84], this can be done proving an *a priori* (meaning independent of the above mentioned coupling) repulsion estimate: under the conditioned random-cluster measure, the clusters naturally move far from each other and never come near each other again. This input will then allow us to use the mixing property of subcritical FK-percolation to derive Theorems 4.1.6 and 4.1.11.

4.1.5 Organization of the paper

We first focus on Theorems 4.1.6 and 4.1.11. As explained previously, the proof consists of two independent tasks: comparing the behaviours of the percolation clusters and of a system of interacting random walks, and then obtaining the scaling limit and fine estimates on such a system of random walks. Our interest mainly being statistical mechanics, we postpone all the results about interacting random walks to Section 4.5, which is independent of the other sections, and may be skipped by readers only interested in the percolation aspects. Section 4.2 consists in a review of the rigorous Ornstein–Zernike results for one single subcritical cluster of FK-percolation. Section 4.3 is devoted to the study of the scaling limit under the *product measure* through a straightforward extension of the Ornstein–Zernike theory to this setting, as discussed before. Finally, Section 4.4 is devoted to the proof of the entropic repulsion estimates, and thus of the announced result.

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4.2 Ornstein–Zernike theory for a single subcritical FK-cluster

In the remainder of the paper, we **fix** $q \geq 1$ and $0 < p < p_c(q)$. Since these parameters will not change throughout the paper, we drop them from the notations and abbreviate $\phi_{p,q} := \phi$.

In this section, we review and discuss the main result of [25] — the Ornstein–Zernike Theorem. Schematically, this result can be described as follows. Under the conditioned measure $\phi[\ \cdot\ | y \in \mathcal{C}_0]$ where y is some vertex far away from 0, the cluster of 0 has a very particular structure. Indeed, it macroscopically looks like the geodesic from 0 to y. Moreover, it exhibits typical Brownian bridge fluctuations around this geodesic, and is confined in a very small tube around this Brownian bridge. The result is precisely stated in Theorem 4.2.7.

Definition 4.2.1 (Directed random walk)**.** A *directed measure* on $\mathbb{Z}^2$ is a probability measure on $\mathbb{N}^* \times \mathbb{Z}$. If $X_1, \dots, X_n, \dots$ are independent and identically distributed random variables sampled according to a directed measure on $\mathbb{Z}^2$, then the distribution of process $S_n = X_1 + \cdots + X_n$ is called a *directed random walk*. We shall call a possible realization of (S_n) a *directed walk* on $\mathbb{Z}^2$.

In the remainder of the paper, we will often interpret trajectories of directed walks as real-valued functions defined on $\mathbb{R}^+$. Indeed, let ν be a directed measure on $\mathbb{Z}^2$ and $(S_n)_{n\geq 0}$ the associated directed random walk. Since $\nu(\mathbb{N}^* \times \mathbb{Z}) = 1$, for any $t \geq 0$, the trajectory of S almost surely intersects the vertical line $\{t\} \times \mathbb{R}$ once. Calling this point $S(t)$ provides us with a continuous and piecewise linear function: moreover this correspondence is one-to-one. We shall often use notations as $\{S \in \mathcal{A}\}$, where $\mathcal{A}$ is a subset of $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$. In that case, S will have to be taken as the continuous function described above. Let $(S_n)_{n\geq 0}$ be a directed random walk on $\mathbb{Z}^2$. If $y \in \mathbb{Z}^2$, introduce the event

$$\mathsf{Hit}_y = \{\exists n \geq 0, S_n = y\}\,.$$

4.2.1 Diamond confinement and diamond decomposition

We need a bit of vocabulary, in order to properly state the confinement property of a long subcritical cluster. Let $\delta > 0, x \in \mathbb{Z}^2$. Following [25], introduce the following subsets of $\mathbb{Z}^2$:

- The δ-forward cone of apex x to be the set $\mathcal{Y}_x^{\delta,+} = x + \big\{(x_1, x_2) \in \mathbb{Z}^2, \delta x_1 \geq |x_2|\big\}\,.$
- The δ-backward cone of apex x to be the set $\mathcal{Y}_x^{\delta,-} = x + \big\{(x_1, x_2) \in \mathbb{Z}^2, \delta x_1 \leq -|x_2|\big\}\,.$
- If $x, y \in \mathbb{Z}^2$ are such that $x_1 < y_1$, the δ-diamond of apexes x, y is the intersection:

$$\mathcal{D}^\delta_{x,y} = \mathcal{Y}_x^{\delta,+} \cap \mathcal{Y}_y^{\delta,-}.$$

  If $x = 0$, we abbreviate the notation by $\mathcal{D}^\delta_y$.

Let G be a finite subgraph of $\mathbb{Z}^2$ containing the vertex 0 (we say that G is a subgraph of $\mathbb{Z}^2$ *rooted at 0*). We say that:

- (G, v) is δ-*left-confined* if there exists $x \in V(G)$ such that $G \subset \mathcal{Y}_x^{\delta,-}$.
- (G, v) is δ-*right-confined* if there exists $x \in V(G)$ such that $G \subset \mathcal{Y}_x^{\delta,+}$.
- G is δ-*diamond-confined* if there exist $y \in V(G)$ such that $G \subset \mathcal{D}_y^\delta$. In that case, we say that $\mathcal{D}_y^\delta$ is the diamond containing G.

Observe that in the previous definitions, if the points x, y do exist, they are necessarily unique. We denote the set of δ-left-confined subgraphs of $\mathbb{Z}^2$ rooted at 0 (resp. δ-right-confined subgraphs of $\mathbb{Z}^2$ rooted at 0, resp δ-diamond-confined subgraphs of $\mathbb{Z}^2$ rooted at 0) by $\mathfrak{C}_L^\delta$ (resp $\mathfrak{C}_R^\delta$, resp $\mathfrak{D}^\delta$).

Definition 4.2.2. We now define the notion of *displacement* along a left-confined, right-confined or diamond-confined subgraph of $\mathbb{Z}^2$.

- Let G be a δ-left-confined subgraph of $\mathbb{Z}^2$ rooted at 0. The *displacement of* G is
$$X^L(G) = x,$$
where x is the unique vertex of G such that $G \subset \mathcal{Y}_x^{-,\delta}$.
- Let G be a δ-right-confined subgraph of $\mathbb{Z}^2$ rooted at 0. The *displacement of* G is
$$X^R(G) = -x,$$
where x is the unique vertex of G such that $G \subset \mathcal{Y}_x^{+,\delta}$.
- Let G be a diamond-confined subgraph of $\mathbb{Z}^2$ rooted at 0. The *displacement of* G is
$$X(G) = y,$$
where y is the only vertex of G such that $G \subset \mathcal{D}_y^\delta$.

In order to properly state what is a diamond decomposition of a cluster, we also need to introduce the operation of concatenation of two confined subgraphs rooted at 0. Let $G_1 \in \mathfrak{C}_L^\delta$ and $G_2 \in \mathfrak{D}^\delta$. The concatenation of G_1 and G_2, called $G_1 \circ G_2$, is defined to be the subgraph
$$G_1 \circ G_2 = G_1 \cup \left(X^L(G_1) + G_2\right).$$
In the same manner, we can concatenate a δ-diamond-confined rooted graph G_1 with a δ-right-confined rooted graph by setting
$$G_1 \circ G_2 = G_1 \cup \left(X^R(G_2) + G_2\right).$$
Finally observe that one can concatenate two δ-diamond-confined rooted graphs by setting
$$G_1 \circ G_2 = G_1 \cup (X(G_2) + G_2).$$

These definitions in hand, we can now define the *diamond decompositions* of a subgraph of $\mathbb{Z}^2$.

Definition 4.2.3 (Diamond decomposition of a subgraph, skeleton of a subgraph)**.** Let G be a finite subgraph of $\mathbb{Z}^2$ rooted at 0. Then, to any decomposition of the type $G = G^L \circ G_1 \circ \cdots \circ G_\ell \circ G^R$ with $G^L \in \mathfrak{C}_L, G^R \in \mathfrak{C}_R, G_i \in \mathfrak{D}^\delta$ for all $i \in \{1, \ldots, \ell\}$, can be associated the concatenation of the confining δ-left cone with all the associated δ-diamonds and the confining δ-right cone. We call such a subset a *diamond decomposition* of G:

$$\mathcal{D}(G) = \mathcal{Y}^{\delta,-}_{X^L(G^L)} \circ \mathcal{D}^\delta_{X(G_1)} \circ \cdots \circ \mathcal{D}^\delta_{X(G_\ell)} \circ \mathcal{Y}^{\delta,+}_{X^R(G^R)}.$$

Let us call $x_0 = 0, x_1 = X^L(G^L), x_k = X(G_{k-1})$ for $2 \leq k \leq \ell+1$, and $x_{\ell+2} = X^R(G^R)$. We then define, for $0 \leq n \leq \ell + 2$,

$$\mathcal{S}(\mathcal{D})(G)_n = \sum_{k=0}^{n} x_k.$$

The process $\mathcal{S}(\mathcal{D})(G)_n$ is called *the skeleton* of the diamond decomposition $\mathcal{D}(G)$.

Remark 4.2.4. Observe that diamond decompositions of G are not unique: as soon as there exists one of them with $\ell \geq 3$, merging inner diamonds allows one to create new (coarser) diamond decompositions of G. However, any finite subgraph rooted at 0 admits a unique *maximal* diamond decomposition: we call it $\mathcal{D}^{\mathsf{max}}(G)$. The skeleton associated to this decomposition will by called $\mathcal{S}^{\mathsf{max}}(G)$ and referred to as the *maximal* skeleton of G.

Remark 4.2.5. Our object of interest will be the skeleton of random diamond decompositions of subgraphs of $\mathbb{Z}^2$. Amongst the properties of the skeleton associated to a diamond decomposition of some rooted subgraph G, observe that the vertices of the skeleton of a diamond decomposition of G are *cone-points* of G, in the sense that for any $n \leq \ell + 2$,

$$G \subset \mathcal{Y}^{\delta,-}_{\mathcal{S}(\mathcal{D})(G)_n} \cup \mathcal{Y}^{\delta,+}_{\mathcal{S}(\mathcal{D})(G)_n}.$$

Furthermore, observe that the skeleton of a diamond decomposition of G is always a finite directed walk, which motivates the terminology introduced in Definition 4.2.1.

Remark 4.2.6. The structure of the diamond decomposition is here given in the direction given by the first coordinate axis. However we see that adapting the definitions of the cones, the diamond decomposition can be defined for any direction $s \in \mathbb{S}^1$. The results of this work naturally adapt to this case, with this slight modification.

4.2.2 Ornstein–Zernike theory for one subcritical cluster

We are ready to state the main result of [25], which we shall refer to as the Ornstein–Zernike Theorem. Set $\mathfrak{G}_0$ to be the set of connected subgraphs of $\mathbb{Z}^2$, rooted at 0.

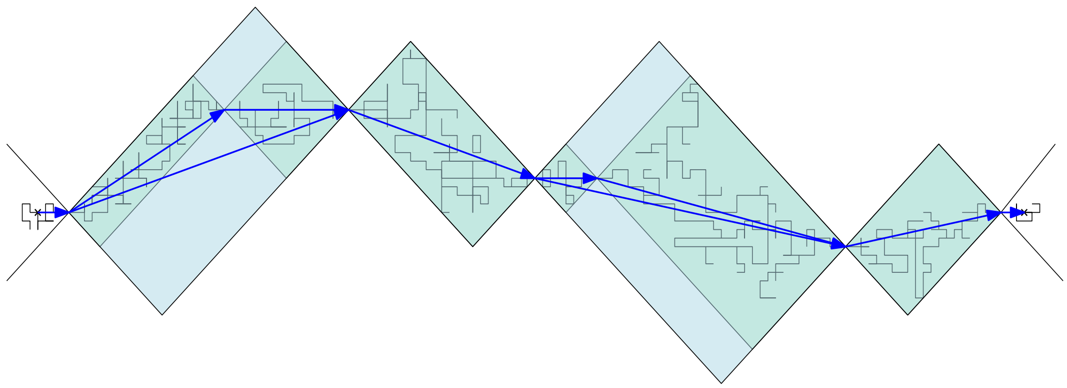

Figure 4.2: Two admissible diamond decompositions susceptible to appear in the Ornstein Zernike Theorem, together with their associated skeleton (in blue). Observe that the darker one is the *maximal* diamond decomposition of the cluster.

Theorem 4.2.7 (Ornstein–Zernike Theorem, [25])**.** *There exist two constants $C, c > 0$ and a positive $\delta > 0$, such that the following holds. There exist two positive finite measures ρ_L, ρ_R on $\mathfrak{C}_L^\delta$ and $\mathfrak{C}_R^\delta$ respectively, and a probability measure $\mathbf{P}$ on $\mathfrak{D}^\delta$ such that for any bounded function $f : \mathfrak{G}_0 \to \mathbb{R}$, any $y \in \mathcal{Y}_0^+$,*

$$\Big| \mathrm{e}^{\tau x_1} \phi\left[f(\mathcal{C}_0)\mathbb{1}_{y\in\mathcal{C}_0}\right] - \sum_{\substack{\ell\geq 0 \\ G^L\in\mathfrak{C}_L^\delta \\ G^R\in\mathfrak{C}_L^\delta \\ G_1,\dots,G_\ell\in\mathfrak{D}^\delta}} \rho_L(G^L)\rho_R(G^R)\mathbf{P}(G_1)\cdots\mathbf{P}(G_\ell)f(G)\Big| \leq C\|f\|_\infty \mathrm{e}^{-c\|y\|_2}, \tag{4.7}$$

where the sum runs over all $G^L \in \mathfrak{C}_L^\delta, G^R \in \mathfrak{C}_R^\delta, G_1, \dots, G_\ell \in \mathfrak{D}^\delta$ satisfying the relation

$$X^L(G^L) + X(G_1) + \cdots + X(G_\ell) + X^R(G^R) = y.$$

We also have written $G = G^L \circ G_1 \circ \cdots \circ G_\ell \circ G^R$ in the argument of f. Moreover, the measures $\rho_L, \rho_R, \mathbf{P}$ have exponential tails with respect to the length of the displacement: there exist $c', C' > 0$ such that

$$\max\left\{\rho_L\left[\left\|X^L(G^L)\right\|_2 > t\right], \rho_R\left[\left\|X^R(G^R)\right\|_2 > t\right], \mathbf{P}\left[\|X(G)\|_2 > t\right]\right\} < C'\mathrm{e}^{-c't}.$$

In the remainder of the paper, we **fix** δ to be equal to the value given by Theorem 4.2.7. In particular, we will not highlight the dependency anymore and we drop it from the notations.

Remark 4.2.8. For any $x \in \mathbb{N}^* \times \mathbb{Z}$, define the following three quantities:

- $\nu_L(x) = \sum_{G^L\in\mathfrak{C}_L, X^L(G^L)=x} \rho_L(G^L)$,

- $\nu_R(x) = \sum_{G^R \in \mathfrak{C}_R, X^R(G^R)=x} \rho_R(G^R)$,
- $\nu(x) = \sum_{G \in \mathfrak{D}, X(G)=x} \mathbf{P}(G)$.

Then, it is clear that ν is a directed probability measure on $\mathbb{Z}^2$, which has exponential tails. We define $\mathbb{P}^{\mathsf{RW}}$ to be the directed random walk measure associated to ν.

These definitions allow us to formulate a second version of Theorem 4.2.7 in terms of a coupling between a percolation cluster conditioned to contain a distant point and a directed random bridge.

Theorem 4.2.9 (Ornstein–Zernike Theorem; coupling version)**.** *Let $y \in \mathcal{Y}_0^+$. There exists a probability space $(\Omega, \mathcal{F}, \Phi_{0\to y})$ supporting a random variable $(\mathcal{C}_0, \mathcal{S})$ such that:*

- *$\mathcal{C}_0$ has the distribution of the cluster of 0 under the measure $\phi[\ \cdot | y \in \mathcal{C}_0]$*, i.e. *if C is a connected subgraph of $\mathbb{Z}^2$ containing 0,*

$$\Phi_{0\to y}[\mathcal{C} = C] = \phi[\mathcal{C}_0 = C | y \in \mathcal{C}_0]$$

- *$\mathcal{S}$ has the distribution of a directed random walk conditioned to hit y, ie for any $\ell \geq 1$, any family $s_1, \dots, s_\ell$ of vertices of $\mathbb{Z}^2$,*

$$\Phi_{0\to y}[\mathcal{S}_1 = s_1, \dots, \mathcal{S}_\ell = s_\ell] \propto \nu_L(s_1)\nu_R(y - s_\ell) \prod_{k=2}^{\ell} \nu(s_k - s_{k-1}),$$

  where the symbol $\propto$ means that one has to normalise the latter quantity to get a proper probability measure,

- *With probability at least $1 - C\mathrm{e}^{-c\|y\|}$, for all $1 \leq k \leq \ell$, $\mathcal{S}_k \in \mathcal{C}_0$ and $\mathcal{S}_k$ is a renewal of $\mathcal{C}_0$. Furthermore, for any $1 \leq k \leq \ell - 1$, the portion of $\mathcal{C}_0$ lying between $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$ is a δ-diamond-confined subgraph of $\mathbb{Z}^2$.*

Proof. Fix some $y \in \mathcal{Y}_0^+$. We define a probability distribution on the space

$$\mathfrak{C}_L \times \bigcup_{l=1}^{+\infty} \left(\prod_{k=1}^{\ell} \mathfrak{D} \right) \times \mathfrak{C}_R$$

by the formula:

$$\phi_y^{\mathsf{Dec}}\left[(G^L, G_1, \dots, G_\ell, G_R)\right] \propto \mathbb{1}_{X_L(G^L)+X(G_1)+\dots+X(G_\ell)+X_R(G^R)=y} \times \sum_{\substack{\ell \geq 0 \\ G^L \in \mathfrak{C}_L^\delta \\ G^R \in \mathfrak{C}_L^\delta \\ G_1,\dots,G_\ell \in \mathfrak{D}^\delta}} \rho_L(G^L)\rho_R(G^R)\mathbf{P}(G_1)\cdots\mathbf{P}(G_\ell).$$

Then, for any percolation event $\mathcal{A}$, we form the ratio of (4.7) with $f = \mathbb{1}_{\mathcal{A}}$ and (4.7) with $f = 1$. We immediately get that the total variation distance between $\phi[\ \cdot\ |y \in \mathcal{C}_0]$ and the pushforward of $\phi_y^{\mathsf{Dec}}[\ \cdot\ ]$ by the concatenation operation is bounded by $C\mathrm{e}^{-c\|x\|_2}$. It is classical that this yields the existence of a maximal coupling between those two measures, *ie* that one can construct a probability space $(\Omega, \mathcal{F}, \Phi_{0\to y})$ supporting $(\mathcal{C}_0^1, \mathcal{C}_0^2)$ such that

- The distribution of $\mathcal{C}_0^1$ is the distribution of the cluster of 0 under $\phi[\ \cdot\ |y \in \mathcal{C}_0]$,
- The distribution of $\mathcal{C}_0^2$ is the distribution of the concatenation $G^L \circ G^1 \circ \cdots \circ G^\ell \circ G^R$ where $(G^L, G^1, \ldots, G^\ell, G^R)$ are sampled according to ϕ_y^{Dec},
- $\Phi_{0\to y}(\mathcal{C}_0^1 \neq \mathcal{C}_0^2) \leq C\mathrm{e}^{-c\|y\|_2}$.

Now consider the random variable $\mathcal{S}$ formed from $(G_L, G^1, \ldots, G^\ell, G_R)$ by the following formula:

$$\mathcal{S}_1 = X_L(G_L) \ \text{ and } \ \mathcal{S}_k = \mathcal{S}_{k-1} + X(G^{k-1}) \ \text{ for } \ 2 \leq k \leq \ell.$$

Then it is immediate that

$$\Phi_{(0,y)}\left[\mathcal{S}_1 = s_1, \ldots, \mathcal{S}_\ell = s_\ell\right] \propto \nu_L(s_1)\nu_R(y - s_\ell)\prod_{k=2}^{\ell}\nu(s_k - s_{k-1}).$$

Moreover by definition, the $\mathcal{S}_k$'s are renewals of $\mathcal{C}_0^2$ and the portions of $\mathcal{C}_0^2$ lying between two consecutive $\mathcal{S}_k$'s are δ-diamond-confined. Thus, $(\Omega, \mathcal{F}, \Phi_{0\to y})$ equipped with the random variable $(\mathcal{C}_0^1, \mathcal{S})$ provides us with the desired coupling. □

For now, we shall only work in the extended probability space $(\Omega, \mathcal{F}, \Phi_{0\to y})$. Thus, each percolation configuration conditioned to contain the distant point y will be sampled together with a directed random walk bridge: we call this directed random bridge **the skeleton of** $\mathcal{C}_0$; the associated diamond decomposition will be called **the diamond decomposition of** $\mathcal{C}_0$. Observe that this enlarged probability space carries extra randomness than the space supporting ϕ: indeed, to a given percolation cluster can be associated several skeletons that are randomly chosen by the measure $\Phi_{0\to y}$(see Figure 4.2). We adopt the terminology of [25] by calling the points of $\mathcal{S}$ *renewals* of the cluster. Observe that due to the latter discussion, all the renewals of $\mathcal{C}$ are cone-points, but the converse is not necessarily true.

In the remainder of this paper, we introduce $\mathbb{P}^{\mathsf{RW}}$, the measure of the directed random walk with independent increments sampled according to ν and started from 0.

Remark 4.2.10. We are often going to be interested in observables of the *skeleton* of a cluster sampled according to $\Phi_{0\to y}$. In that case, Theorem 4.2.9 reads as follows: let f be a bounded function of the set of directed random walks. Then,

$$\left|\Phi_{0\to y}\left[f(\mathcal{S})\right]-\sum_{x_L,x_R}\nu_L(x_L)\nu_R(y-x_R)\mathbb{E}^{\mathsf{RW}}\left[f(x_L\circ S\circ x_R)|S\in\mathsf{Hit}_{x_R-x_L}\right]\right| \leq C\|f\|_\infty \mathrm{e}^{-c\|y\|_2}. \quad (4.8)$$

In the latter writing, the notation $x_R\circ S\circ x_L$ stands for the directed walk obtained by the concatenation of x_L, the trajectory of S, and x_R. In the writing $\mathbb{E}^{\mathsf{RW}}\left[f(x_L\circ S\circ x_R)|S\in\mathsf{Hit}_{x_R-x_L}\right]$, only S is random - and has law $\mathbb{P}^{\mathsf{RW}}\left[\ \cdot\ |S\in\mathsf{Hit}_{x_R-x_L}\right]$.

Finally, the unconditionnal version of (4.2.10) is the following.

$$\left|\mathrm{e}^{\tau y_1}\Phi_{0\to y}\left[f(\mathcal{S})\right]-\sum_{x_L,x_R}\nu_L(x_L)\nu_R(y-x_R)\mathbb{E}^{\mathsf{RW}}\left[f(S)\mathbb{1}_{\mathsf{Hit}_{x_R-x_L}}\right]\right|\leq C\|f\|_\infty \mathrm{e}^{-c\|y\|_2}. \quad (4.9)$$

The following lemma states when looking at certain families of observables of the trajectories of directed walks, it is sufficient to study the measure $\mathbb{P}^{\mathsf{RW}}$ started from 0 rather than the intricate second summand of the left-hand side of (4.2.10)

Lemma 4.2.11. *There exists a constant $\varsigma>0$ such that for any $y\in\mathcal{Y}_0^+$, any two sequences a_n,b_n of positive numbers going to infinity, any bounded function $f:\mathcal{C}([0,x_1],\mathbb{R})\to\mathbb{R}$ continuous with respect to the Skorokhod topology (see [14] for the definition and properties of this topology),*

$$\left|\mathrm{e}^{-\tau b_n y_1}\Phi_{0\to b_n y}\left[f\left(a_n^{-1}\mathcal{S}(\lfloor b_n t\rfloor)\right)_{t\geq 0}\right]-\varsigma\mathbb{E}^{\mathsf{RW}}\left[f\left(a_n^{-1}S(\lfloor b_n t\rfloor)\right)_{t\geq 0}\mathbb{1}_{S\in\mathsf{Hit}_{b_n y}}\right]\right| \xrightarrow[n\to\infty]{} 0.$$

We have used the interpretation of directed walks as real-valued functions explained above.

Proof. Set $\varsigma=\nu_L(\mathbb{Z}^2)\nu_R(\mathbb{Z}^2)$. By (4.8), it sufficient to prove that

$$\Big|\sum_{x_L,x_R\in\mathbb{Z}^2}\nu_L(x_L)\nu_R(x_R)\mathbb{E}^{\mathsf{RW}}_{x_L}\left[f\left(a_n^{-1}(x_L\circ S\circ x_R)(\lfloor b_n t\rfloor)\right)_{t\geq 0}\mathbb{1}_{S\in\mathsf{Hit}_{x_L-x_R}}\right] - \varsigma\mathbb{E}^{\mathsf{RW}}\left[f\left(a_n^{-1}S(\lfloor b_n t\rfloor)\right)_{t\geq 0}\mathbb{1}_{S\in\mathsf{Hit}_{b_n y}}\right]\Big| \xrightarrow[n\to\infty]{} 0.$$

The right-hand side can be dominated by

$$\sum_{x_L,x_R\in\mathbb{Z}^2}\nu_L(x_L)\nu_R(x_R)\mathbb{E}^{\mathsf{RW}}\Big[\Big|f\left(a_n^{-1}(x_L\circ(S+x_L)\circ x_R)(\lfloor b_n t\rfloor)\right)_{t\geq 0}\mathbb{1}_{S\in\mathsf{Hit}_{x_R-x_L}} - f\left(a_n^{-1}S(\lfloor b_n t\rfloor)\right)_{t\geq 0}\mathbb{1}_{S\in\mathsf{Hit}_{b_n y}}\Big|\Big].$$

Now we take advantage of the exponential tails of ν_L and ν_R by splitting the sum in two parts, the first one running over $x_L, x_R \in B(0, \log(\min(a_n, b_n)))$, and the remaining one. Thanks to the exponential tails of ν_L and ν_R, the remaining one can be bounded by $2\|f\|_\infty \min(a_n, b_n)^{-c}$, which indeed goes to 0. The first part is shown to go to 0 by noticing that when $x_L, x_R \in B(0, \log(\min(a_n, b_n)))$, the Skorokhod distance between the two considered functions goes to 0. We conclude by continuity of f and dominated convergence. □

We state two byproducts of Theorem 4.2.7:

Corollary 4.2.12. *There exists three constants $c, C, K > 0$ such that for any $y \in \mathcal{Y}_0^+$ with $\|y\|_2$ sufficiently large,*

$$\phi\left[\mathcal{C}_0 \text{ has less than } K\|y\|_2 \text{ renewal points} | y \in \mathcal{C}_0\right] < C\mathrm{e}^{-c\|y\|_2}.$$

Corollary 4.2.13. *There exists a constant $K > 0$, such that for any $y \in \mathcal{Y}_0^+$,*

$$\Phi_{0\to y}\Big[\max_{\substack{\mathcal{D}\subset\mathcal{D}(\mathcal{C}_0)\\ \mathcal{D} \text{ diamond}}} \mathit{Vol}\,(\mathcal{D}) > K(\log y_1)^2\Big] < C\,\|y\|_2^{-c\log\|y\|_2},$$

where Vol denotes the Euclidean volume, and where the max *is taken over all the diamonds appearing in the diamond decomposition of the cluster of 0 under the measure $\Phi_{(0,y)}$.*

Note that the latter bound decays faster than the inverse of any polynomial in $\|y\|_2$.

4.2.3 Ornstein–Zernike in a strip with boundary conditions

We import a few facts about Ornstein–Zernike theory that will be useful later on in our analysis. They deal with the uniformity of the Ornstein–Zernike formula in the boundary conditions and are directly imported from [24, 68]. For $y = (y_1, y_2) \in \mathcal{Y}_0^+$, let us call Strip_y the strip $\mathsf{Strip}_y = [0, y_1] \times \mathbb{Z}$. In the following proposition, the probability measure $\mathbf{P}$ is the same object as in Theorem 4.2.7.

Proposition 4.2.14 (Uniform OZ formula in a strip)**.** *Let $y = (y_1, y_2) \in \mathcal{Y}_0^+$. Let $C^L_{\mathsf{EXT}} \ni 0$ be a finite connected subset of edges of $\mathcal{Y}_0^-$ and $C^R_{\mathsf{EXT}} \ni y$ be a finite connected subset of edges of $\mathcal{Y}_y^+$. Then there exist two positive and bounded measures $\rho_L^{\mathsf{EXT}}, \rho_R^{\mathsf{EXT}}$ on $\mathcal{C}_L$ and $\mathfrak{C}_R$ respectively such that for any bounded function $f : \mathfrak{G}_0 \to \mathbb{R}$,*

$$\Big|\mathrm{e}^{\tau y_1}\phi\left[f(\mathcal{C}_0)\mathbb{1}\left(\{\mathcal{C}_0 \cap \mathsf{Strip}_y^c = C^L_{\mathsf{EXT}} \sqcup C^R_{\mathsf{EXT}}\}\right)\right] - \sum_{\substack{\ell\geq 0\\ G^L\in\mathfrak{C}_L\\ G^R\in\mathfrak{C}_L\\ G_1,\dots,G_\ell\in\mathfrak{D}}} \rho_L^{\mathsf{EXT}}(G^L)\rho_R^{\mathsf{EXT}}(G^R)\mathbf{P}(G_1)\cdots\mathbf{P}(G_\ell)f(G)\Big| \leq C\|f\|_\infty \mathrm{e}^{-c\|y\|_2}, \quad (4.10)$$

where the sum holds over all $G^L \in \mathfrak{C}_L, G^R \in \mathfrak{C}_R, G_1, \dots, G_k \in \mathfrak{D}$ satisfying the relation

$$X_0^L(G^L) + X(G_1) + \cdots + X(G_k) + X_x^R(G^R) = y,$$

and where we have written $G = G^L \circ G_1 \circ \cdots \circ G_k \circ G^R$. Moreover, the measures ρ_L^{EXT} and ρ_R^{EXT} have exponential tails, uniformly in the sets $C_{\mathsf{EXT}}^L, C_{\mathsf{EXT}}^R$ satisfying the above-stated assumptions: indeed, there exist $c', C' > 0$ such that for any $t > 0$,

$$\sup_{C_{\mathsf{EXT}}^L, C_{\mathsf{EXT}}^R} \max \left\{ \rho_L^{\mathsf{EXT}}(X(G^L) > t), \rho_R^{\mathsf{EXT}}(X(G^R) > t) \right\} < C' \mathrm{e}^{-c't}.$$

Observe that in the latter formula, the event $\{\mathcal{C}_0 \cap \mathsf{Strip}_y^c = C_{\mathsf{EXT}}^L \sqcup C_{\mathsf{EXT}}^R\}$ implies that $y \in \mathcal{C}_0$.

Remark 4.2.15. As done previously, for any $y \in \mathcal{Y}_0^+$, we shall call

$$\nu_L^{\mathsf{EXT}}(x) = \sum_{G^L \in \mathfrak{C}_L^\delta, X_0^L(G^L) = x} \rho_L^{\mathsf{EXT}}(G^L) \text{ and } \nu_R^{\mathsf{EXT}}(x) = \sum_{G^R \in \mathfrak{C}_R^\delta, X_x^R(G^R) = x} \rho_R^{\mathsf{EXT}}(G^R)$$

We simply sketch the proof of the proposition, since it is a simple byproduct of the analysis of [25]

Proof of Proposition 4.2.14. Apply the Ornstein–Zernike formula (4.7) to the function $g(\mathcal{C}_0) = f(\mathcal{C}_0) \mathbb{1}\left\{\mathcal{C}_0 \cap \mathsf{Strip}_y^c = C_{\mathsf{EXT}}^L \sqcup C_{\mathsf{EXT}}^R\right\}$. Thus, one has that ρ_L^{EXT} (resp. ρ_{EXT}^R) is the restriction of ρ_L (resp. ρ_R) to pieces of clusters compatible with C_{EXT}^L (resp. C_{EXT}^R). The announced exponential decay is a byproduct of the exponential tails of ρ_L and ρ_R. □

A non-trivial consequence of the latter proposition is the following estimate, appearing in [68, Equation (2.19)].

Proposition 4.2.16 (Ornstein–Zernike decay uniform in the boundary conditions)**.** *There exists $\chi > 0$ such that for any sets $C_{\mathsf{EXT}}^L, C_{\mathsf{EXT}}^R$ satisfying the assumptions of Proposition 4.2.14,*

$$\frac{1}{\chi} \frac{\mathrm{e}^{-\tau n}}{\sqrt{n}} \leq \phi \left[0 \overset{\mathsf{Strip}_n}{\longleftrightarrow} (n, 0) \middle| \begin{array}{l} \mathcal{C}_0 \cap (\mathbb{Z}^- \times \mathbb{Z}) = C_{\mathsf{EXT}}^L \text{ and} \\ \mathcal{C}_{(n,0)} \cap ([n, +\infty) \times \mathbb{Z}) = C_{\mathsf{EXT}}^R \end{array} \right] \leq \chi \frac{\mathrm{e}^{-\tau n}}{\sqrt{n}}. \tag{4.11}$$

4.3 Scaling limit for the product measure

For our purposes, we need to develop an analog of the Ornstein–Zernike theory for r non-intersecting clusters of FK-percolation. However, there is a supplementary difficulty, namely that these non-intersecting clusters are not independent, beyond the obvious interaction introduced by the conditioning. If we consider *a product measure*, we can readily extend the Ornstein–Zernike Theorem to r clusters sampled independently according to ϕ. This is the goal of the present section. Even though it might seem a bit strange to consider the conditioned product measure $\phi^{\otimes r}$ instead of the real conditioned random-cluster measure, we are going to see in Section 4.4 that because of the conditioning, these two measures behave similarly "in the bulk". This is a consequence of the spatial mixing property of the subcritical random-cluster measure combined with an a priori repulsion estimate.

In what follows, $\phi^{\otimes r}$ will always denote the measure consisting in the product of r random-cluster measures ϕ. Moreover, $\mathcal{C}_i$ will denote $\mathcal{C}_i(\omega_i)$. In particular, if $\mathcal{A}$ is an event measurable with respect to $(\mathcal{C}_1, \ldots, \mathcal{C}_r)$, we have:

$$\phi^{\otimes r}\left[\mathcal{A}\right] = \mathbb{P}\left[(\mathcal{C}_1(\omega_1), \ldots, \mathcal{C}_r(\omega_r)) \in \mathcal{A}\right],$$

where $\omega_1, \ldots, \omega_r$ are **independent** percolation configurations sampled according to ϕ.

The main goal of this section is the following proposition:

Proposition 4.3.1. *Recall the definition of the envelopes of a cluster $\Gamma^{\pm}(\mathcal{C})$ introduced in Def 4.1.10, and their natural parametrization. Then there exists $\sigma > 0$ such that:*

$$\frac{1}{\sqrt{n}}\left(\Gamma^{+}(\mathcal{C}_1)(nt), \ldots, \Gamma^{+}(\mathcal{C}_r)(nt)\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} \left(\sigma \mathsf{BW}_t^{(r)}\right)_{0\leq t\leq 1},$$

where the percolation configuration is sampled under the measure $\phi^{\otimes r}\left[\,\cdot\,\middle|\mathsf{Con},\mathsf{NI}\right]$, and the convergence occurs in the space $\mathcal{C}^r([0,1])$ equipped with the topology of uniform convergence. Moreover, almost surely, for any $1 \leq i \leq r$,

$$\frac{1}{\sqrt{n}}\|\Gamma_i^{+} - \Gamma_i^{-}\|_\infty \xrightarrow[n\to+\infty]{} 0.$$

The strategy of the proof is the following: we start to state an analog of the Ornstein–Zernike Theorem in the case of a product measure in Section 4.3.1, and use this coupling to compare the skeletons of a system of r non-intersecting clusters with a system of r non-intersecting directed random walks. However, there is a small difficulty while implementing this program: indeed, conditioning on the event that the *clusters* do not intersect is not the same as conditioning on the event that the *skeletons* of the clusters do not intersect. Moreover, while the latter event is very well described in terms of the Ornstein–Zernike coupling, it is not a priori clear how the former acts on the coupled walks. For that reason, we shall show first that under the conditioning on $\mathsf{NI} \cap \mathsf{Con}$, the clusters very soon get far from each other (this is the goal of Subsection 4.3.2), and thanks to this input we will be able to prove that "in the bulk" of the system, the conditioning of non-intersection for the clusters or for the skeletons of the clusters yield the same scaling limit. Thus we shall apply the invariance principle of Theorem 4.5.3 to conclude in Section 4.3.3.

4.3.1 Definition of the product measure and multidimensional version of Ornstein–Zernike Theorem

We first state a r-dimensional version of Theorem 4.2.7 (the Ornstein–Zernike formula). Indeed, if $\omega^1, \dots, \omega^r$ are r independent configurations of law ϕ and $f : \mathfrak{G}^r \to \mathbb{R}$ is a bounded function, then it is an easy consequence of Theorem 4.2.7 that when $n \in \mathbb{N}$ is sufficiently large,

$$\left| \mathrm{e}^{\tau r n} \phi^{\otimes r} \left[f(\mathcal{C}_1, \dots, \mathcal{C}_r) \mathbb{1}_{(\omega^1, \dots, \omega^r) \in \mathsf{Con}} \right] - \sum_{\substack{G_1^L \in \mathfrak{C}_L \\ \cdots \\ G_r^L \in \mathfrak{C}_L}} \sum_{\substack{G_1^R \in \mathfrak{C}_R \\ \cdots \\ G_r^R \in \mathfrak{C}_R}} \sum_{\substack{k_1 \geq 0 \\ \cdots \\ k_r \geq 0}} \sum_{G_1^1, \dots, G_1^{k_1} \in \mathfrak{D}} \cdots \sum_{G_r^1, \dots, G_r^{k_r} \in \mathfrak{D}} \right.$$
$$\left. \left(\prod_{i=1}^r \rho_L(G_i^L) \rho_R(G_i^R) \mathbf{P}(G_i^1) \cdots \mathbf{P}(G_i^{k_i}) \right) f(G_1, \dots, G_r) \right| \leq Cr \|f\|_\infty \mathrm{e}^{-cn}. \quad (4.12)$$

where we sum over all the $G_1^L, \dots, G_r^L \in \mathfrak{C}_L$, the $G_1^R, \dots, G_r^R \in \mathfrak{C}_R$, the $G_1^1, \dots, G_1^{k_1}, \dots, G_r^1, \dots, G_r^{k_r} \in \mathfrak{D}$ such that for any $1 \leq i \leq r$,

$$X_0^L(G_i^L) + X(G_1^i) + \cdots + X(G_i^{k_i}) + X_x^R(G_i^R) = n(y_i - x_i).$$

The coupling stated in Theorem 4.2.9 is available in this context: we call it $\Phi^{\otimes r}_{(0,x) \to (n,y)}$. It simply consists in the product of the r couplings $\Phi_{(0,x_j) \to (n,y_j)}$ given by Theorem 4.2.9. Its main feature is that for any bounded function of the skeletons of a system of r clusters,

$$\left| \Phi^{\otimes r}_{(0,x) \to (n,y)} \left[f\left(\mathcal{S}_1, \dots, \mathcal{S}_r \right) \right] - \right.$$
$$\left. \sum_{\substack{x_L^1, \dots, x_L^r \\ x_R^1, \dots, x_R^r}} \left(\prod_{i=1}^r \nu_L(x_L^i) \nu_R(x_R^i) \right) \left(\mathbb{E}^{\mathsf{RW}} \right)^{\otimes r} \left[f(x_L^1 \circ S^1 \circ x_R^1, \dots, x_L^r \circ S^r \circ x_R^r) \mathbb{1}_{S \in \mathsf{Hit}_{x_R - x_L}} \right] \right|$$
$$< Cr \|f\|_\infty \mathrm{e}^{-cn}, \quad (4.13)$$

where we denoted by $(\mathbb{E}^{\mathsf{RW}})^{\otimes r}$ the expectation under the measure of r independent directed walks $(S^1, \dots, S^r)$ started from 0, and $\mathsf{Hit}_{x_R - x_L}$ the event that each S^i ever hits $x_R^i - x_L^i$. Note that the uniform Ornstein–Zernike coupling introduced in Proposition 4.2.14 also holds in this context. Moreover, Lemma 4.2.11 is also true in its r-dimensional version, so that it will be sufficient to study $(\mathbb{E}^{\mathsf{RW}})^{\otimes r}$ when estimating probabilities for *scaled* random walks.

Before working on the repulsion estimates as announced, we lower bound the probability of non-intersection and connection in the product measure.

Lemma 4.3.2. *Let $x, y \in W \cap \mathbb{Z}^r$. Then, there exists $c > 0$ such that*

$$\phi^{\otimes r}\left[\mathsf{NI}, \mathsf{Con}\right] \geq cV(x)V(y)n^{-\frac{r^2}{2}}\mathrm{e}^{-\tau r n}, \tag{4.14}$$

where V is the function introduced in Theorem 4.5.5.

Proof. We use the Ornstein–Zernike coupling given by (4.3.1): indeed, up to exponential terms due to the coupling, and using the diamond confinement property,

$$\begin{aligned}
&\phi^{\otimes r}\left[\mathsf{NI}, \mathsf{Con}\right] \\
&= \mathrm{e}^{-\tau r n}\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[(\mathcal{C}_1, \ldots, \mathcal{C}_r) \in \mathsf{NI}\right] \\
&\geq \mathrm{e}^{-\tau r n}\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\bigcap_{1\leq i\neq j\leq r}\{\mathcal{D}(\mathcal{S}^i)\cap\mathcal{D}(\mathcal{S}^j) = \emptyset\}\right] \\
&= \mathrm{e}^{-\tau r n}\sum_{\substack{x_L^1,\ldots,x_L^r \\ x_R^1,\ldots,x_R^r}}\left(\prod_{i=1}^r \nu_L(x_L^i)\nu_R(x_R^i)\right) \\
&\qquad\times c\left(\mathbb{P}^{\mathsf{RW}}\right)^{\otimes r}\left[\bigcap_{1\leq i\neq j\leq r}\{\mathcal{D}\left(x_L^i\circ S^i\circ x_R^i\right)\cap\mathcal{D}\left(x_L^j\circ S^j\circ x_R^j\right) = \emptyset\}, \mathsf{Hit}_{(n,y-x)}\right].
\end{aligned}$$

Hence the result boils down to lower bound the probability of non-intersection and connection for r independent *decorated* directed random walks. This is precisely the content of Lemma 4.5.16. By finiteness of the measures ν_L, ν_R, we conclude that

$$\phi^{\otimes r}\left[\mathsf{NI}, \mathsf{Con}\right] \geq cV(x)V(y)n^{-\frac{r^2}{2}}\mathrm{e}^{-\tau r n}.$$

□

Observe that the latter bound reads as follows on the coupling measure:

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\mathsf{NI}\right] \geq cV(x)V(y)n^{-\frac{r^2}{2}}. \tag{4.15}$$

Thanks to Proposition 4.2.14, the same analysis holds for deriving the analog of the latter result in a strip with boundary conditions.

Corollary 4.3.3. *Let $x, y \in W$. Let $C^L_{i,\mathsf{EXT}} \ni x_i, C^R_{i,\mathsf{EXT}} \ni y_i$, and assume that the family $C = \left(C^L_{i,\mathsf{EXT}}, C^R_{i,\mathsf{EXT}}\right)_{1\leq i\leq r}$ satisfies the assumptions of the uniform Ornstein–Zernike coupling given by Proposition 4.2.14. Then, there exists a uniform constant $\chi > 0$ such that*

$$\phi^{\otimes r}\left[\mathsf{Con}, \mathsf{NI}\,\middle|\,\begin{matrix}\mathcal{C}_{(0,x_i)}\cap(\mathbb{Z}^-\times\mathbb{Z}) = C^L_{i,\mathsf{EXT}}, \\ \mathcal{C}_{(n,y_i)}\cap([n,+\infty]\times\mathbb{Z}) = C^R_{i,\mathsf{EXT}}, \quad \forall 1\leq i\leq r\end{matrix}\right] \geq \frac{c}{\chi}V(x)V(y)n^{-\frac{r^2}{2}}\mathrm{e}^{-\tau r n}.$$

4.3.2 Edge repulsion

The goal of this section is to prove Lemma 4.13 which we refer to as the "edge repulsion" lemma for the independent system. Beforehand we introduce the important notion of *synchronized skeleton* of a system of long clusters.

Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_r)$ be sampled according to $\Phi^{\otimes r}_{(0,x)\to(n,y)}$. We say that $k \in \mathbb{N}$ is a *synchronization time* for $\mathcal{C}$ if there exists $(s^1_k, \dots, s^r_k) \in \mathbb{Z}^r$ such that for any $1 \le i \le r$, one has $(k, s^i_k) \in \mathcal{S}^i$. In other words, k is a synchronization time for $\mathcal{C}$ if and only if each one of the r skeletons of $\mathcal{C}$ contains a point of x-coordinate k. We define the set of synchronization times of $\mathcal{C}$ by

$$\mathsf{ST}(\mathcal{C}) = \{0 \le k_1 < k_2 < \dots < k_l \le n\}.$$

The *synchronized skeleton* of $\mathcal{C}$, called $\check{\mathcal{S}}$ is now defined to be the process defined on $\mathsf{ST}(\mathcal{C})$, taking its values in $\mathbb{Z}^r$, such that for any $k \in \mathsf{ST}(\mathcal{C})$,

$$\check{\mathcal{S}}_k = (\mathcal{S}^1(k), \dots, \mathcal{S}^r(k)).$$

As previously, we extend this process as a function of $\mathbb{R}^+$ to $\mathbb{R}$ by linear interpolation. Let us observe an important property of this process (which the reason of its introduction) before turning to Lemma 4.3.7.

Claim 4.3.4. Under $\Phi^{\otimes r}_{(0,x)\to(n,y)}$,

$$\{(\mathcal{C}_1, \dots, \mathcal{C}_r) \in \mathsf{NI}\} \subseteq \left\{ \left(\check{\mathcal{S}}(\mathcal{C}_1), \dots, \check{\mathcal{S}}(\mathcal{C}_r)\right) \in \mathcal{W}_{[0,n]} \right\}. \tag{4.16}$$

Proof. This is an immediate consequence of the Intermediate Value Theorem. □

Remark 4.3.5. Observe that this inclusion would not be true when replacing $\check{\mathcal{S}}$ by $\mathcal{S}$ (see Figure 4.3).

Moreover, due to the exponential tails of the increments of $\mathcal{S}$, it is clear that the increments of $\check{\mathcal{S}}$ also have exponential tails. Thus, $\check{\mathcal{S}}$ falls into the class of *synchronized directed random walks*, studied in Section 4.5.1.

Next lemma indicates that in a time less than $n - o(n)$ the clusters have been far away from at least n^ε at least once. A convenient notion for stating this result is the gap of a point $x \in W$.

Definition 4.3.6 (Gap of a point)**.** Let $x \in W$. We define its *gap* to be the following quantity:

$$\mathsf{Gap}(x) = \min_{1 \le i \le r-1} (x_{i+1} - x_i).$$

Observe that due to the fact that x lies in W, $\mathsf{Gap}(x)$ is always a positive quantity.

We are ready to state the edge repulsion result for the independent system.

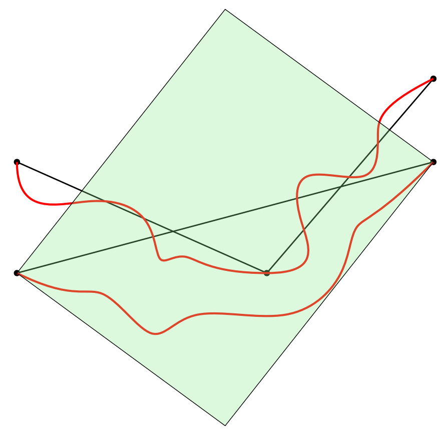

Figure 4.3: Illustration of the necessity of considering the process of synchronized renewals: here, the red clusters do not intersect while their associated skeletons do intersect.

Lemma 4.3.7 (Edge repulsion for the independent system)**.** *There exists an $\varepsilon > 0$ such that the following holds. Let T_1 and T_2 be the following random variables:*

$$T_1 = \inf\left\{k \geq 0, \mathsf{Gap}(\check{\mathcal{S}}_k) > n^{\varepsilon}\right\}.$$

and

$$T_2 = \sup\left\{k \geq 0, \mathsf{Gap}(\check{\mathcal{S}}_k) > n^{\varepsilon}\right\}$$

Then, there exist $C, c > 0$ such that when $n \geq 0$ is large enough,

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\left\{T_1 > n^{1-\varepsilon}\right\} \cup \left\{T_2 < n - n^{1-\varepsilon}\right\} | \mathsf{NI}\right] < \frac{1}{c}\exp(-cn^{\varepsilon}). \tag{4.17}$$

Proof. We prove that $\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[T_1 > n^{1-\varepsilon} | \mathsf{NI}\right] < \exp(-cn^{\varepsilon})$. By time reversal and a basic union bound, it will be sufficient to conclude. We roughly upper bound:

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[T_1 > n^{1-\varepsilon} | \mathsf{NI}\right] \leq \frac{\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[T_1 > n^{1-\varepsilon}\right]}{\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\mathsf{NI}\right]}. \tag{4.18}$$

We are going to separately bound the numerator and the denominator. We start by the numerator of (4.14). Since T_1 is measurable with respect to the synchronized skeleton of $\mathcal{C}$ (which itself is measurable with respect to $\mathcal{S}$), we use the fact that the law of $\mathcal{S}$ under $\Phi^{\otimes r}_{(0,x)\to(n,y)}$ is that of a system of directed random bridges to write - up to an exponential correction due to the coupling:

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[T_1 > n^{1-\varepsilon}\right] = \mathbb{P}^{\mathsf{RW}}_{(0,x)}\left[T_1(\check{S}) > n^{1-\varepsilon} | \mathsf{Hit}_{(n,y)}\right].$$

We then are exactly in the context of entropic repulsion for synchronized directed random bridges, and we refer to Corollary 4.5.10, which asserts that the latter probability is upper bounded by $c^{-1}\exp(-cn^{\varepsilon})$ for some constant $c > 0$.

By (4.11), we have a polynomial lower bound on the denominator. Hence, up to slightly changing the value of c, we obtained

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[T_1 > n^{1-\varepsilon}|\mathsf{NI}\right] < \frac{1}{c}\exp(-cn^{\varepsilon}),$$

which achieves the proof. □

By the synchronized renewal time property, for any $1 \leq i \leq r$, the cluster $\mathcal{C}_i$ intersects the line $\{T_1\}\times\mathbb{Z}$ (resp. $\{T_2\}\times\mathbb{Z}$) at a unique vertex of $\mathbb{Z}^2$, whose y-coordinate shall be called $\mathscr{X}_i$ (resp. $\mathscr{Y}_i$). Thus, $\mathscr{X}$ and $\mathscr{Y}$ are elements of $\mathbb{Z}^r$ satisfying $\mathscr{X}_1 < \cdots < \mathscr{X}_r$ (resp. $\mathscr{Y}_1 < \cdots < \mathscr{Y}_r$). Introduce the following *edge-regularity* condition:

Definition 4.3.8 (Edge-regularity property)**.** Let $\omega \in \mathsf{Con}\cap\mathsf{NI}$ be a percolation configuration. We call ω *edge-regular* an abbreviate this event in $\mathsf{EdgeReg}$ if it satisfies the following properties:

(i) $T_1 < n^{1-\varepsilon}$ and $T_2 > n - n^{1-\varepsilon}$,

(ii) $\|\mathscr{X}\|_2 \leq n^{1/2-\varepsilon/4}$ and $\|\mathscr{Y}\|_2 \leq n^{1/2-\varepsilon/4}$

where $\varepsilon > 0$ is given by Lemma 4.3.7.

We then prove that a percolation configuration sampled under $\Phi^{\otimes r}_{(0,x)\to(n,y)}[\,\cdot\,|\mathsf{NI}]$ is typically edge-regular.

Lemma 4.3.9. *There exists a small constant $c > 0$ such that*

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\mathsf{EdgeReg}^c|\mathsf{NI}\right] < \frac{1}{c}\exp(-cn^{\frac{\varepsilon}{2}}).$$

Proof. Let us work conditionally on the event $T_1 < 2n^{1-\varepsilon}$, as it has been proved to occur with exponentially large probability in Lemma 4.3.7. As previously, we use the rough upper bound

$$\begin{aligned}\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\|\mathscr{X}\|_2 > n^{1/2-\varepsilon/4}|\mathsf{NI}\right] &= \Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\|\mathcal{S}(T_1)\|_2 > n^{1/2-\varepsilon/4}|\mathsf{NI}\right]\\ &= \frac{\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\|\mathcal{S}(T_1)\|_2 > n^{1/2-\varepsilon/4}\right]}{\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\mathsf{NI}\right]}.\end{aligned}$$

As previously we use the lower bound (4.11) to argue that the denominator is at least polynomial, while we are going to produce a stretched-exponential upper bound on the numerator. First, observe that

$$\begin{aligned}\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\|\mathscr{X}\|_2 > n^{1/2-\varepsilon/4}\right] &= \mathbb{P}^{\mathsf{RW}}_{(0,x)}\left[\|S(T_1)\|_2 > n^{1/2-\varepsilon/4}|\mathsf{Hit}_{(n,y)}\right] \\ &\leq \frac{\mathbb{P}^{\mathsf{RW}}_{(0,x)}\left[\|S(T_1)\|_2 > n^{1/2-\varepsilon/4}\right]}{\mathbb{P}^{\mathsf{RW}}_{(0,x)}\left[\mathsf{Hit}_{(n,y)}\right]}.\end{aligned}$$

By Theorem 4.5.5, the denominator is at least polynomial. Now observe that the classical theory of large deviations for random walks allows us to produce a stretched-exponential upper bound on the numerator (remember that we work conditionally on $T_1 < n^{1-\varepsilon}$): there exists $c > 0$ such that

$$\mathbb{P}^{\mathsf{RW}}_{(0,x)}\left[\|S(T_1)\|_2 > n^{\frac{1-\varepsilon}{2}+\varepsilon/4}\right] \leq \exp(-cn^{\frac{\varepsilon}{2}}).$$

This proves, up to some small change in the constant c, that

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\|\mathscr{X}\|_2 > n^{1/2-\varepsilon/4}|T_1 \leq n^{1-\varepsilon}, \mathsf{NI}\right] \leq \exp(-cn^{\frac{\varepsilon}{2}}).$$

We conclude writing (the factor 2 comes from the terms in T_2 and $\mathscr{Y}$ that are handled by symmetry):

$$\begin{aligned}&\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\mathsf{EdgeReg}^c|\mathsf{NI}\right] \leq \\ &2\left(2\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[T_1 > n^{1-\varepsilon}|\mathsf{NI}\right] + \Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\|\mathscr{X}\|_2 > n^{1/2-\varepsilon/4}|T_1 \leq n^{1-\varepsilon}, \mathsf{NI}\right]\right).\end{aligned}$$

Thus,

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\mathsf{EdgeReg}^c|\mathsf{NI}\right] \leq \frac{1}{c}\exp(-cn^{\frac{\varepsilon}{2}}).$$

□

4.3.3 Convergence towards the Brownian watermelon

As we shall explain here, the edge repulsion stated in Lemma 4.3.7 is the main ingredient needed to show that the rescaled system, *sampled under the product measure* and conditioned both on the mutual avoidance of the clusters and on the connection event converges in distribution towards the Brownian watermelon.

The technique of proof will be used several times through the paper. Basically, it consists in splitting the system of clusters sampled under the measure $\phi^{\otimes r}[\cdot|\mathsf{Con}, \mathsf{NI}]$ in two different parts (for the sake of exposition, we explain the splitting on the first half of the cluster "near the starting point" - of course, one has to do the symmetric splitting "near the arrival point"). The first part will be given by the random time T_1 introduced in Lemma 4.3.7. At this time, the clusters are far from each other, sufficiently far for the conditioning on the non-intersection of the *clusters* to be asymptotically equivalent to the conditioning on the non-intersection of the *skeletons* of the clusters. This allows us to implement the Ornstein–Zernike coupling given by (4.3.1) for the section of the clusters which is after T_1 (taking into account the boundary conditions enforced by the configuration outside of the strip thank to Proposition 4.2.14). We conclude by applying the invariance principle for directed random walks derived in Section 4.5.3.

Due to the fact that we work between the random times T_1 and T_2 we need a technical input that allows us to extend the convergence as a process of the interval $(0,1)$ to the convergence as a process defined on $[0,1]$.

Lemma 4.3.10. *Let G_n be random sequence of functions of the space $\mathcal{C}([0,1],\mathbb{R}^r)$ and G be a continuous stochastic process of $\mathcal{C}([0,1],\mathbb{R}^r)$. Assume that:*

(i) For any $\delta > 0$, for any bounded and continuous function $f : \mathcal{C}([\delta,1-\delta],\mathbb{R}^r) \to \mathbb{R}$,

$$\mathbb{E}\left[f(G_n|_{[\delta,1-\delta]})\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[f(G|_{[\delta,1-\delta]})\right],$$

(ii) For all $\varepsilon > 0$

$$\lim_{t\to 0}\sup_{n\geq 0}\mathbb{P}\left[\left|G_n(t) - G_n(0)\right| > \varepsilon\right] = 0$$

and

$$\lim_{t\to 1}\sup_{n\geq 0}\mathbb{P}\left[\left|G_n(t) - G_n(1)\right| > \varepsilon\right] = 0$$

Then, G_n converges in distribution towards G in the space $\mathcal{C}([0,1],\mathbb{R}^r)$.

Sketch of proof of Lemma 4.3.10. The proof of Lemma 4.3.10 relies on very classical arguments and we refer to [14] for details. Observe that hypothesis (i) together with the fact that the family $[\delta, 1-\delta]$ is a compact exhaustion of $(0,1)$ yields the convergence of G_n towards G as processes from $(0,1)$ to $\mathbb{R}^r$. The equicontinuity of G_n at 0 and 1 (hypothesis (ii)) then yields the desired convergence by the Arzelà-Ascoli Theorem. □

This technical tool in hand, we can prove the main result of this section.

Proof of Proposition 4.3.1. In what follows, introduce the scaled version of $\mathcal{S}$, for any $0 \leq t \leq 1$:

$$\mathcal{S}_n(t) = \tfrac{1}{\sqrt{n}}\mathcal{S}(nt).$$

We are going to implement the strategy given by Lemma 4.3.10 to show that under the measure $\Phi^{\otimes r}_{(0,x)\to(n,y)}[\ \cdot\ |\mathsf{NI}]$, the scaled system of skeletons $\mathcal{S}_n$ converges towards the Brownian watermelon as random functions of $\mathcal{C}([0,1],\mathbb{R}^r)$.

We start with the proof of point (i) (the crucial part of the proof). Fix $\delta > 0$. Fix $f^\delta : \mathcal{C}([\delta, 1-\delta], \mathbb{R}^r) \to \mathbb{R}$, continuous and bounded. Our goal is to show that there exists $\sigma > 0$, independent of δ, such that

$$\Phi^{\otimes r}_{(0,x)\to(n,y)} \left[f^\delta(\mathcal{S}_n|_{[\delta,1-\delta]})|\mathsf{NI}\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[f^\delta(\sigma \mathsf{BW}^r|_{[\delta,1-\delta]})\right]. \tag{4.19}$$

For sake of notational simplicity, the restrictions of the functions $\mathcal{S}_n$ and $\mathsf{BW}^{(r)}$ to the interval $[\delta, 1-\delta]$ will not be made explicit anymore.

We first observe that, by Lemma 4.3.9, and using the fact that f^δ is bounded,

$$\Phi^{\otimes r}_{(0,x)\to(n,y)} \left[f^\delta(\mathcal{S}_n)|\mathsf{NI}\right] = (1+o(1))\Phi^{\otimes r}_{(0,x)\to(n,y)} \left[f^\delta(\mathcal{S}_n)|\mathsf{NI}, \mathsf{EdgeReg}\right].$$

Hence, it is sufficient to establish the convergence (4.15) for the measure conditioned on the configuration to be edge-regular. We recall that under this conditioning, there exist T_1 and T_2 such that:

(i) $T_1 < n^{1-\varepsilon}, T_2 > n - n^{1-\varepsilon}$ and T_1 and T_2 are synchronized renewal times of $\mathcal{C}$.

(ii) $\mathsf{Gap}(\mathscr{X}), \mathsf{Gap}(\mathscr{Y}) > n^\varepsilon$.

(iii) $\|\mathscr{X}\|_2, \|\mathscr{Y}\|_2 < n^{1/2-\varepsilon/4}$.

We chose n large enough so that $n\delta > T_1$ and $n(1-\delta) < T_2$[1].

We call $\mathsf{Strip} := [T_1, T_2] \times \mathbb{Z}$. We are going to use an exploration argument, by conditioning on the portion of the clusters $\mathcal{C}$ that lies outside of Strip. To that end, for an edge-regular percolation configuration $\omega \in \mathsf{NI} \cap \mathsf{Con}$, introduce the following sets of vertices:

$$\mathsf{EXT}_i = (\mathcal{C}_i \cup \partial_{\mathsf{ext}}\mathcal{C}_i) \cap \mathsf{Strip}^c \qquad \text{and} \qquad \mathsf{EXT} = \bigcup_i \mathsf{EXT}_i.$$

Now observe that summing over all the possible exterior edge-regular configurations yields

$$\begin{aligned}\Phi^{\otimes r}_{(0,x)\to(n,y)} &\left[f^\delta(\mathcal{S}_n)|\mathsf{NI}, \mathsf{EdgeReg}\right] = \\ &\sum_{\mathsf{Ext}} \Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})} \left[f^\delta(\mathcal{S}_n)|\mathsf{NI}, \mathsf{EXT} = \mathsf{Ext}\right] \\ &\qquad \times \Phi^{\otimes r}_{(0,x)\to(n,y)} \left[\mathsf{EXT} = \mathsf{Ext}|\mathsf{NI}, \mathsf{EdgeReg}\right].\end{aligned}$$

We would like to conclude using Theorem 4.5.3. For that, we need to understand how the measure changes when switching the conditioning from NI to $\mathsf{NonIntDiam}(\mathcal{S})$ (this is

[1]This is the only place where we use the fact that our functions are defined on $[\delta, 1-\delta]$ and it is the reason why we follow the strategy given by Lemma 4.3.10 instead of directly working on $[0, 1]$.

the event appearing in the statement of Theorem 4.5.3 that the *decorated* skeletons do not intersect).

Fix such an admissible edge-regular Ext. We use the following important input from Section 4.5.3. By edge-regularity of Ext, usual properties of the coupling and Lemma 4.5.18 ensure that

$$\Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}\left[\inf_{t\in[T_1,T_2]}\mathsf{Gap}(\mathcal{S}(t))\leq(\log n)^3|\mathsf{NonIntDiam}(\mathcal{S}),\mathsf{EXT}=\mathsf{Ext}\right]\xrightarrow[n\to\infty]{}0.$$

Now under the complementary event, the diamond confinement property given by the Ornstein–Zernike coupling ensures that (Δ here means the symmetric difference) the event $\mathsf{NonIntDiam}(\mathcal{S})\Delta\{\mathcal{C}\in\mathsf{NI}\}$ can occur only if one of the diamonds appearing in the diamonds decompositions of the $\mathcal{C}_i$ has a volume larger than $(\log^3 n)^2$. This event has been shown in Corollary 4.2.13 to occur with probability going to 0. Thus, we proved that:

$$\Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}\left[\mathsf{NonIntDiam}(\mathcal{S})\Delta\left\{\mathcal{C}\in\mathsf{NI}\right\}|\mathsf{EXT}=\mathsf{Ext}\right]\xrightarrow[n\to\infty]{}0.$$

It is an easy consequence that:

$$\Bigg|\Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}\left[f^\delta(\mathcal{S}_n)|\mathsf{NI},\mathsf{EXT}=\mathsf{Ext}\right]-\\ \Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}\left[f^\delta(\mathcal{S}_n)|\mathsf{NonIntDiam}(\mathcal{S}),\mathsf{EXT}=\mathsf{Ext}\right]\Bigg|\xrightarrow[n\to\infty]{}0.$$

The right-hand summand of the latter formula is measurable with respect to $\mathcal{S}$, except the conditioning on $\mathsf{EXT}=\mathsf{Ext}$. Thanks to the uniform Ornstein–Zernike formula stated in Proposition 4.2.16 and Lemma 4.2.11 to get rid of the boundary conditions, we obtain that - uniformly on Ext being edge-regular:

$$\Bigg|\Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}\left[f^\delta(\mathcal{S}_n)|\mathsf{NonIntDiam}(\mathcal{S}),\mathsf{EXT}=\mathsf{Ext}\right]-\\ \left(\mathbb{P}^{\mathsf{RW}}_{(T_1,\mathscr{X})}\right)^{\otimes r}\left[f^\delta(S)\;\middle|\mathsf{NonIntDiam}(S),\mathsf{Hit}_{(T_2,\mathscr{Y})}\right]\Bigg|\xrightarrow[n\to\infty]{}0$$

The main input of Section 4.5, namely Theorem 4.5.3 then allows us to conclude that

$$\Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}\left[f^\delta(\mathcal{S}_n)|\mathsf{NonIntDiam}(\mathcal{S}),\mathsf{EXT}=\mathsf{Ext}\right]\xrightarrow[n\to\infty]{}\mathbb{E}\left[f^\delta(\sigma\mathsf{BW}^{(r)})\right],$$

for some $\sigma>0$ that depends on the distribution $\mathbf{P}$ in (4.2.14), but of course not on δ. This concludes the proof of point (i) of Lemma 4.3.10.

The point (ii) - the equicontinuity of the family $\mathcal{S}_n$ at 0 and 1 - is an easy consequence of classical large deviations estimates combined with the arguments above.

By Lemma 4.3.10, we thus proved that $\mathcal{S}_n$ converges in distribution towards $\mathsf{BW}^{(r)}$ when sampled under the distribution $\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[f^\delta(\mathcal{S}_n)|\mathcal{S}\in\mathcal{W}_{[T_1,T_2]},\mathsf{EXT}=\mathsf{Ext}\right]$. Now, observe that the diamond confinement property and the volume estimate stated in Lemma 4.2.13 yield that

$$\Phi^{\otimes r}_{(0,x)\to(n,y)}\left[\sup_{0\le t\le n}\left|\Gamma^{\pm}(t)-\mathcal{S}(t)\right|>\log^3 n\right]\le\exp(-c\log^2 n).$$

This concludes the proof, by the usual observation that this decay is faster than any polynomial. ☐

4.4 Brownian watermelon asymptotics for the random-cluster measure

Now that the convergence of the rescaled clusters towards the Brownian watermelon is established in the case of the product measure, the goal of the following section is to transfer this convergence to the rescaled clusters sampled under the "real" random-cluster measure, and thus to achieve our journey towards Theorems 4.1.6 and 4.1.11. The strategy looks similar to the precedent section: indeed, we shall first prove an edge repulsion lemma in Section 4.4.1. Then we shall prove in Section 4.4.2 that the clusters remain far away from each other in the bulk. This will finally allow us to conclude that in the bulk, the conditioned random-cluster measure is close to the conditioned product measure thanks to a mixing argument, and to import the results of the precedent section to conclude the proofs in Section 4.4.3. The main difficulty and the reason why we needed to introduce and study the measure $\phi^{\otimes r}$ is that a coupling such as $\Phi^{\otimes r}_{x\to y}$ is not available in this setting. Hence, the random diamond decomposition and its associated skeleton given by the coupling $\Phi^{\otimes r}_{x\to y}$ do not exist anymore. We then work with the *maximal* diamond decomposition and *maximal* skeletons of the clusters (see remark 4.2.4). We draw the attention of the reader on the fact that this maximal skeleton *does not* behave like a process with independent increments as it was the case in Section 4.3.

Introduce the following notation: if $\mathcal{E}$ is a set of edges of $\mathbb{Z}^2$ and η,ω are two percolation configurations, we set

$$\{\omega\overset{\mathcal{E}}{=}\eta\}=\{\forall e\in\mathcal{E},\omega(e)=\eta(e)\}.$$

The first easy comparison between the infinite volume and the product measures is given by the following lemma:

Lemma 4.4.1. *Let $\mathcal{E}$ be an arbitrary subset of $E(\mathbb{Z}^2)$ and η an arbitrary percolation configuration on $\mathbb{Z}^2$. Then,*

$$\phi\left[\mathsf{Con},\mathsf{NI}|\omega\overset{\mathcal{E}}{=}\eta\right]\ge\phi^{\otimes r}\left[\mathsf{Con},\mathsf{NI}|\omega_1\overset{\mathcal{E}}{=}\dots\overset{\mathcal{E}}{=}\omega_r\overset{\mathcal{E}}{=}\eta\right].\tag{4.20}$$

Proof. It is a simple consequence of the FKG inequality applied to $\phi[\cdot|\omega \overset{\mathcal{E}}{=} \eta]$, which is a random-cluster measure on the graph $\mathbb{Z}^2 \setminus \mathcal{E}$ with some boundary conditions imposed by the configuration η. If C is a connected set of edges of $\mathbb{Z}^2$, introduce its edge exterior boundary by:

$$\partial_{\text{ext}} C = \big\{\{x, y\} \in E(\mathbb{Z}^2) \cap C^c, x \text{ is the endpoint of an edge of } C\big\}.$$

Then, we write, summing over all the potential realizations of $\mathcal{C}_1, \ldots, \mathcal{C}_r$ such that $\mathsf{Con} \cap \mathsf{NI}$ occurs:

$$\begin{aligned}
\phi\big[\mathsf{NI}, \mathsf{Con}|\omega \overset{\mathcal{E}}{=} \eta\big] &= \sum_{C_1,\ldots,C_r} \phi\big[\bigcap_{i=1}^{r}\{\mathcal{C}_i = C_i\}|\omega \overset{\mathcal{E}}{=} \eta\big] \\
&= \sum_{C_1,\ldots,C_r} \phi\big[\bigcap_{i=1}^{r}\{C_i \text{ is open }, \partial_{\text{ext}} C_i \text{ is closed}\}|\omega \overset{\mathcal{E}}{=} \eta\big] \\
&= \sum_{C_1,\ldots,C_r} \phi\big[\bigcap_{i=1}^{r}\{\partial_{\text{ext}} C_i \text{ is closed}\}|\omega \overset{\mathcal{E}}{=} \eta\big] \prod_{i=1}^{r} \phi^0_{C_i}\big[C_i \text{ is open } | \ \omega \overset{\mathcal{E}\cap C_i}{=} \eta\big] \\
&\geq \sum_{C_1,\ldots,C_r} \prod_{i=1}^{r} \phi\big[\partial_{\text{ext}} C_i \text{ is closed } |\omega \overset{\mathcal{E}}{=} \eta\big] \phi_{C_i}\big[C_i \text{ is open } |\omega \overset{\mathcal{E}\cap C_i}{=} \eta\big] \\
&= \sum_{C_1,\ldots,C_r} \prod_{i=1}^{r} \phi\big[\mathcal{C}_i = C_i|\omega \overset{\mathcal{E}}{=} \eta\big] \\
&= \phi^{\otimes r}\big[\mathsf{NI}, \mathsf{Con}|\omega \overset{\mathcal{E}}{=} \eta\big],
\end{aligned}$$

where the inequality comes from the positive association property (FKG) of the measure $\phi\big[\,\cdot\,|\omega \overset{\mathcal{E}}{=} \eta\big]$.

□

Remark 4.4.2. The lemma above together with Lemma 4.3.2 immediately yields that for any $x, y \in W \cap \mathbb{Z}^r$,

$$\phi\,[\mathsf{NI}, \mathsf{Con}] \geq c V(x) V(y) n^{-\frac{r^2}{2}} \mathrm{e}^{-\tau r n}. \tag{4.21}$$

This will be of particular interest later - and is the first half of the proof of Theorem 4.1.6.

While the latter bound is optimal (up to a constant), we also import a rough non-optimal upper bound.

Lemma 4.4.3. *Let $x, y \in W \cap \mathbb{Z}^2$. Then,*

$$\phi\,[\mathsf{Con}, \mathsf{NI}] \leq \mathrm{e}^{-\tau r n}.$$

Before turning to the proof of Lemma 4.4.3, we introduce a useful notation for the rest of the paper. When $x, y \in W \cap \mathbb{Z}^r$, if $1 \leq i \leq r$ we write $(\mathsf{Con}, \mathsf{NI})_{\neq i}$ for the event that $\mathcal{C}_1, \ldots, \mathcal{C}_{i-1}, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_r$ realize the connection event and are non-intersecting. Observe that whenever $1 \leq i \leq r$, $(\mathsf{Con}, \mathsf{NI}) \subset (\mathsf{Con}, \mathsf{NI})_{\neq i}$, while the opposite inclusion is obviously not true.

Proof of Lemma 4.4.3. We proceed by induction on r. For $r = 1$, the statement to prove is

$$\phi\left[(0, x) \leftrightarrow (n, y)\right] \leq \mathrm{e}^{-\tau r n},$$

which is the consequence of a well-known subbaditivity argument.

If the statement is established with r clusters, let $x, y \in \mathbb{Z}^{r+1}$. Then, observe that if $\mathsf{Con}, \mathsf{NI}$ occurs, then $(\mathsf{Con}, \mathsf{NI})_{\neq r+1}$ has to occur. Summing over all the potential realizations of $\mathcal{C}_1, \ldots, \mathcal{C}_r$ under $\mathsf{Con}, \mathsf{NI}$, we get:

$$\begin{aligned}
\phi\left[\mathsf{Con}, \mathsf{NI}\right] &= \sum_{C_1, \ldots, C_r} \phi\Big[(0, x_{r+1}) \overset{(C_1 \sqcup \cdots \sqcup C_r)^c}{\longleftrightarrow} (n, y_{r+1}) | \mathcal{C}_i = C_i, \forall 1 \leq i \leq r\Big] \\
&\qquad\qquad\qquad\qquad \times \phi\left[\mathcal{C}_i = C_i, \forall 1 \leq i \leq r\right] \\
&= \sum_{C_1, \ldots, C_r} \phi^0_{(C_1 \sqcup \cdots \sqcup C_r)^c}\left[(0, x_{r+1}) \leftrightarrow (n, y_{r+1})\right] \phi\left[\mathcal{C}_i = C_i, \forall 1 \leq i \leq r\right] \\
&\leq \phi\left[(0, x_{r+1}) \leftrightarrow (n, y_{r+1})\right] \phi\left[(\mathsf{NI}, \mathsf{Con})_{\neq r+1}\right] \\
&\leq \mathrm{e}^{-\tau n} \phi\left[(\mathsf{NI}, \mathsf{Con})_{\neq r+1}\right],
\end{aligned}$$

where we used (SMP) in the second line, (CBC) in the third line, and the case $r = 1$ in the last line. The statement follows by the induction hypothesis. □

We next state another consequence of these two bounds, observing that they allow us to derive a diamond confinement property for the infinite volume measure conditioned on $\mathsf{Con} \cap \mathsf{NI}$, the exact analog of Corollary 4.2.13 for the conditioned measure. We formulate it for a rather particular class of boundary conditions, in order to be able to apply it later: however the reader should think about the measure ϕ on the strip Strip_n with boundary conditions given by the trace of a subcritical cluster outside of the strip. We recall that $\mathcal{D}^{\mathsf{max}}(\mathcal{C})$ denotes the *maximal* diamond decomposition of the cluster $\mathcal{C}$, and introduce the following events:

$$\mathsf{BigDiam}_i = \{\max_{\substack{\mathcal{D} \subset \mathcal{D}^{\mathsf{max}}(\mathcal{C}_i) \\ \mathcal{D} \text{ diamond}}} \mathrm{Vol}(\mathcal{D}) \geq \log^2 n\} \quad \text{and} \quad \mathsf{BigDiam} = \bigcup_{i=1}^{r} \mathsf{BigDiam}_i.$$

Lemma 4.4.4 (Diamond confinement)**.** *There exists a constant $c > 0$ such that the following occurs. Let Ext be a finite set of edges such that $\mathsf{Ext} \cap E(\mathsf{Strip}_n) = \emptyset$. Then for any n large enough,*

$$\phi^0_{\mathsf{Ext}^c}\Big[\mathsf{BigDiam} \;\Big|\mathsf{Con}, \mathsf{NI}\Big] \leq \exp\Big(-c\left(\log n\right)^2\Big).$$

Proof. We write

$$\phi^0_{\mathsf{Ext}^c}\Big[\mathsf{BigDiam}\,\Big|\mathsf{Con},\mathsf{NI}\Big] \le \sum_{i=1}^{r} \phi^0_{\mathsf{Ext}^c}\Big[\mathsf{BigDiam}_i\,\Big|\mathsf{Con},\mathsf{NI}\Big].$$

We fix an $i \in \{1,\ldots,r\}$. Now we shall focus on the numerator of the latter probability, namely on estimating the quantity $\phi^0_{\mathsf{Ext}^c}\Big[\mathsf{BigDiam}_i,\mathsf{Con},\mathsf{NI}\Big]$. Summing over all the potential clusters $C_1,\ldots,C_{i-1},C_{i+1},\ldots,C_r$ under Con,NI,

$$\begin{aligned}\phi^0_{\mathsf{Ext}^c}&[\mathsf{BigDiam}_i,\mathsf{Con},\mathsf{NI}]\\ &= \sum_{C_1,\ldots,C_{i-1},C_{i+1},\ldots C_r} \phi^0_{\mathsf{Ext}^c}[\mathsf{BigDiam}_i,\mathsf{Con},\mathsf{NI}|\mathcal{C}_j=C_j]\phi^0_{\mathsf{Ext}^c}[\mathcal{C}_j=C_j],\end{aligned} \tag{4.22}$$

where the event in the conditioning is shorthand for $\mathcal{C}_j = C_j$ for all $1 \le j \ne i \le r$. Fix such a system of clusters $C_1,\ldots,C_{i-1},C_{i+1},\ldots,C_r$, and call by convenience $\widetilde{\mathsf{Ext}} = \mathsf{Ext} \cup \left(\bigcup_{1\le j\ne i\le r} C_j \cup \partial_{\mathsf{ext}} C_j\right)$. We then observe that - thanks to (SMP),

$$\phi^0_{\mathsf{Ext}^c}\Big[\mathsf{BigDiam}_i,\mathsf{Con},\mathsf{NI}\,\Big|\mathcal{C}_j=C_j\Big] = \phi^0_{\widetilde{\mathsf{Ext}}^c}\Big[\mathsf{BigDiam}_i,(n,y_i)\in\mathcal{C}_i\Big].$$

Moreover, since the events $\{(n,y_i)\in\mathcal{C}_i\}$ and $\mathsf{BigDiam}_i$ are increasing, we obtain:

$$\phi^0_{\widetilde{\mathsf{Ext}}^c}\Big[\mathsf{BigDiam}_i,(n,y_i)\in\mathcal{C}_i\Big] \le \phi\Big[\mathsf{BigDiam}_i,(n,y_i)\in\mathcal{C}_i\Big]$$

But we are now in the setting of Corollary 4.2.13, which ensures that

$$\phi\Big[\mathsf{BigDiam}_i,(n,y_i)\in\mathcal{C}_i\Big] \le \mathrm{e}^{-\tau n} n^{-c\log n}.$$

Indeed, the diamonds appearing in the *maximal* diamond decomposition are always contained in the ones appearing in the diamond decomposition given by the Ornstein–Zernike coupling. Coming back to (4.4), we proved that

$$\phi^0_{\mathsf{Ext}^c}\Big[\mathsf{BigDiam}_i,\mathsf{Con},\mathsf{NI}\Big] \le \mathrm{e}^{-\tau n - c\log^2 n}\phi^0_{\mathsf{Ext}^c}\left[(\mathsf{Con},\mathsf{NI})_{\ne i}\right]$$

Using the rough upper bound given by Lemma 4.4.3, we obtain:

$$\phi^0_{\mathsf{Ext}}\Big[\mathsf{BigDiam}_i,\mathsf{Con},\mathsf{NI}\Big] \le \mathrm{e}^{-(\tau r n + c\log^2 n)}.$$

Now, thanks to Remark 4.4.2 and the uniform Ornstein–Zernike decay (4.9), we bound the denominator:

$$\phi^0_{\mathsf{Ext}^c}\left[\mathsf{Con},\mathsf{NI}\right] \ge c n^{-\frac{r^2}{2}} \mathrm{e}^{-\tau r n}.$$

Combining the two bounds above, we find

$$\phi^0_{\mathsf{Ext}^c}\Big[\mathsf{BigDiam}_i\,\Big|\mathsf{Con},\mathsf{NI}\Big] \le \tfrac{1}{c} n^{\frac{r^2}{2}} \exp(-c(\log n)^2).$$

Applying the union bound yields the result for an amended value of c. □

4.4.1 Edge repulsion

The goal of this section is to prove the equivalent of Lemma 4.3.7 for the measure $\phi[\,\cdot\,|\mathsf{Con},\mathsf{NI}]$. We need an alternative definition of the random times T_1 and T_2, since $\mathcal{S}$ is not available anymore. Recall the definition of the upper and lower interfaces of a cluster $\Gamma^{\pm}(t)$.

Definition 4.4.5. Fix $\varepsilon > 0$. We define the two following random variables.

$$T_1' = \min\Big\{t \geq 0, \min_{*,\star\in\pm}\min_{1\leq i<j\leq r}\big|\Gamma_i^\star(t) - \Gamma_j^*(t)\big| > n^\varepsilon\Big\} \quad \text{and}$$
$$T_2' = \max\Big\{t \geq 0, \min_{*,\star\in\pm}\min_{1\leq i<j\leq r}\big|\Gamma_i^\star(t) - \Gamma_j^*(t)\big| > n^\varepsilon\Big\}.$$

The analogous of Lemma 4.3.7 is the following:

Lemma 4.4.6 (Edge repulsion)**.** *There exists $\varepsilon > 0$ and $c > 0$ such that for any n large enough,*

$$\phi\left[\left\{T_1' > n^{1-\varepsilon}\right\} \cup \left\{T_2' < n - n^{1-\varepsilon}\right\}|\mathsf{Con},\mathsf{NI}\right] < 2\exp\left(-cn^{1-3\varepsilon}\right). \qquad (4.23)$$

The value of $\varepsilon > 0$ given by Lemma 4.4.6 will be fixed in the rest of the paper.

Proof. Let $\varepsilon > 0$, its value will be determined at the end of the proof. By symmetry, we focus on proving the following bound

$$\phi\left[T_1' > n^{1-\varepsilon}|\mathsf{Con},\mathsf{NI}\right] \leq \exp(-cn^{1-3\varepsilon}).$$

As in the proof of Lemma 4.3.7, we will conclude by time reversal and an easy union bound. For $2 \leq i \leq r$, we define the event MLCP_i (meaning "many left-close points") by

$$\mathsf{MLCP}_i = \{\#\{k \in \{0,\dots,n^{1-\varepsilon}\}, |\Gamma_i^-(k) - \Gamma_{i-1}^+(k)| < n^\varepsilon\} \geq \frac{1}{r}n^{1-\varepsilon}\}.$$

The reason for the introduction of this event is the following inclusion (that is a simple consequence of the pigeonhole principle):

$$\{T_1' > n^{1-\varepsilon}\} \subset \bigcup_{i=2}^r \mathsf{MLCP}_i.$$

Thus, by union bound

$$\phi\left[T_1' > n^{1-\varepsilon}|\mathsf{Con},\mathsf{NI}\right] \leq \sum_{i=2}^r \phi\left[\mathsf{MLCP}_i|\mathsf{Con},\mathsf{NI}\right].$$

Fix some $i \in \{2, \ldots, r\}$. We upper bound $\phi\,[\mathsf{MLCP}_i|\mathsf{Con},\mathsf{NI}]$ by separately bounding the numerator and the denominator of this fraction. We start with the numerator, and we write, conditioning over all the possible clusters $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_r$ under Con,NI:

$$\phi\,[\mathsf{MLCP}_i, \mathsf{Con}, \mathsf{NI}] = \sum_{C_1,\ldots,C_{i-1},C_{i+1},\ldots,C_r} \phi\,[\mathsf{MLCP}_i, (n, y_i) \in \mathcal{C}_i | \mathcal{C}_j = C_j]\, \phi\,[\mathcal{C}_j = C_j]\,, \tag{4.24}$$

where the event in the conditioning is shorthand for $\mathcal{C}_j = C_j$ for all $1 \le j \neq i \le r$. Let us fix $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_r$ that can appear in the sum (4.19). As in the precedent proof, we define the following set of edges

$$\widetilde{\mathsf{Ext}} = \bigcup_{1 \le j \neq i \le r} (C_j \cup \partial_{\mathsf{ext}} C_j).$$

Following the previous computation and using the spatial Markov property (SMP), we observe that

$$\phi\,[\mathsf{MLCP}_i, (n, y_i) \in \mathcal{C}_i | \mathcal{C}_j = C_j] = \phi^0_{\widetilde{\mathsf{Ext}}^c}\,[\mathsf{MLCP}_i, (n, y_i) \in \mathcal{C}_i]\,,$$

where

$$\phi^0_{\widetilde{\mathsf{Ext}}^c}\,[\mathsf{MLCP}_i, (n, y_i) \in \mathcal{C}_i] =$$
$$\phi^0_{\widetilde{\mathsf{Ext}}^c}\left[(n, y_i) \in \mathcal{C}_i, \#\{k \in \{0, \ldots, n^{1-\varepsilon}\}, |\Gamma_i^-(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon\} \ge \tfrac{1}{r} n^{1-\varepsilon}\right].$$

The event appearing on the right-hand side of the latter equation is increasing (the connection event is always increasing, and adding edges to the configuration can only push Γ_i^- down, rendering it closer to $\Gamma^+(C_{i-1})$). Thus, by (CBC), we obtain:

$$\phi\,[\mathsf{MLCP}_i, (n, y_i) \in \mathcal{C}_i | \mathcal{C}_j = C_j] \le$$
$$\phi\left[(n, y_i) \in \mathcal{C}_i, \#\{k \in \{0, \ldots, n^{1-\varepsilon}\}, |\Gamma_i^-(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon\} \ge \frac{1}{r} n^{1-\varepsilon}\right]. \tag{4.25}$$

We are now in the framework of the classical one-cluster Ornstein–Zernike theory - and we are going to conclude using Lemma 4.5.20. Indeed, observe that if $k \in \{0, \ldots, n^{1-\varepsilon}\}$ satisfies $|\Gamma_i^-(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon$, then one can invoke Lemma 4.4.4 applied to both $\mathcal{C}_i$ and to C_{i-1} to argue that with very large probability, there exists some time t_k (measurable with respect to $\mathcal{C}_i$: one can choose the next renewal after k) such that t_k is a renewal time for $\mathcal{C}_i$ and $|\mathcal{S}(t_k) - \Gamma^+(C_{i-1})(t_k)| < 2n^\varepsilon$. The map $k \mapsto t_k$ is not one-to-one but still due to Lemma 4.4.4 is can be at most $\log^2 n$-to-1 with high probability. Thus we get that for some $\alpha < 1$,

$$\phi\left[(n, y_i) \in \mathcal{C}_i, \#\{k \in \{0, \ldots, n^{1-\varepsilon}\}, |\Gamma_i^-(k) - \Gamma^+(C_{i-1})(k)| < n^\varepsilon\} \ge \frac{1}{r} n^{1-\varepsilon}\right]$$
$$\le \mathrm{e}^{-\tau n} \Phi_{(0,x_i)\leftrightarrow(n,y_i)}\left[\#\{0 \le k \le n^{1-\varepsilon}, |\mathcal{S}^i(k) - \Gamma^+(C_{i-1})(k)| < 2n^\varepsilon\} \ge \frac{\alpha}{r} n^{1-\varepsilon}\right],$$

In words, the trajectory of $\mathcal{S}$ stays confined during a large amount of time close to the function $\Gamma^+(C_{i-1})$. Since $\Gamma^+(C_{i-1})$ is the graph of a fixed function we can apply Lemma 4.5.20 [2] to argue that

$$\Phi_{(0,x_i)\leftrightarrow(n,y_i)}\left[\#\{k\in\{0,\dots,n^{1-\varepsilon}\},|\mathcal{S}^i(k)-\Gamma^+(C_{i-1})(k)|<n^\varepsilon\}\geq\frac{1}{r}n^{1-\varepsilon}\right] \leq \exp(-cn^{1-3\varepsilon}),$$

provided that $\varepsilon>0$ is small enough. Coming back to (4.19), we proved that

$$\phi\left[\mathsf{MLCP^i(r)},\mathsf{Con},\mathsf{NI}\right]\leq\exp(-cn^{1-3\varepsilon}-\tau n)\phi\left[(\mathsf{Con},\mathsf{NI})_{\neq i}\right].$$

Finally using the rough bound given by Lemma 4.4.3 we proved that

$$\phi\left[T_1'>n^{1-\varepsilon},\mathsf{Con},\mathsf{NI}\right]\leq\exp(-(\tau rn+cn^{1-3\varepsilon})).$$

We conclude by using the lower bound on $\phi\left[\mathsf{Con},\mathsf{NI}\right]$ given by (4.17). We obtain

$$\phi\left[T_1'>n^{1-\varepsilon}|\mathsf{Con},\mathsf{NI}\right]\leq n^{\frac{r^2}{2}}\exp(-cn^{1-3\varepsilon}),$$

which yields the result, up to slightly changing the value of the constant c.

□

Observe that, by their definition, T_1' and T_2' are not necessarily synchronization times. We chose to define them as such so as to obtain an increasing event in (4.4.1); otherwise the proof of Lemma 4.4.6 would be more complicated. Thus let us define the actual random variables T_1 and T_2 by the following:

$$T_1=\min\{k\geq T_1',k\text{ is a synchronization time for }\mathcal{S}^{\mathsf{max}}\}\quad\text{and}$$
$$T_2=\max\{k\leq T_2',k\text{ is a synchronization time for }\mathcal{S}^{\mathsf{max}}\}.$$

Similarly to Section 4.3.2, Lemma 4.4.6 will be used to produce edge-regular configurations with large probability. We recall and modify slightly the notion of edge-regular configurations: a percolation configuration $\omega\in\mathsf{Con},\mathsf{NI}$ is called *edge-regular* (also written $\omega\in\mathsf{EdgeReg}$) if the following set of conditions is satisfied:

(i) $T_1<n^{1-\varepsilon}$ and $T_2>n-n^{1-\varepsilon}$.

(ii) $\|\mathscr{X}\|_2\leq n^{1/2-\varepsilon/4}$ and $\|\mathscr{Y}\|_2\leq n^{1/2-\varepsilon/4}$.

(iii) $\mathsf{Gap}(\mathscr{X})>\frac{1}{2}n^\varepsilon$ and $\mathsf{Gap}(\mathscr{Y})>\frac{1}{2}n^\varepsilon$.

[2]Lemma 4.5.20 is stated for unconditioned random walks, but as usual due to the Local Limit Theorem, being a bridge has a polynomial probability which is always beaten by the quantity $\exp(-cn^{1-3\varepsilon})$

Then typical configurations are edge-regular under the conditioning on $\mathsf{Con}, \mathsf{NI}$.

Lemma 4.4.7. *There exists a constant $c > 0$ such that*

$$\phi\left[\mathsf{EdgeReg}^c | \mathsf{Con}, \mathsf{NI}\right] \leq \frac{1}{c}\exp(-cn^{\varepsilon/2}).$$

We only briefly sketch the proof since it is very similar to the one of Lemma 4.3.9.

Proof. By Lemma 4.4.4, it is easy to see that:

$$\phi\left[\max\{|T_1 - T_1'|, |T_2 - T_2'|\} > (\log n)^3 | \mathsf{Con}, \mathsf{NI}\right] \leq \mathrm{e}^{-c(\log n)^2} \tag{4.26}$$

for some small constant $c > 0$. Indeed, one can use Lemma 4.4.4 together with the fact that being a cone-point is a decreasing event. This fact established, the proof is essentially the same as the proof of Lemma 4.3.9: in a time smaller than $(\log n)^2$, the clusters cannot move to a polynomial distance of their starting point by a basic large deviations estimate. □

4.4.2 Global repulsion

In this section, we work under the measure $\phi[\,\cdot\,|\mathsf{Con}, \mathsf{NI}]$, and we want to prove that between a time $o(n)$ and $n - o(n)$, the minimal gap between the clusters diverges with n. The event that we are going to estimate is the following "global repulsion" event:

$$\mathsf{GlobRep} := \{T_1 < n^{1-\varepsilon}\} \cap \{T_2 > n - n^{1-\varepsilon}\} \cap \Big\{ \min_{\substack{1<i\leq r\\ t\in[T_1,T_2]}} \left|\Gamma_i^-(t) - \Gamma_{i-1}^+(t)\right| > (\log n)^2 \Big\}.$$

The goal of this section is to prove the following statement.

Proposition 4.4.8 (Global repulsion estimate)**.** *There exists $\beta > 0$, depending only on r, and $c > 0$ such that*

$$\phi\left[\mathsf{GlobRep} | \mathsf{Con}, \mathsf{NI}\right] \geq 1 - cn^{-\beta}. \tag{4.27}$$

This lemma will be the main ingredient for the proofs of Theorems 4.1.6 and 4.1.11. The rest of the section is dedicated to the proof of Proposition 4.4.8. We first use Lemma 4.4.7 to write:

$$\begin{aligned}
\phi\left[\mathsf{GlobRep}^c | \mathsf{NI}, \mathsf{Con}\right] &\leq \phi[\mathsf{EdgeReg}^c | \mathsf{NI}, \mathsf{Con}] + \phi[\mathsf{GlobRep}^c | \mathsf{NI}, \mathsf{Con}, \mathsf{EdgeReg}]\\
&\leq \tfrac{1}{c}\exp\left(-cn^{\varepsilon/2}\right) + \phi[\mathsf{GlobRep}^c | \mathsf{NI}, \mathsf{Con}, \mathsf{EdgeReg}].
\end{aligned}$$

We will focus on bounding the second term in the right-hand side of the above. We will do so by conditioning on T_1, T_2 and the shape of the clusters before T_1 and after T_2.

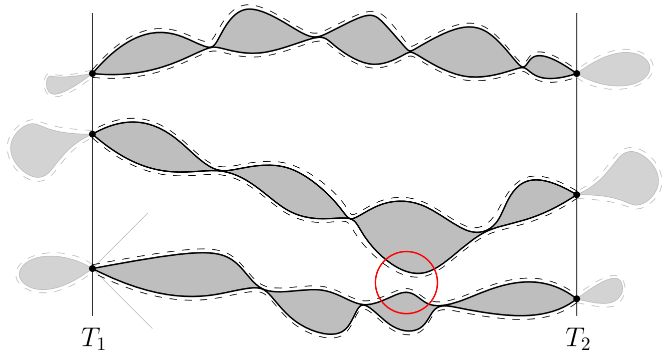

Figure 4.4: The region Strip is the vertical strip between T_1 and T_2; EXT is the configuration outside of this strip. The events NI, Con and Diam are realised by the three traversing grey clusters. Here GlobRep is violated since the bottom two clusters come close to each-other in the marked region.

As previously, write EXT for the trace of the clusters and their boundaries outside of the strip Strip $= [T_1, T_2] \times \mathbb{Z}$. Recall the notation $\mathscr{X}$ and $\mathscr{Y}$ for the vertical positions of the renewals at the times T_1 and T_2. Then $\phi[\cdot|\mathsf{Con}, \mathsf{NI}, \mathsf{EXT}]$[3] on the complement of EXT is the measure $\phi^0_{\mathsf{EXT}^c}$ conditioned on $\mathscr{X}$ being connected to $\mathscr{Y}$ by disjoint clusters contained in the respective diamonds. Write Diam_i for the fact that the i-th connection above occurs indeed in the corresponding diamond, and set $\mathsf{Diam} = \bigcap_{1 \le i \le r} \mathsf{Diam}_i$. Then

$$\begin{aligned}
&\phi\big[\mathsf{GlobRep}^c \big| \mathsf{NI}, \mathsf{Con}, \mathsf{EdgeReg}\big] \\
&\quad = \sum_{\mathsf{Ext}} \phi\left[\mathsf{GlobRep}^c \middle| \mathsf{NI}, \mathsf{Con}, \mathsf{EXT} = \mathsf{Ext}\right] \phi\left[\mathsf{EXT} = \mathsf{Ext} \middle| \mathsf{Con}, \mathsf{NI}, \mathsf{EdgeReg}\right] \\
&\quad = \sum_{\mathsf{Ext}} \frac{\phi\left[\mathsf{GlobRep}^c, \mathsf{NI}, \mathsf{Con} | \mathsf{EXT} = \mathsf{Ext}\right]}{\phi\left[\mathsf{NI}, \mathsf{Con} | \mathsf{EXT} = \mathsf{Ext}\right]} \phi\left[\mathsf{EXT} = \mathsf{Ext} \middle| \mathsf{Con}, \mathsf{NI}, \mathsf{EdgeReg}\right] \qquad (4.29)
\end{aligned}$$

where the sum runs over all possible edge-regular realizations Ext of EXT. We are going to focus on bounding the ratio above uniformly over Ext. This ratio may be written as

$$\frac{\phi\left[\mathsf{GlobRep}^c, \mathsf{NI}, \mathsf{Con} | \mathsf{EXT} = \mathsf{Ext}\right]}{\phi\left[\mathsf{NI}, \mathsf{Con} | \mathsf{EXT} = \mathsf{Ext}\right]} = \frac{\phi^0_{\mathsf{Ext}^c}\left[\mathsf{GlobRep}^c, \mathsf{NI}, \mathsf{Con}, \mathsf{Diam}\right]}{\phi^0_{\mathsf{Ext}^c}\left[\mathsf{NI}, \mathsf{Con}, \mathsf{Diam}\right]}. \qquad (4.30)$$

Lemma 4.4.1 and Corollary 4.3.3 allow us to lower bound the denominator as

$$\phi^0_{\mathsf{Ext}^c}\left[\mathsf{NI}, \mathsf{Con}, \mathsf{Diam}\right] \ge \tfrac{1}{\chi} V(\mathscr{X}) V(\mathscr{Y}) (T_2 - T_1)^{-\frac{r^2}{2}} e^{-\tau r (T_2 - T_1)}, \qquad (4.31)$$

where χ and $V(\cdot)$ were described in the aforementioned lemmas. We claim the following upper bound on the numerator.

[3]The conditioning on EXT contains implicitly the fact that T_1 and T_2 are renewals.

Lemma 4.4.9. *There exist constants $\beta > 0$ and $C > 0$ such that, for any edge-regular* Ext

$$\phi^0_{\mathsf{Ext}^c}[\mathsf{GlobRep}^c, \mathsf{NI}, \mathsf{Con}, \mathsf{Diam}] \leq CV(\mathscr{X})V(\mathscr{Y})(T_2 - T_1)^{-r^2/2-\beta}e^{-\tau r(T_2-T_1)}. \quad (4.32)$$

The lemma above is the main difficulty in the proof of Proposition 4.4.8; we postpone its proof and finish that of the proposition.

Proof of Proposition 4.4.8. Lemma 4.4.9 together with the estimate (4.23) yield that for any edge-regular Ext, the following holds:

$$\frac{\phi^0_{\mathsf{Ext}^c}[\mathsf{GlobRep}^c, \mathsf{NI}, \mathsf{Con}, \mathsf{Diam}]}{\phi^0_{\mathsf{Ext}^c}[\mathsf{NI}, \mathsf{Con}, \mathsf{Diam}]} \leq \frac{C}{\chi}(T_2 - T_1)^{-\beta}.$$

By edge-regularity of Ext, we know that $(T_2 - T_1) \geq n - 2n^{1-\varepsilon}$, so that $(T_2 - T_1)^{-\beta} = n^{-\beta}(1 + o(1))$. Thus, inserting this estimate into (4.4.2), we proved that

$$\phi[\mathsf{GlobRep}^c | \mathsf{NI}, \mathsf{Con}, \mathsf{EdgeReg}] \leq \frac{C}{\chi}n^{-\beta}.$$

Proposition 4.4.8 is obtained by applying equations (4.22), (4.4.2) and (4.4.2). □

We now turn to the proof of Lemma 4.4.9. Fix some edge-regular Ext. When NI, Con and Diam occur, write Γ_i for the top-most path of the cluster of $\mathscr{X}_i$. This is a non-simple path of open edges contained in Strip (due to Diam) connecting $\mathscr{X}_i$ to $\mathscr{Y}_i$. For any such path, write $\partial\Gamma_i$ for all the edges of Strip adjacent to and above Γ_i. Finally, write Γ for the r-tuple of paths $\Gamma_1, \dots, \Gamma_r$.

The idea of the proof of Lemma 4.4.9 is to bound the probability of $\{\Gamma = \gamma\}$ for any potential realisation γ of Γ by the probability of a suitable event in the product measure. Then we use the Ornstein–Zernike theory to estimate the latter probability. The first step is contained by the following statement, which constitutes the core of the proof.

Lemma 4.4.10. *There exists $\varepsilon > 0$ such that for any possible realization γ of Γ,*

$$\phi^0_{\mathsf{Ext}^c}[\Gamma = \gamma, \mathsf{NI}, \mathsf{Con}, \mathsf{Diam}] \leq (1-\exp(-n^\varepsilon))^{-r}(\phi^0_{\mathsf{Ext}^c})^{\otimes r}\Big[\bigcap_{i=1}^r \{\gamma_i \textit{ is open}, \partial\gamma_i \textit{ is closed}\}\Big].$$

Remark 4.4.11. The careful reader might think that this lemma is in contradiction with the lower bound given by Lemma 4.4.1. However, observe that while the events Diam and $\{\Gamma = \gamma\}$ do imply the events Con and NI in the measure $\phi^0_{\mathsf{Ext}^c}$, it is not the case for the measure $(\phi^0_{\mathsf{EXT}^c})^{\otimes r}$.

Proof of Lemma 4.4.10. Fix γ as in the statement. The paths $\gamma_1, \ldots, \gamma_r$ all cross Strip horizontally, are disjoint and are in increasing vertical order. The same holds for their upper boundaries $\partial\gamma_1, \ldots, \partial\gamma_r$. Moreover,

$$\begin{aligned}\phi^0_{\mathsf{Ext}^c}\left[\Gamma = \gamma, \mathsf{NI}, \mathsf{Con}, \mathsf{Diam}\right] &= \prod_{i=1}^{r} \phi^0_{\mathsf{Ext}^c}\Big[\Gamma_i = \gamma_i, \mathsf{Diam_i}\Big| \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\Big] \\ &\leq \prod_{i=1}^{r} \phi^0_{\mathsf{Ext}^c}\Big[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\Big| \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\Big].\end{aligned}$$

Our goal is now to prove that for any $i \in \{1, \ldots, r\}$,

$$\begin{aligned}&\phi^0_{\mathsf{Ext}^c}\Big[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\Big| \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\Big] \\ &\qquad \leq (1-\exp(-n^{\varepsilon}))^{-1}\phi^0_{\mathsf{Ext}^c}\left[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\right]. \end{aligned} \tag{4.33}$$

Indeed, assuming that the above inequality is true, we will have proved that

$$\begin{aligned}\phi^0_{\mathsf{Ext}^c}\big[\Gamma = \gamma, \mathsf{NI}, \mathsf{Con}, \mathsf{Diam}\big] &\leq \prod_{i=1}^{r}(1-\exp(-n^{\varepsilon}))^{-1}\phi^0_{\mathsf{Ext}}\big[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\big] \\ &= (1-\exp(-n^{\varepsilon}))^{-r}(\phi^0_{\mathsf{Ext}^c})^{\otimes r}\Big[\bigcap_{i=1}^{r}\{\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\}\Big].\end{aligned}$$

We thus focus on (4.4.2). The bound is obviously true for $i = 1$. Fix next $i > 1$. Write:

$$\begin{aligned}&\phi^0_{\mathsf{Ext}^c}\Big[\gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\Big| \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\Big] \\ &= \phi^0_{\mathsf{Ext}^c}\Big[\gamma_i \text{ open}\Big| \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\Big]\phi^0_{\mathsf{Ext}^c}\Big[\partial\gamma_i \text{ closed}\Big|\gamma_i \text{ open}, \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\Big].\end{aligned} \tag{4.34}$$

The first factor is easy to upper bound, as the conditioning decreases the probability for γ_i to be open. Indeed, explore the clusters of $\mathscr{X}_1, \ldots, \mathscr{X}_{i-1}$ together with their boundaries, and call Expl the set of explored edges. Because of the conditioning on $\mathsf{Diam}_1, \ldots, \mathsf{Diam}_{i-1}$ and of the disjointness of the paths of γ, γ_i is disjoint from the explored edges $\mathsf{Expl} \cup \mathsf{Ext}$. Furthermore, the measure induced on the complement of these clusters by this exploration procedure is $\phi^0_{(\mathsf{Expl}\cup\mathsf{Ext})^c}$. By (CBC),

$$\phi^0_{(\mathsf{Ext}\cup\mathsf{Expl})^c}\left[\gamma_i \text{ is open}\right] \leq \phi^0_{\mathsf{Ext}^c}\left[\gamma_i \text{ is open}\right],$$

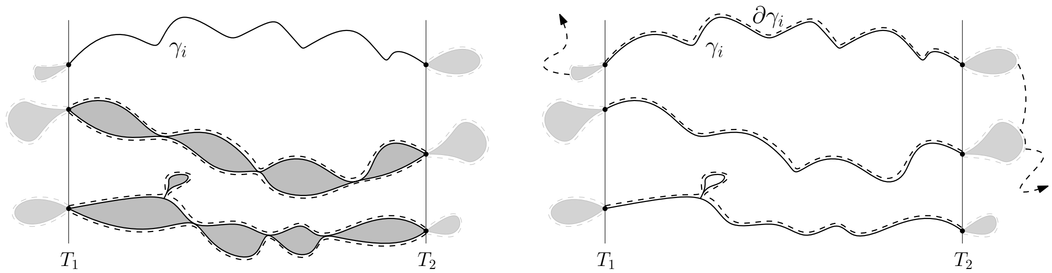

Figure 4.5: *Left:* To upperbound $\phi^0_{\mathsf{Ext}^c}[\gamma_i \text{ open}|\bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}]$ it suffices to explore the clusters of Γ_k for $k < i$ and observe that any such instance induces negative information on the rest of the space. *Right:* When bounding $\phi^0_{\mathsf{Ext}^c}[\partial\gamma_i \text{ closed}|\gamma_i \text{ open}, \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}]$, the conditioning may increase the probability for $\partial\gamma_i$ to be closed, but not more so than the occurrence of $\mathcal{H}^{\mathsf{L}} \cap \mathcal{H}^{\mathsf{R}}$. The latter events are ensured by the existence of the infinite dual paths on the left and right of the strip.

which in turn implies

$$\phi^0_{\mathsf{Ext}^c}\Big[\gamma_i \text{ is open}\Big|\bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\Big] \le \phi^0_{\mathsf{Ext}^c}[\gamma_i \text{ is open}]\,. \tag{4.35}$$

We turn to the second factor in the right-hand side of (4.4.2). Upper bounding this term is slightly more subtle, since the the boundary conditions induced by the conditioning may *a priori* help $\partial\gamma_i$ to be closed. Introduce the following events

$$\mathcal{H}^{\mathsf{L}} = \big\{(T_1, \mathscr{Y}_i)^* \text{ is connected to } \infty \text{ by a dual open path lying in } \mathsf{Strip}^c\big\} \quad \text{and}$$
$$\mathcal{H}^{\mathsf{R}} = \big\{(T_2, \mathscr{Y}_i)^* \text{ is connected to } \infty \text{ by a dual open path lying in } \mathsf{Strip}^c\big\}.$$

We now claim that

$$\phi^0_{\mathsf{Ext}^c}\Big[\partial\gamma_i \text{ closed}\Big|\gamma_i \text{ open}, \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\Big] \le \phi^0_{\mathsf{Ext}^c}\big[\partial\gamma_i \text{ closed}\big|\mathcal{H}^{\mathsf{L}} \cap \mathcal{H}^{\mathsf{R}} \cap \{\gamma_i \text{ open}\}\big]. \tag{4.36}$$

Indeed, the conditioning on $\bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}$ may induce negative information, which improves the probability of $\partial\gamma_i$ to be closed. Nevertheless, this influence is weaker than that of the decreasing events $\mathcal{H}^{\mathsf{L}}$ and $\mathcal{H}^{\mathsf{R}}$. A formal proof of the above is obtained by conditioning on the lowest paths producing $\mathcal{H}^{\mathsf{L}}$ and $\mathcal{H}^{\mathsf{R}}$; we do not give additional details here.

The following claim will be particularly convenient to handle the right-hand side of the above.

Claim 4.4.12. There exist constants $\varepsilon > 0$ and $c > 0$ such that

$$\phi^0_{\mathsf{Ext}^c}\left[\mathcal{H}^{\mathsf{L}} \cap \mathcal{H}^{\mathsf{R}} \middle| \gamma_i \text{ is open}\right] \geq 1 - \exp\left(-cn^{\varepsilon}\right).$$

Proof of Claim 4.4.12. The proof is a standard argument using the properties of the subcritical regime. We will prove that

$$\phi^0_{\mathsf{Ext}^c}\left[\mathcal{H}^{\mathsf{L}} \middle| \gamma_i \text{ is open}\right] \geq 1 - \tfrac{1}{2}\exp\left(-cn^{\varepsilon}\right), \tag{4.37}$$

and the claim will follow by the union bound.

The event $\mathcal{H}^{\mathsf{L}}$ is decreasing and the conditioning only depends on the configuration in Strip. Thus (CBC) implies

$$\phi^0_{\mathsf{Ext}^c}\left[\mathcal{H}^{\mathsf{L}} \middle| \gamma_i \text{ is open}\right] \geq \phi^1_{\mathsf{Strip}^c}\left[\mathcal{H}^{\mathsf{L}}\right].$$

For $\mathcal{H}^{\mathsf{L}}$ to fail, there must exist at least one index $k \geq 0$ such that $(-k, x_i)$ is connected to the vertical axis $\{T_1\} \times \mathbb{Z}$ by a (primal) open path lying in the half-plane $(-\infty, T_1] \times \mathbb{Z}$. Thus

$$\phi^1_{\mathsf{Strip}^c}\left[(\mathcal{H}^{\mathsf{L}})^c\right] \leq \sum_{k \geq 0} \phi^1_{\mathsf{Strip}^c}\left[(-k, x_i) \leftrightarrow \{T_1\} \times \mathbb{Z}\right]. \tag{4.38}$$

It is well-known that the exponential decay of the primal cluster applies also within wired boundary conditions [52], and therefore the terms in the sum above are bounded above by e^{-ck} for some $c > 0$ and all k. Summing over $k \geq 0$ we find

$$\phi^1_{\mathsf{Strip}^c}\left[(\mathcal{H}^{\mathsf{L}})^c\right] \leq C\mathrm{e}^{-cT_1}.$$

By the cone confinement property $T_1 \geq \frac{n^{\varepsilon}}{2\delta}$. This proves (4.4.2) after altering the constants. □

We are now ready to conclude. The claim along with (4.4.2) imply that

$$\phi^0_{\mathsf{Ext}^c}\left[\partial\gamma_i \text{ closed}\middle|\gamma_i \text{ open}, \bigcap_{k=1}^{i-1}\{\Gamma_k = \gamma_k, \mathsf{Diam}_k\}\right] \leq \left(1 - \mathrm{e}^{-cn^{\varepsilon}}\right)^{-1}\phi^0_{\mathsf{Ext}^c}\left[\partial\gamma_i \text{ closed}\middle|\gamma_i \text{ open}\right].$$

The above, together with (4.25) may be inserted into (4.4.2) to obtain (4.4.2). As already mentioned, this concludes the proof of Lemma 4.4.10

□

We turn to the second step of the proof of Lemma 4.4.9.

Proof of Lemma 4.4.9. Recall the definition of the family of paths Γ, defined when NI, Con and Diam occur. Define the set $\mathsf{ClosePath}$ of realisations γ of Γ for which there exists $2 \le i \le r$ and $T_1 \le t \le T_2$ such that

$$\mathrm{dist}\big[\gamma_{i-1} \cap (\{t\} \times \mathbb{Z}),\, \gamma_i \cap (\{t\} \times \mathbb{Z})\big] < (\log n)^3.$$

We first observe that

$$\begin{aligned}
&\phi^0_{\mathsf{Ext}^c}\left[\mathsf{GlobRep}^c, \mathsf{Con}, \mathsf{NI}, \mathsf{Diam}\right] \\
&\quad \le \phi^0_{\mathsf{Ext}^c}\left[\mathsf{GlobRep}^c, \mathsf{BigDiam}^c, \mathsf{Con}, \mathsf{NI}, \mathsf{Diam}\right] + \phi^0_{\mathsf{Ext}^c}\big[\mathsf{BigDiam}, \mathsf{Con}, \mathsf{NI}\big] \\
&\quad \le \sum_{\gamma \in \mathsf{ClosePath}} \phi^0_{\mathsf{Ext}}\left[\Gamma = \gamma, \mathsf{Con}, \mathsf{NI}, \mathsf{Diam}\right] + \mathrm{e}^{-c(\log n)^2 - \tau r(T_2 - T_1)}. \qquad (4.39)
\end{aligned}$$

Indeed, the second inequality is true term by term. For the first term, due to $\mathsf{BigDiam}^c$, the clusters are entirely within distance $(\log n)^2$ of the corresponding paths Γ_i. Thus, for $\mathsf{GlobRep}^c$ to occur, the paths Γ_i need to come within distance $(\log n)^2 + 2(\log n)^2$ of each other, and in particular need to belong to $\mathsf{ClosePath}$. The bound on the second term is a direct consequence of Lemma 4.4.4 and (4.2). The second term obviously satisfies the upper bound in (4.24), and we may focus on bounding the first term.

By (4.4.2) and Lemma 4.4.10,

$$\begin{aligned}
&\phi^0_{\mathsf{Ext}^c}\big[\mathsf{GlobRep}^c, \mathsf{BigDiam}^c, \mathsf{Con}, \mathsf{NI}, \mathsf{Diam}\big] \\
&\le (1 + o(1)) \sum_{\gamma \in \mathsf{ClosePath}} (\phi^0_{\mathsf{Ext}^c})^{\otimes r}\Big[\bigcap_{i=1}^{r} \{\gamma_i \text{ open}, \partial\gamma_i \text{ closed}\}\Big] \\
&\le (1 + o(1))\, (\phi^0_{\mathsf{Ext}^c})^{\otimes r}\big[\exists \gamma \in \mathsf{ClosePath} \text{ s.t. } \forall i \in \{1, \dots, r\}, \gamma_i \text{ open}, \partial\gamma_i \text{ closed}\big]. \qquad (4.40)
\end{aligned}$$

The last upper bound is obtained by observing that when the last event is satisfied, at most one family of paths of $\mathsf{ClosePath}$ can achieve it (due to the event Diam). We will bound the last term using the Ornstein–Zernike coupling $\Phi^{0,\otimes r}_{\mathsf{Ext}^c,(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}[\ \cdot\ ]$ and the random skeleton system $\mathcal{S}$ given by this coupling. This argument will only be sketched as it already appeared in the proof of Proposition 4.3.1.

Under the event in the last line of (4.4.2), the paths $\gamma_1, \dots, \gamma_r$ contain all the renewal points of the clusters $\mathcal{C}_i$ in Strip, and therefore the synchronised skeleton $\check{\mathcal{S}}$ is guaranteed to be non-intersecting. In addition, due to the diamond confinement property and since $\gamma \in \mathsf{ClosePath}$, $\inf_{t \in [T_1, T_2]} \mathsf{Gap}(\check{\mathcal{S}}(t)) \le 3\log^2 n + (\log n)^2$ with probability going to 1. Finally we conclude that

$$\begin{aligned}
&(\phi^0_{\mathsf{Ext}^c})^{\otimes r}\big[\exists \gamma \in \mathsf{ClosePath} \text{ such that } \forall i \in \{1, \dots, r\}, \gamma_i \text{ is open}, \partial\gamma_i \text{ is closed}\big] \\
&\le (1 + o(1))\, \mathrm{e}^{-\tau r(T_2 - T_1)} \Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}\left[\check{\mathcal{S}} \in \mathcal{W}_{[T_1,T_2]}, \mathsf{Gap}(\check{\mathcal{S}}(t)) \le 4\log^2 n \,\middle|\, \mathsf{EXT} = \mathsf{Ext}\right].
\end{aligned}$$

We now use once the local limit Theorem 4.5.5 and then Lemma 4.5.12 (the assumptions of the lemma are satisfied due to the edge-regularity of Ext) to conclude that

$$\Phi^{\otimes r}_{(T_1,\mathscr{X})\to(T_2,\mathscr{Y})}\left[\check{\mathcal{S}}\in\mathcal{W}_{[T_1,T_2]},\mathsf{Gap}(\check{\mathcal{S}}(t))\le 4\log^2 n\big|\mathsf{EXT}=\mathsf{Ext}\right]$$
$$\le CV(\mathscr{X})V(\mathscr{Y})(T_2-T_1)^{-\frac{r^2}{2}}(T_2-T_1)^{-\beta}.$$

Putting everything together and coming back to (4.4.2), we obtain:

$$\phi^0_{\mathsf{Ext}^c}\left[\mathsf{GlobRep}^c,\mathsf{Con},\mathsf{NI},\mathsf{Diam}\right]\le C\,V(\mathscr{X})V(\mathscr{Y})\,(T_2-T_1)^{-\frac{r^2}{2}-\beta}\mathrm{e}^{-\tau r(T_2-T_1)},$$

which concludes the proof. □

4.4.3 The mixing argument and the proof of Theorems 4.1.6 and 4.1.11

We start with the proof of Theorem 4.1.6.

Proof of Theorem 4.1.6. Recall that one bound, namely

$$\phi\,[\mathsf{NI},\mathsf{Con}]\ge c\,V(x)V(y)\,n^{-\frac{r^2}{2}}\mathrm{e}^{-\tau rn} \tag{4.41}$$

has already been proved in Remark 4.4.2. Our goal here is to prove a matching upper bound.

Running the argument used in the proof of Lemma 4.4.9, but summing over all realisations of Γ rather than only those in $\mathsf{ClosePath}$ yields

$$\phi^0_{\mathsf{Ext}^c}\left[\mathsf{NI},\mathsf{Con}\right]\le CV(\mathscr{X})V(\mathscr{Y})(T_2-T_1)^{-\frac{r^2}{2}}\mathrm{e}^{-\tau r(T_2-T_1)}. \tag{4.42}$$

The above does not match the desired bound since T_1 is larger than 0 and T_2 smaller than n by a polynomial quantity. Moreover, $V(\mathscr{X})$ and $V(\mathscr{Y})$ are also of polynomial order. To obtain the upper bound matching (4.27), we will run the same argument with T_1 and T_2 replaced with random times of finite order.

Let $\tilde{T}_1$ (resp. $\tilde{T}_2$) be the first (resp. the last) synchronization point of the maximal skeletons of the clusters after 0 (resp. before n). We also call $\tilde{\mathscr{X}}$ (resp. $\tilde{\mathscr{Y}}$) the unique vector such that $(\tilde{T}_1,\tilde{\mathscr{X}}_i)\in\mathcal{C}_i$ (resp. $(\tilde{T}_2,\tilde{\mathscr{Y}}_i)\in\mathcal{C}_i$). We already argued in the proof of Lemma 4.4.4 that $\tilde{T}_1$ and $\tilde{T}_2$ have exponential tails: for any $t\ge 0$ large enough,

$$\phi\left[\max\{\tilde{T}_1,\tilde{T}_2\}>t|\mathsf{Con},\mathsf{NI}\right]\le\mathrm{e}^{-ct}.$$

We also already argued that

$$\phi\left[\max\{\|\tilde{\mathscr{X}}-x\|,\|\tilde{\mathscr{Y}}-y\|\}>t|\mathsf{Con},\mathsf{NI}\right]\le\mathrm{e}^{-ct},$$

for a possibly different value of $c > 0$. For the rest of this proof fix t so that $e^{-ct} \leq 1/2$. Then, we upper bound:

$$\begin{aligned}\phi\big[\mathsf{Con},\mathsf{NI}\big] &= \phi\big[\mathsf{Con},\mathsf{NI},\max\{\|\tilde{\mathscr{X}}-x\|,\|\tilde{\mathscr{Y}}-y\|\}>t\big] \\ &\quad + \phi\big[\mathsf{Con},\mathsf{NI},\max\{\|\tilde{\mathscr{X}}-x\|,\|\tilde{\mathscr{Y}}-y\|\}<t\big] \\ &\leq \phi\big[\mathsf{Con},\mathsf{NI}\big|\max\{\|\tilde{\mathscr{X}}-x\|,\|\tilde{\mathscr{Y}}-y\|\}<t\big] + \tfrac{1}{2}\phi\big[\mathsf{Con},\mathsf{NI}\big].\end{aligned}$$

Hence, we obtain that

$$\phi\left[\mathsf{Con},\mathsf{NI}\right] \leq 2\,\phi\big[\mathsf{Con},\mathsf{NI}\big|\max\{\|\tilde{\mathscr{X}}-x\|,\|\tilde{\mathscr{Y}}-y\|\}<t\big]. \tag{4.43}$$

We focus on upper bounding $\phi[\mathsf{Con},\mathsf{NI}|\max\{\|\tilde{\mathscr{X}}-x\|,\|\tilde{\mathscr{Y}}-y\|\}<t]$, and will do so using the method of Lemma 4.4.10. As in the proof of Lemma 4.4.10, condition on the shape of the clusters outside of $\mathsf{Strip} = [\tilde{T}_1,\tilde{T}_2]\times\mathbb{Z}$ to write:

$$\begin{aligned}\phi\left[\mathsf{Con},\mathsf{NI}|\max\{\|\tilde{\mathscr{X}}-x\|,\|\tilde{\mathscr{Y}}-y\|\}<t\right] &= \sum_{\mathsf{Ext}}\phi^0_{\mathsf{Ext}^c}\left[\mathsf{Con},\mathsf{NI},\mathsf{Dian}\right] \\ &\quad\times\phi\left[\mathsf{Ext}=\mathsf{Ext}|\max\{\|\tilde{\mathscr{X}}-x\|,\|\tilde{\mathscr{Y}}-y\|\}<t\right].\end{aligned}$$

As previously, the conditioning on EXT contains the fact that $\tilde{T}_1$ and $\tilde{T}_2$ are renewals.

Fix some Ext appearing in the sum above. Recall the definition of the top-most path Γ_i of the cluster $\mathcal{C}_i$ and its upper boundary $\partial\Gamma_i$. Then

$$\begin{aligned}\phi^0_{\mathsf{Ext}^c}\big[\mathsf{Con},\mathsf{NI},\mathsf{Diam}\big] &\leq \sum_{\gamma}\prod_{i=1}^{r}\phi^0_{\mathsf{Ext}^c}\Big[\gamma_i\text{ open}\Big|\bigcap_{k=1}^{i-1}\{\Gamma_k=\gamma_k,\mathsf{Diam}_k\}\Big] \\ &\quad\times\phi^0_{\mathsf{Ext}^c}\Big[\partial\gamma_i\text{ closed}\Big|\gamma_i\text{ open},\bigcap_{k=1}^{i-1}\{\Gamma_k=\gamma_k,\mathsf{Diam}_k\}\Big],\end{aligned} \tag{4.44}$$

where the sum is over all possible realisations γ of Γ that induce disjoint connections between $\tilde{\mathscr{X}}$ and $\tilde{\mathscr{Y}}$.

The two terms in the right-hand side of the above may be bounded as in Lemma 4.4.10 by

$$\begin{aligned}\phi^0_{\mathsf{Ext}^c}\big[\gamma_i\text{ open}\big|\bigcap_{k=1}^{i-1}\{\Gamma_k=\gamma_k,\mathsf{Diam}_k\}\big] &\leq \phi^0_{\mathsf{Ext}^c}\big[\gamma_i\text{ open}\big]\text{ and} \\ \phi^0_{\mathsf{Ext}^c}\big[\partial\gamma_i\text{ closed}\big|\gamma_i\text{ open},\bigcap_{k=1}^{i-1}\{\Gamma_k=\gamma_k,\mathsf{Diam}_k\}\big] &\leq C\phi^0_{\mathsf{Ext}^c}\big[\partial\gamma_i\text{ closed}\big|\gamma_i\text{ open}\big]\end{aligned} \tag{4.45}$$

This is the only place where the proof differs a little from that of Lemma 4.4.10. Indeed, in the second bound above, we will use that

$$\phi^0_{\mathsf{Ext}^c}\big[\mathcal{H}^{\mathsf{L}} \cap \mathcal{H}^{\mathsf{R}} \big| \gamma_i \text{ open}\big] \geq c. \tag{4.46}$$

for some universal constant c, where $\mathcal{H}^{\mathsf{L}}$ and $\mathcal{H}^{\mathsf{R}}$ are defined as in Claim 4.4.12. The proof of (4.4.3) is easier than that of Claim 4.4.12: it relies simply on the fact that, in the subcritical regime, the dual percolates even in a half-plane with wired boundary conditions.

Now, injecting (4.4.3) back into (4.4.3), we find that

$$\phi^0_{\mathsf{Ext}^c}\big[\mathsf{Con}, \mathsf{NI}, \mathsf{Diam}\big] \leq C\,(\phi^0_{\mathsf{Ext}^c})^{\otimes r}\big[\exists \gamma \text{ s.t. } \forall i \in \{1, \dots, r\}, \gamma_i \text{ open}, \partial\gamma_i \text{ closed}\big] \tag{4.47}$$

We conclude using the Ornstein–Zernike coupling for the product measure. Indeed, the event on the right-hand side of (4.4.3) implies that, in the product measure, the connection event occurs and the synchronized skeletons are non-intersecting. Thus,

$$\phi^0_{\mathsf{Ext}^c}\big[\mathsf{Con}, \mathsf{NI}, \mathsf{Diam}\big] \leq C\,\mathrm{e}^{-\tau r(\tilde{T}_2 - \tilde{T}_1)} \phi^{0,\otimes r}_{(\tilde{T}_1, \tilde{\mathscr{X}}) \to (\tilde{T}_2, \tilde{\mathscr{Y}})}\big[\check{\mathcal{S}} \in \mathcal{W}_{[\tilde{T}_1, \tilde{T}_2]} \big| \mathsf{Ext} = \mathsf{Ext}\big].$$

We make use of the Local limit Theorem 4.5.5 to upper bound the right-hand side probability by $C\,V(\tilde{\mathscr{X}})V(\tilde{\mathscr{Y}})(\tilde{T}_2 - \tilde{T}_1)^{-\frac{r^2}{2}}$. Using the assumption on Ext, we then very roughly upper bound

$$\max\{V(\tilde{\mathscr{X}}), V(\tilde{\mathscr{Y}})\} \leq (2\max\{\|\tilde{\mathscr{X}} - x\|, \|\tilde{\mathscr{Y}} - y\|\})^{\frac{r(r-1)}{2}} V(x)V(y) \leq V(x)V(y)(2t)^{\frac{r(r-1)}{2}}.$$

Gathering everything together, we conclude that for all Ext satisfying $\max\{\|\tilde{\mathscr{X}}\|, \|\tilde{\mathscr{Y}}\|\} \leq t$ and $\max\{\tilde{T}_1, n - \tilde{T}_2\} \leq t$

$$\phi^0_{\mathsf{Ext}^c}\big[\mathsf{Con}, \mathsf{NI}\big] \leq C\,(2t)^{\frac{r(r-1)}{2}} V(x)V(y)\mathrm{e}^{-\tau r(n-2t)}(n-2t)^{\frac{r^2}{2}} \leq C' V(x)V(y)\mathrm{e}^{-\tau r n} n^{\frac{r^2}{2}}.$$

Summing over all such Ext and using (4.29) we find that

$$\phi\big[\mathsf{Con}, \mathsf{NI}\big] \leq C'\,V(x)V(y)\mathrm{e}^{-\tau r n} n^{-\frac{r^2}{2}},$$

where the value of C' has been increased between every equation, but does not depend on n. This concludes the proof.

□

We then turn to the proof of Theorem 4.1.11. It follows from the repulsion estimate of Proposition 4.4.8 and the convergence of the product system stated in Proposition 4.3.1. The observation is that when GlobRep occurs, then it is a consequence of the mixing property of the random-cluster measure (MIX) that the distribution of the system of clusters is very close to the one of an *independent* system of clusters.

Proof of Theorem 4.1.11. We follow the same pattern as in the proof of Proposition 4.3.1, and use the strategy given by Lemma 4.3.10. Fix some $\delta > 0$ and arbitrary signs for the r envelopes which we denote by $\pm$. As previously, define the scaled process $\Gamma_n^{\pm}(t) := \frac{1}{\sqrt{n}}\Gamma^{\pm}(nt)$, and consider a function $f : \mathcal{C}([\delta, 1-\delta], \mathbb{R}^r) \to \mathbb{R}$, continuous and bounded. As in the proof of Proposition 4.3.1 we shall omit to write the restriction to the interval $[\delta, 1-\delta]$ when writing $f^\delta(\Gamma_n^{\pm})$. We start by arguing that due to the boundedness of f^δ and to Lemma 4.4.7,

$$\phi\big[f^\delta(\Gamma_n^{\pm})\big|\mathsf{NI}, \mathsf{Con}\big] = (1+o(1))\phi\big[f^\delta(\Gamma_n^{\pm})\big|\mathsf{NI}, \mathsf{Con}, \mathsf{EdgeReg}\big].$$

Next, we condition on T_1, T_2 and the edge-regular shape of the clusters outside of $\mathsf{Strip} := \mathsf{Strip}_{[T_1,T_2]}$. Due to the edge-regularity condition, we can chose n large enough so that $n\delta > T_1$ and $n(1-\delta) < T_2$. We find

$$\phi\big[f^\delta(\Gamma_n^{\pm})\big|\mathsf{NI}, \mathsf{Con}, \mathsf{EdgeReg}\big] = \sum_{\mathsf{Ext}}\phi\big[f^\delta(\Gamma_n^{\pm})\big|\mathsf{NI}, \mathsf{Con}, \mathsf{EXT} = \mathsf{Ext}\big] \times \phi\big[\mathsf{EXT} = \mathsf{Ext}\big|\mathsf{NI}, \mathsf{Con}, \mathsf{EdgeReg}\big].$$

Fix some Ext which is edge-regular. We make use of Proposition 4.4.8 to argue that:

$$\phi\left[f^\delta(\Gamma_n^{\pm})|\mathsf{NI}, \mathsf{Con}, \mathsf{EXT} = \mathsf{Ext}\right] = (1+o(1))\,\phi\left[f^\delta(\Gamma_n^{\pm})|\mathsf{NI}, \mathsf{Con}, \mathsf{EXT} = \mathsf{Ext}, \mathsf{GlobRep}\right].$$

Now, we claim that the mixing property (MIX) implies that

$$\left|\frac{\phi\big[f^\delta(\Gamma_n^{\pm})|\mathsf{NI}, \mathsf{Con}, \mathsf{EXT} = \mathsf{Ext}, \mathsf{GlobRep}\big]}{\phi^{\otimes r}\big[f^\delta(\Gamma_n^{\pm})|\mathsf{NI}, \mathsf{Con}, \mathsf{EXT} = \mathsf{Ext}, \mathsf{GlobRep}\big]} - 1\right| < \mathrm{e}^{-2(\log n)^2}. \tag{4.48}$$

Indeed, decompose the term $\phi\big[f^\delta(\Gamma_n^{\pm})|\mathsf{NI}, \mathsf{Con}, \mathsf{EXT} = \mathsf{Ext}, \mathsf{GlobRep}\big]$ as follows:

$$\phi\big[f^\delta(\Gamma_n^{\pm})|\mathsf{NI}, \mathsf{Con}, \mathsf{EXT} = \mathsf{Ext}, \mathsf{GlobRep}\big] = \sum_{C_1,\dots,C_r} f^\delta(\Gamma_n^{\pm})\frac{\phi^0_{\mathsf{Ext}^c}\big[\mathcal{C}_1 = C_1, \dots, \mathcal{C}_r = C_r\big]}{\phi^0_{\mathsf{Ext}^c}\big[\mathsf{NI}, \mathsf{Con}, \mathsf{Globrep}\big]},$$

where the sum runs over the possible realisations $C_1, \dots, C_r$ of the clusters of $\mathscr{X}$ under the measure $\phi^0_{\mathsf{Ext}^c}[\cdot|\mathsf{NI}, \mathsf{Con}, \mathsf{GlobRep}]$. The point is that those sets are almost surely finite and have a mutual distance larger than $\delta(\log n)^3$ by the diamond confinement property. We can then apply (MIX) to both the numerator and the denominator of the fraction to obtain (4.30).

The last thing to notice is that the entropic repulsion estimate (4.21) also holds for the product measure:

$$\phi^{\otimes r}\left[\mathsf{GlobRep}|\mathsf{Con}_{\mathscr{X},\mathscr{Y}}, \mathsf{NI}, \mathsf{EXT} = \mathsf{Ext}\right] \geq 1 - cn^{-\beta},$$

because of the usual Ornstein–Zernike coupling (4.2.14) and the entropic repulsion for random walks given by Lemma 4.5.8. In conclusion, we proved that:

$$\phi\left[f^{\delta}(\Gamma_n^{\pm})|\mathsf{NI},\mathsf{Con},\mathsf{EXT}=\mathsf{Ext}\right]=(1+o(1))\phi^{\otimes r}\left[f^{\delta}(\Gamma_n^{\pm})|\mathsf{NI},\mathsf{Con},\mathsf{EXT}=\mathsf{Ext}\right].$$

Finally, due to the assumed edge-regularity of Ext and to Proposition 4.3.1, we know that the RHS converges towards $\mathbb{E}\left[f^{\delta}(\sigma\mathsf{BW}^{(r)})\right]$. Hence,

$$\phi\left[f^{\delta}(\Gamma_n^{\pm})|\mathsf{NI},\mathsf{Con}\right]\xrightarrow[n\to\infty]{}\mathbb{E}\left[f^{\delta}(\sigma\mathsf{BW}^{(r)})\right],$$

and so is established point (i) of Lemma 4.3.10. As previously the equicontinuity at 0 and 1 is an easy consequence of basic large deviations estimates. This observation achieves the proof. □

4.5 Local statistics of directed non-intersecting random bridges

As seen before, by the Ornstein–Zernike theory and the entropic repulsion, a system of clusters subject to the non-intersection conditioning resembles a system of non-intersecting directed random walks.

Non-intersecting random walks, and more largely random walks in cones have a very rich combinatorial and probabilistic structure. They have been studied widely throughout the last 50 years. The seminal work is the paper of Karlin and McGregor [89] which proves a determinantal formula for the probability of r random walks to intersect. Their approach only applies to a very specific class of walks, and is combinatorial by nature; it lead to remarkable developments around integrable systems of walks (see [67, 88]).

A more probabilistic treatment has been started in [56], [43, 44, 54]. Indeed, in [56] a definition of the random walk conditioned to stay in a cone was given in terms of a Doob h-transform by a harmonic function vanishing on the boundary of the cone, allowing the authors to obtain Local Limit Theorems and invariance principles for a much broader class of random walks. We briefly summarize the definitions and construction of the concerned objects.

The goal of this section is then to study the properties of such systems of walks, especially their behaviour under the diffusive scaling. Let us introduce the relevant object to study.

Definition 4.5.1 (Directed system of random walks)**.** Let $r\geq 1$ be an integer, and for $1\leq i\leq r$, let $(\theta_n^i,X_n^i)_{n\geq 1}$ be an independent and identically distributed family of independent and identically distributed random variables on $\mathbb{N}^*\times\mathbb{Z}$. We assume that it satisfies the following properties:

- Both θ_1^1 and X_1^1 have an exponential moment.

- Conditionally on θ_n^i, X_n^i is centered.

Call:

$$\mathbf{T}_n^i = \sum_{k=1}^{n} \theta_k^i \qquad \text{and} \qquad \mathbf{Z}_n^i = \sum_{k=1}^{n} X_k^i$$

Then the system

$$(\mathbf{S}_n)_{n\geq 0} := \big((\mathbf{T}_n^1, \mathbf{Z}_n^1), \dots, (\mathbf{T}_n^r, \mathbf{Z}_n^r)\big)_{n\geq 0}$$

is called a *system of directed random walks*. For any $(k_i, x_i)_{1\leq i\leq r} \in (\mathbb{N} \times \mathbb{Z})^r$, we write $\mathbf{P}_{(k,x)}$ for the law of the r-directed random walk with $\mathbf{S}_0^i = (k_i, x_i)$ – this is defined as above, with the addition of an initial offset. When all the k_i are equal, which will often be the case, we make a slight abuse of notation by writing $\mathbf{P}_{(k,x)}$ with $k \in \mathbb{Z}, x \in \mathbb{Z}^r$.

As observed in the precedent sections, a subcritical percolation cluster can be roughly described as the trajectory of a directed random walk *decorated* with δ-confined clusters of edges. For that reason, it is convenient to study directed system of non-intersecting random bridges carrying δ-diamonds around their steps. We then make the following assumption:

Assumption 4.5.2. There exists a $\delta > 0$ such that almost surely,

$$(\theta_1^1, X_1^1) \in \mathcal{Y}_0^{+,\delta}. \tag{4.49}$$

Recall the definition of the diamonds from Section 4.2. If $(\mathbf{S}_n)_{n\geq 0}$ is a system of directed random walks, we introduce

$$\mathcal{D}_{i,k}^{\delta} = \mathcal{D}^{\delta}_{(\mathbf{T}_k^i, \mathbf{Z}_k^i),(\mathbf{T}_{k+1}^i, \mathbf{Z}_{k+1}^i)} \qquad \text{and} \qquad \mathcal{D}(\mathbf{S}^i) := \bigcup_{k\geq 0} \mathcal{D}_{i,k}^{\delta}.$$

We also introduce the diamond-decorated walks analogs of the events Con and NI (see Figure 4.6).

For $y \in \mathbb{R}^r, n \geq 0$, the hitting event is defined by:

$$\mathsf{Hit}_{(n,y)} = \left\{\exists k_1, \dots, k_r \geq 0, \forall i \in \{1, \dots, r\}, (\mathbf{T}_{k_i}^i, \mathbf{S}_{k_1}^1) = (n, y_i)\right\}.$$

The non-intersection of diamond event is defined by

$$\mathsf{NonIntDiam}(\mathbf{S}) = \bigcap_{1\leq i\neq j\leq r} \{\mathcal{D}(\mathbf{S}^i) \cap \mathcal{D}(\mathbf{S}^j) = \emptyset\}.$$

The goal of this section is the proof of the following result:

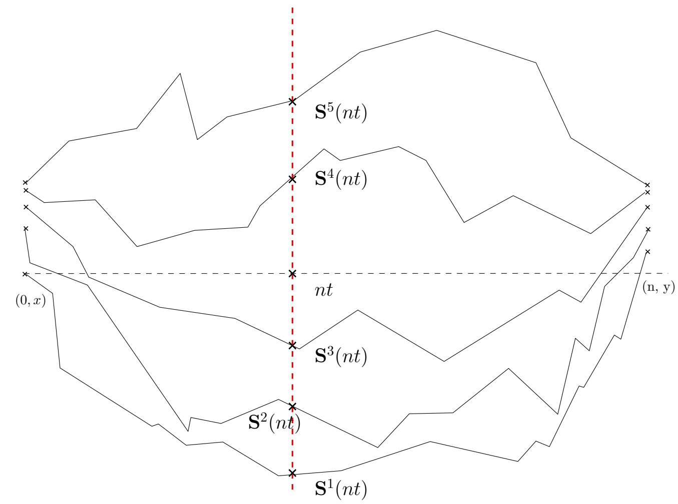

Figure 4.6: A depiction of a system of $r = 5$ non-synchronized random walks under the event $\left\{\mathbf{S} \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right\}$.

Theorem 4.5.3 (Invariance principle for directed random walks)**.** *Let S be a system of r directed random walks sampled according to $\mathbf{P}_{(0,x)}\left[\cdot|\mathbf{S} \in \mathsf{Hit}_{(n,y)}, \mathsf{NonIntDiam}(\mathbf{S})\right]$. Then, there exists $\sigma > 0$ such that*

$$\left(\frac{1}{\sqrt{n}}\mathbf{S}\left(nt\right)\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} \left(\sigma \mathsf{BW}_t^{(r)}\right)_{0\leq t\leq 1},$$

where the convergence holds in the space $\mathcal{C}([0,1],\mathbb{R}^r)$ equipped with the topology of the uniform convergence. Moreover, x and y can depend on n in the statement, as long as they both have norms that are $o(\sqrt{n})$.

This theorem, as well as Theorem 4.5.5, has already been proved in the setting of *regular* random walks (that is when $\theta_1^1 = 1$ almost surely), and replacing the conditioning over the non-intersection of *diamonds* by a conditioning of non-intersection of their *spatial trajectories*. Our goal here is simply to extend this to the setting of directed random walks decorated with diamonds, and state some properties tailored to our needs. The key object to derive this statement is the *embedded synchronized system of directed random walks*. We define it in the next section and derive the key input for the study, which the Local Limit Theorem 4.5.5.

4.5.1 Synchronized directed random walks

As observed in Sections 4.3 and 4.4 our arguments are often soft enough to boil down to the study of a *synchronized* system of walks, where the time reference is still random but common to every walk.

Definition 4.5.4 (Synchronized directed random walk)**.** Let $r \geq 1$ be an integer. We consider a sequence of independent and identically distributed random variables $(\theta_k, X_k^1, \dots, X_k^r)_{k\geq 0}$ taking values in $\mathbb{N}^* \times \mathbb{Z}^r$. Moreover we assume that

- Both θ_1 and X_1^1 have an exponential moment
- Conditionally on θ_1, $X_1^1, \dots, X_1^r$ are centered, independent and identically distributed.

We call:

$$T_n = \sum_{k=1}^{n} \theta_k \quad \text{and} \quad Z_n^i = \sum_{k=1}^{n} X_k^i$$

Then the system

$$(S_n)_{n\geq 0} = \left(T_n, Z_n^1, \dots, Z_n^r\right)_{n\geq 0}$$

is called a *synchronized system of directed random walks*. In what follows, we see it as a random object of $\mathbb{N} \times \mathbb{Z}^r$, which we will refer to as $(S_n)_{n\geq 0} = (T_n, Z_n)_{n\geq 0}$. For any $(k, x) \in \mathbb{N} \times \mathbb{Z}^r$, we will denote by $\mathbb{P}_{(k,x)}$ the law of the synchronized r-random walk started from the point (k, x), *i.e.* the law of $((k, x) + S_n)_{n\geq 0}$.

We also assume for convenience that Assumption 4.5.2 holds.

Introduce the following hitting event, for any $(n, y) \in \mathbb{N} \times \mathbb{Z}^r$:

$$\mathsf{Hit}_{(n,y)} = \{\exists k \geq 0, S_k = (n, y)\},$$

and the stopping time

$$H_{(n,y)} = \min\{k \geq 0, S_k = (n, y)\}.$$

Moreover, ρ will denote the stopping time corresponding to the first exit of the Weyl chamber:

$$\rho = \min\{n \geq 0, S_n \notin W\}$$

The key results of this section are the following:

Theorem 4.5.5 (Local limit Theorem for synchronized, non-intersecting directed random walks)**.** *Let $(S_n)_{n\geq 0}$ be a synchronized system of random walks. There exists a function $V : W \to \mathbb{R}_+^*$ and a constant $C_1 > 0$ such that for any pair of sequences $(x_n)_{n\geq 0}, (y_n)_{n\geq 0}$ taking values in W such that $\|x_n\|_2, \|y_n\|_2 = o(\sqrt{n})$, when $n \to \infty$,*

$$\mathbb{P}_{(0,x_n)}\left[H_{(n,y_n)} < \rho, \mathsf{Hit}_{(n,y_n)}\right] = C_1 \frac{V(x_n)V(y_n)}{n^{r^2/2}}(1 + o(1)).$$

Furthermore, the function V satisfies the following set of properties:

1. *If $x, y \in W$ are such that $|y_{i+1} - y_i| > |x_{i+1} - x_i|$ for any $1 \leq i \leq r-1$, then*

$$V(y) \geq V(x).$$

2. *When* $\mathsf{Gap}(x) \to \infty$*, then* $\frac{V(x)}{\Delta(x)} \to 1$.

3. *There exists a positive* $c > 0$ *such that*

$$V(x) \le c \prod_{1 \le i < j \le r} |1 + x_j - x_i|$$

The second local limit result is the analog of Gnedenko's Local Limit Theorem. It corresponds to [44, Thm. 5].

Theorem 4.5.6. *Let* $(S_n)_{n \ge 0}$ *be a synchronized system of random walks. Then, there exists a constant* $\varkappa > 0$ *such that for any fixed* $x \in W$*,*

$$\sup_{y \in W} \left| n^{\frac{r(r+1)}{4}} \mathbb{P}_{(0,x)} \left[S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)} \right] - \varkappa V(x) \Delta\left(\frac{y}{\sqrt{n}}\right) \mathrm{e}^{-\frac{\|y\|_2^2}{2n}} \right| \xrightarrow[n\to\infty]{} 0.$$

The last theorem of this section is the invariance principle stating that a synchronized system of random walks conditioned on the events $\{H_{(n,y_n)} < \rho\}$ and $\{\mathsf{Hit}_{(n,y_n)}\}$ converges towards the Brownian watermelon.

Theorem 4.5.7 (Invariance principle for synchronized, non-intersecting random walks)**.** *Let* $(S_n)_{n \ge 0}$ *be a system of* r *synchronized random walks. We study the trajectory of* S *on* $[0, H_{(n,y)}]$ *under the measure*

$$\mathbb{P}_{(0,x)} \left[\ \cdot \ | H_{(n,y)} < \rho, \mathsf{Hit}_{(n,y)} \right].$$

Let $\mathfrak{T}$ *be the linear interpolation between the points* $(T_1, S_1), \ldots, (n, y)$*, and* $S(t)$ *be the almost surely unique intersection* $\mathfrak{T} \cap (\{t\} \times \mathbb{R}^r)$*. Then, there exists* $\sigma > 0$ *such that:*

$$\left(\frac{1}{\sqrt{n}} S(nt)\right)_{0 \le t \le 1} \xrightarrow[n\to\infty]{} \left(\sigma \mathsf{BW}_t^{(r)}\right)_{0 \le t \le 1},$$

The convergence occurs in the space $\mathcal{C}([0,1], \mathbb{R}^r)$ *endowed with the topology of uniform convergence. Moreover, the convergence holds when* x, y *depend on* n*, still as long as their norm is* $o(\sqrt{n})$*.*

Theorems 4.5.5, 4.5.6 and 4.5.7 have already been derived in the works [56], [43, 44] and most importantly in [54] in the case of *regular* random walks, meaning that $\theta_1 = 1$ almost surely. Moreover it has been explained in great detail how to adapt the proofs of these articles to the case of *directed* walks in [84]. For that reason, a very brief sketch of proof of these three important results is deferred to the Appendix.

We also import fast repulsion estimates that are going to be useful later on

Lemma 4.5.8 (Edge repulsion for synchronized random walks)**.** *There exists $\varepsilon > 0$ such that the following holds. Let $\left(S_n^i\right)_{n\geq 0, 1\leq i\leq r}$ be a synchronized system of directed random walks. Let*

$$\eta_n = \min\left\{k \geq 0, \min_{1\leq i<j\leq r}\left|S_k^i - S_k^j\right| > n^{\varepsilon}\right\}. \tag{4.51}$$

Then there exists $c > 0$ such that for any $x \in \mathbb{R}^r$, when n is sufficiently large,

$$\mathbb{P}_{(0,x)}\left[\eta_n > n^{1-\varepsilon}\right] < \tfrac{1}{c}\exp(-cn^{\varepsilon}). \tag{4.52}$$

Proof. This fact has been proved in [43, Lemma 7] in the case of regular random walks, with a stronger statement: indeed, the n^{ε} in the definition of η_n is replaced by $n^{\frac{1}{2}-\varepsilon}$ in the latter paper. We briefly explain how to derive the result in our setting. First, condition on the time increments $(\theta_k)_{k\geq 0}$. The spatial increments become a sequence of independent (though non identically distributed) random variables. However one can check that the proof of [43, Lemma 7] can be *mutatis mutandi* repeated in that setting. □

Remark 4.5.9. Since this probability in (4.34) is stretch-exponentially small, the bound also holds - up to a change in the constant c - when conditioning the synchronized system of random walks on an event of polynomial probability. In particular, the next corollary follows from (4.34) and Theorem 4.5.5 (which will be proved shortly without the use of the statement below).

Corollary 4.5.10. *There exists $\varepsilon > 0$ such that the following holds. Let $\left(S_n^i\right)_{1\leq i\leq r, n\geq 0}$ be a synchronized system of directed random walks. Then there exists $c > 0$ such that for any $x, y \in \mathbb{R}^r$, when n is sufficiently large,*

$$\mathbb{P}_{(0,x)}\left[\eta_n > n^{1-\varepsilon} | \mathsf{Hit}_{(n,y)}\right] < \tfrac{1}{c}\exp(-cn^{\varepsilon}).$$

Using the input given by the Local Limit Theorem 4.5.5, we are now able to derive the essential bulk repulsion for non-intersecting synchronized random walks in the next two lemmas.

Lemma 4.5.11. *Let S be a system of directed synchronized random walks and fix $\varepsilon > 0$. Then, for any $\delta > 0$ sufficiently small, any points $x, y \in W$ satisfying $\|x\|_2, \|y\|_2 = o(\sqrt{n})$, there exist $\beta > 0, C > 0$ such that for any $n \geq 0$ sufficiently large,*

$$\mathbb{P}_{(0,x)}\left[\exists t \in [n^{\varepsilon}, n - n^{\varepsilon}], \mathsf{Gap}(S(t)) \leq n^{\delta} \middle| H_{(n,y)} < \rho, \mathsf{Hit}_{(n,y)}\right] \leq Cn^{-\beta}. \tag{4.53}$$

Proof. First, notice that one can actually examine only integer values of t in (4.35) since the minimal distance between two synchronized piecewise linear functions is achieved at a slope change time, which by definition of S is an integer. Introduce the following kernel:

$$q_n(x, y) = \mathbb{P}_{(0,x)}\left[S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right]. \tag{4.54}$$

By the union bound it is sufficient to prove that

$$\mathbb{P}_{(0,x)}\left[\exists k \in \{n^\varepsilon, \dots, n-n^\varepsilon\}, |S^i(k) - S^{i-1}(k)| \le n^\delta \,\middle|\, S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \le Cn^{-\beta}.$$

for any $2 \le i < r$. Fix such an i and introduce the following subset of W

$$W_{n,\delta} = \left\{u \in W, |u_i - u_{i-1}| < n^\delta\right\}.$$

We make use of Theorems 4.5.5 and 4.5.6. Indeed, choose n large enough so that for $n^\varepsilon < k < n - n^\varepsilon$, one has that for any $u \in W_{n,\delta}$:

$$\begin{cases} q_n(x,y) & \ge (1-\varepsilon)V(x)V(y)n^{-\frac{r^2}{2}} \\ q_k(x,u) & \le 2V(x)\Delta\left(\frac{u}{\sqrt{k}}\right)k^{-\frac{r(r+1)}{4}}\mathrm{e}^{-\frac{\|u\|_2^2}{2k}} \\ q_{n-k}(u,y) & \le 2V(y)\Delta\left(\frac{u}{\sqrt{n-k}}\right)(n-k)^{-\frac{r(r+1)}{4}}\mathrm{e}^{-\frac{\|u\|_2^2}{2(n-k)}}. \end{cases}$$

Then, a union bound over k yields:

$$\begin{aligned} &\mathbb{P}_{(0,x)}\left[\exists k \in \{n^\varepsilon, \dots, n-n^\varepsilon\}, \left|S^{i+1}(k) - S^i(k)\right| \le n^\delta \,\middle|\, S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \\ \le\ & \sum_{k=n^\varepsilon}^{n-n^{1-\varepsilon}} \sum_{u \in W_{n,\delta}} \frac{q_k(x,u)q_{n-k}(u,y)}{q_n(x,y)} \\ \le\ & \tfrac{4}{1-\varepsilon} \sum_{k=n^\varepsilon}^{n-n^{1-\varepsilon}} \sum_{u \in W_{n,\delta}} n^{\frac{r^2}{2}}(k(n-k))^{-\frac{r(r+1)}{4}}\Delta\left(\tfrac{u}{\sqrt{k}}\right)\Delta\left(\tfrac{u}{\sqrt{n-k}}\right)\mathrm{e}^{-\frac{\|u\|_2^2}{2}\left(\frac{1}{k}+\frac{1}{n-k}\right)}. \end{aligned}$$

We make two observations: the first is that this sum is actually symmetric around $\frac{n}{2}$, so that it is sufficient to bound it for k going from n^ε to $\frac{n}{2}$. The second is that since $u \in W_{n,\delta}$, we have:

$$\Delta\left(\tfrac{u}{\sqrt{k}}\right) \le 2\,\|u\|_2^{\frac{r(r-1)}{2}-1}\, n^\delta\, k^{-\frac{r(r-1)}{4}}.$$

Then,

$$\begin{aligned} &\mathbb{P}_{(0,x)}\left[\exists k \in \{n^\varepsilon, \dots, n-n^\varepsilon\}, |S^{i+1}(k) - S^i(k)| \le n^\delta \,\middle|\, \mathsf{Hit}_{(n,y)}, \tau > H_{(n,y)}\right] \\ &\qquad\qquad \le \tfrac{32}{1-\varepsilon} \sum_{k=n^\varepsilon}^{\frac{n}{2}} \left(\tfrac{n}{k(n-k)}\right)^{\frac{r^2}{2}} n^{2\delta} \underbrace{\sum_{u \in W_{n,\delta}} \|u\|_2^{r(r-1)-2}\,\mathrm{e}^{-\frac{\|u\|_2^2}{2k}}}_{I}. \end{aligned}$$

We then evaluate the order of the sum I. Indeed, let us write:

$$I = \sum_{\ell \ge 0} \sum_{\substack{u \in W_{n,\delta} \\ \|u\|_2 = \ell}} \ell^{r(r-1)-2}\mathrm{e}^{-\frac{\ell^2}{2k}} \lesssim \sum_{\ell \ge 0} n^\delta \ell^{r-2}\ell^{r(r-1)-2}\mathrm{e}^{-\frac{\ell^2}{2k}},$$

where we have used the fact that when $\ell \to \infty$, if $B_\ell(0)$ denotes the $\|\cdot\|_2$ ball of $\mathbb{R}^r$ centered at 0 and of radius ℓ, then

$$|W_{n,\delta} \cap \partial B_\ell(0)| \lesssim n^\delta \ell^{r-2}. \tag{4.55}$$

We then compare the latter sum with the integral $\int_{x=0}^{\infty} x^{r^2-4} e^{-\frac{x^2}{2k}} \mathrm{d}x$, which after the change of variables $t = \frac{x^2}{2k}$, can be explicitly evaluated:

$$\int_{x=0}^{\infty} x^{r^2-4} e^{-\frac{x^2}{2k}} \mathrm{d}x = \sqrt{2}^{r^2-5} \Gamma\left(\tfrac{r^2-3}{2}\right) k^{\frac{r^2-3}{2}}.$$

Inserting this into our previous computation we find

$$\mathbb{P}_{(0,x)}\left[\exists k \in \{n^\varepsilon, \dots, n-n^\varepsilon\}, |S^{i+1}(k) - S^i(k)| \le n^\delta \,\middle|\, S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right]$$
$$\le C \sum_{k=n^\varepsilon}^{n/2} \left(\tfrac{n}{k(n-k)}\right)^{\frac{r^2}{2}} n^{3\delta} k^{\frac{r^2-3}{2}} \le C n^{3\delta} \sum_{k=n^\varepsilon}^{\infty} k^{-\frac{3}{2}} \le C n^{3\delta} n^{-\frac{\varepsilon}{2}}.$$

Hence, whenever $\delta < \frac{\varepsilon}{6}$, this probability decays polynomially, as announced. □

In the proofs of Sections 4.3 and 4.4, we used this lemma under a slightly different form that we state now.

Lemma 4.5.12. *Let S be a directed system of synchronized random walks and $\varepsilon > 0$. Let x_n, y_n two sequences of elements of W such that*

$$\min\{\mathsf{Gap}(x_n), \mathsf{Gap}(y_n)\} \ge n^\varepsilon \quad \text{and} \quad \|x_n\|_2, \|y_n\|_2 = o(\sqrt{n}).$$

Then, for any $\delta > 0$ sufficiently small, there exist $\beta > 0$ and $C > 0$ such that for $n \ge 0$ large enough,

$$\mathbb{P}_{(0,x_n)}\left[\inf_{0 \le t \le n} \mathsf{Gap}(S(t)) \le n^\delta \,\middle|\, S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y_n)}\right] \le C n^{-\beta}.$$

Proof. All the work has been done in Lemma 4.5.11. Indeed, we already know that

$$\mathbb{P}_{(0,x_n)}\left[\inf_{n^\varepsilon \le t \le n-n^\varepsilon} \mathsf{Gap}(S(t)) \le n^\delta \,\middle|\, S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y_n)}\right] \le C n^{-\beta}.$$

It remains to control the range of indexes $k \in \{1, \dots, n^\varepsilon\} \cup \{n-n^\varepsilon, \dots, n\}$ (observe that we cannot make use of the local limit theorems in this range). However it is a basic large deviations estimate: let us write it for $k \in \{0, \dots, n^\varepsilon\}$. We roughly bound

$$\mathbb{P}_{(0,x_n)}\left[\inf_{0 \le k \le n^\varepsilon} \mathsf{Gap}(S(k)) \le n^\delta \,\middle|\, S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y_n)}\right]$$
$$\le \frac{\mathbb{P}_{(0,x_n)}\left[\inf_{0 \le k \le n^\varepsilon} \mathsf{Gap}(S(k)) \le n^\delta\right]}{\mathbb{P}_{(0,x_n)}\left[S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y_n)}\right]}.$$

Now observe that for the event of the numerator to occur, one of the walks has to travel at a distance at least $\frac{1}{2}(n^\varepsilon - n^\delta)$ of its starting point in a time n^ε, which by large deviations occurs with stretched exponentially small probability as soon as $\delta < \varepsilon$. Additionally, by Theorem 4.5.5, the denominator is of order at most polynomial. Thus

$$\mathbb{P}_{(0,x_n)}\left[\inf_{0\le k\le n^\varepsilon} \mathsf{Gap}(S(k)) \le n^\delta \,\middle|\, S\in\mathcal{W}_n, \mathsf{Hit}_{(n,y_n)}\right] \le C\mathrm{e}^{-cn^\varepsilon},$$

for constants $c, C > 0$. The same holds for $k \in \{n - n^\varepsilon, \dots, n\}$, and the union bound provides the desired result. □

4.5.2 Synchronized systems of random walks with random decorations

To prove Theorem 4.5.3, we are going to compare a system of decorated non-intersecting random bridges with a system of decorated non-intersecting synchronized random bridges. This motivates us to study the properties of such a system. Recall the definition of $\mathcal{D}(\mathbf{S}^i)$ form the precedent section. When S is a synchronized system of directed walks, we simply set

$$\mathcal{D}^\delta_{i,k} = \mathcal{D}^\delta_{(T_k,S^i_k),(T_{k+1},S^i_{k+1})}.$$

and

$$\mathcal{D}(S^i) = \bigcup_{k\ge 0} \mathcal{D}^\delta_{i,k}.$$

The crucial result of this section is the following lemma - adapted from [103, Lemma 2.7]

Lemma 4.5.13. *Let $\delta > 0$, and $x, y \in W$. Then, there exists $c > 0$ such that:*

$$\mathbb{P}_{(0,x)}\left[S\in\mathsf{Hit}_{(n,y)}, \mathsf{NonIntDiam}(S)\right] > c\mathbb{P}_{(0,x)}\left[S\in\mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right].$$

Proof. We need to prove that there exists some $c < 1$ such that

$$\mathbb{P}_{(0,x)}\left[\exists 1\le i<j\le r, \mathcal{D}(S^i)\cap\mathcal{D}(S^j)\neq\emptyset | S\in\mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \le c.$$

By the union bound, the latter probability is lesser or equal than

$$\sum_{1\le i\le r-1} \mathbb{P}_{(0,x)}\left[\mathcal{D}(S^i)\cap\mathcal{D}(S^{i+1})\neq\emptyset | S\in\mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right],$$

and we now focus on the terms of this sum. Introduce the following family of events (recall that $\theta_k = T_{k+1} - T_k$) :

$$\mathcal{L}_k = \left\{\left|S^{i+1}_k - S^i_k\right| < 2\delta\theta_{k+1}\right\}.$$

Observe that due to the cone-confinement property, if $\left\{\mathcal{D}(S^{i+1}) \cap \mathcal{D}(S^i) \neq \emptyset\right\}$, then one of the $\mathcal{L}_k$ must occur. Now we call N the total number of steps. There exists a constant $\mu := \mathbb{E}[\theta_1]^{-1}$ such that $N \in [(\mu-\varepsilon)n, (\mu+\varepsilon)n]$ with exponentially large probability in n. For sake of simplicity, we continue the computation assuming that $N = \mu n$. Formally one should sum over all the possible values of N in the latter range, but it makes no difference in the proof. We even only treat the special case $\mu = 1$, as a general μ would only modify the constants inside our estimates but not the dependency in n. Let $T > 0$ be a large integer, that will be fixed later. We first argue that there exists a constant $c_1 > 0$ which only depends on δ such that

$$\mathbb{P}_{(0,x)}\left[\bigcup_{k=1}^{n} \mathcal{L}_k | S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \leq \mathrm{e}^{+c_1 T} \mathbb{P}_{(0,x)}\left[\bigcup_{k=T}^{n-T} \mathcal{L}_k | S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right].$$

This is a finite-energy property, the fact that c_1 is uniform over T comes from the cone-confinement property. Then by union bound, let us write:

$$\begin{aligned}
& \mathbb{P}_{(0,x)}\left[\bigcup_{k=T}^{n-T} \mathcal{L}_k | S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \\
\leq\ & \sum_{k=T}^{n-T} \mathbb{P}_{(0,x)}\left[\mathcal{L}_k | S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \\
\leq\ & \sum_{k=T}^{n-T}\sum_{\ell=1}^{n-k} \mathbb{P}_{(0,x)}\left[\mathcal{L}_k, \theta_{k+1} = \ell | S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \\
\leq\ & \sum_{k=T}^{n-T}\sum_{\ell=1}^{n-k} \mathbb{P}_{(0,x)}\left[\left|S_k^{i+1} - S_k^i\right| < 2\delta\ell, \theta_{k+1} = \ell | S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \\
\leq\ & \sum_{k=T}^{n-T}\sum_{\ell=1}^{n-k}\sum_{u \in W_{\delta\ell}}\sum_{v \in W} \mathrm{e}^{-c_1\ell}\mathrm{e}^{-c_2\|u-v\|_2} \frac{q_k(x,u)q_{n-l-k}(v,y)}{q_n(x,y)},
\end{aligned}$$

where as in the proof of Lemma 4.5.12, we have introduced the kernel

$$q_n(x,y) = \mathbb{P}_{(0,x)}\left[S \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right],$$

and the notation

$$W_{\delta\ell} = \{u \in W, |u_{i+1} - u_i| < \delta\ell\}.$$

Moreover, we also used the property that both the random variables θ_k and $\check{X}_k$ have an exponential moment. We now use the same technique as in Lemma 4.5.11 and choose $T > 0$ large enough (uniformly of everything else) to upper bound the latter quantity, using Theorems 4.5.5 and 4.5.6:

$$\mathbb{P}_{(0,x)}\left[\bigcup_{k=T}^{n-T}\mathcal{L}_k|S\in\mathcal{W}_n,\mathsf{Hit}_{(n,y)}\right]$$
$$\leq \frac{2C}{1-\varepsilon}\sum_{k=T}^{n/2}\left(\frac{n}{k(n-k)}\right)^{\frac{r^2}{2}}\sum_{\ell=1}^{n-k}\mathrm{e}^{-c_1\ell}(\delta\ell)^2\underbrace{\sum_{u\in W_{\delta\ell}}\|u\|_2^{r(r-1)-2}\,\mathrm{e}^{-\frac{\|u\|_2^2}{2k}}}_{I}.$$

As in the proof of Lemma 4.5.11, we now estimate the sum I. Here, we will use crucially the fact that we sum over $W_{\delta\ell}$ and not over W. We write

$$\begin{aligned} I &= \sum_{s\geq 0}\sum_{\substack{u\in W_{\delta\ell}\\ \|u\|_2=s}} s^{r(r-1)-2}\mathrm{e}^{-\frac{r^2}{2k}} \\ &\leq C\delta\ell\sum_{s\geq 0}s^{r-2}s^{r(r-1)-2}e^{-\frac{r^2}{2k}}. \end{aligned}$$

We used once again the estimation (4.37) for the volume of the set we are summing over. As before, we compare this sum to the integral $I_2=\int_{x=0}^{\infty}x^{r^2-4}\mathrm{e}^{-\frac{x^2}{2k}}\,\mathrm{d}x$, which, after the appropriate change of variables $t=\frac{x^2}{2k}$, can be explicitly computed, yielding

$$I_2=\sqrt{2}^{r^2-5}\Gamma(\frac{r^2-3}{2})k^{\frac{r^2-3}{2}}.$$

Continuing our previous computation, we obtain that:

$$\mathbb{P}_{(0,x)}\Big[\bigcup_{k=T}^{n-T}\mathcal{L}_k|S\in\mathcal{W}_n,\mathsf{Hit}_{(n,y)}\Big]$$
$$\begin{aligned} &\leq \frac{2\widetilde{C}}{1-\varepsilon}\sum_{k=T}^{n/2}\left(\frac{n}{k(n-k)}\right)^{\frac{r^2}{2}}k^{\frac{r^2-3}{2}}\sum_{\ell=1}^{n-k}\mathrm{e}^{-c_1\ell}(\delta\ell)^3 \\ &\leq \frac{2\widetilde{C}}{1-\varepsilon}\sum_{k=T}^{n/2}k^{-\frac{r^2}{2}+\frac{r^2}{2}-\frac{3}{2}} \\ &\leq \frac{2\widetilde{C}}{1-\varepsilon}T^{-\frac{1}{2}}. \end{aligned}$$

Chose $T>0$ large enough so that quantity is smaller than $\frac{1}{2}$. We then showed that:

$$\mathbb{P}\left[\bigcap_{k=1}^{n}\mathcal{L}_k^c|S\in\mathcal{W}_n,\mathsf{Hit}_{(n,y)}\right]\geq\frac{1}{2}\mathrm{e}^{-c_1T},$$

which conclude the proof, since $T>0$ has been chosen uniformly of n. □

4.5.3 Non-intersecting systems of decorated directed random walks

The goal of this section is to transmit the result of Section 4.5.1 to the setting of *non-synchronized* random walks. For that, we will interpret such a system as an embedded synchronized random walk carrying random decorations, and use the results of the precedent section.

Before diving into the proof, we introduce the "embedded system of synchronized random walks" of a system of random walks.

Definition 4.5.14 (Embedded system of synchronized random walks)**.** Let $(\mathbf{S}_n) = (\mathbf{T}_n, \mathbf{Z}_n)$ be a system of non-synchronized directed random walks. Introduce the random set of synchronization times:

$$\mathsf{ST} = \left\{ \ell \geq 0, \exists k_1(\ell), \dots, k_p(\ell) \geq 0, \mathbf{T}^1_{k_1} = \dots = \mathbf{T}^r_{k_p} = \ell \right\}.$$

Writing $\mathsf{ST} = \{\ell_1 < \dots < \ell_r < \dots\}$, we define the "embedded system of synchronized random walks" to be the process:

$$(\check{\mathbf{S}}_n)_{n\geq 0} = \left(\ell_n, \mathbf{Z}^1_{k_1(\ell_n)}, \dots, \mathbf{Z}^r_{k_p(\ell_n)}\right)_{n\geq 0}.$$

Observe that in particular the trajectory of $\check{\mathbf{S}}$ is a subset of the trajectory of $\mathbf{S}$, and that by definition the system $\check{\mathbf{S}}$ is synchronized.

Lemma 4.5.15. *The process $\check{\mathbf{S}}$ is a synchronized system of random walks (recall Definition 4.5.4). Moreover, for any $i \in \{1, \dots, r\}$, $\mathcal{D}(\mathbf{S}^i) \subset \mathcal{D}(\check{\mathbf{S}}^i)$.*

Proof. All the statements are easy to check, the exponential tails of the length being a consequence of the Renewal Theorem of [57]. □

Lemma 4.5.16. *There exists a positive $c > 0$ such that for any fixed $x, y \in W$,*

$$\mathbf{P}_{(0,x)}\left[\mathsf{NonIntDiam}(\mathbf{S}), \mathsf{Hit}_{(n,y)}\right] > c\frac{V(x)V(y)}{n^{\frac{r^2}{2}}}.$$

Proof. Observe that

$$\mathbf{P}_{(0,x)}\left[\mathsf{NonIntDiam}(\mathbf{S}), \mathsf{Hit}_{(n,y)}\right] \geq \mathbb{P}_{(0,x)}\left[\mathsf{NonIntDiam}(\check{\mathbf{S}}), \mathsf{Hit}_{(n,y)}\right],$$

so that we focus on lower bounding the right-hand side. We are in the setting of Lemma 4.5.13, allowing us to write:

$$\begin{aligned}\mathbb{P}_{(0,x)}\left[\mathsf{NonIntDiam}(\check{\mathbf{S}}), \mathsf{Hit}_{(n,y)}\right] &\geq c\mathbb{P}_{(0,x)}\left[\check{\mathbf{S}} \in \mathcal{W}_n, \mathsf{Hit}_{(n,y)}\right] \\ &\geq cV(x)V(y)n^{-\frac{r^2}{2}},\end{aligned}$$

where the first inequality comes from Lemma 4.5.13 and the second one comes from the fact that $\check{S}$ has the distribution of a synchronized system of random walks, so that Theorem 4.5.5 applies. ☐

Remark 4.5.17. The exact same technique of proof can be used to show an analog of Lemma 4.5.12 for non-synchronized random walks. Indeed the probability of two non-synchronized random walks coming close one from each other can be upper bounded by the probability of two decorated synchronized random walks coming close one from each other. Making use of Lemmas 4.5.12 and 4.5.16 we obtain:

Lemma 4.5.18. *Let $\mathbf{S}$ be a system of non-synchronized random walks. Let x, y two sequences of elements of W such that*

$$\mathsf{Gap}(x), \mathsf{Gap}(y) \geq n^{\varepsilon}$$

and

$$\|x\|_2, \|y\|_2 = o(\sqrt{n}).$$

Then for any $\delta > 0$ sufficiently small,, there exists $\beta > 0$, $C > 0$ such that for $n \geq 0$ large enough,

$$\mathbf{P}_{(0,x)}\left[\inf_{1\leq k\leq n} \mathsf{Gap}(\mathbf{S}_k) \leq n^{\delta} \mid \mathbf{S} \in \mathsf{Hit}_{(n,y)}, \mathsf{NonIntDiam}(\mathbf{S})\right] \leq Cn^{-\beta}.$$

The next step in our way to the proof of Theorem 4.5.3 is then to show a fast repulsion estimate near the starting and ending points stated in Lemma 4.5.8 in the setting of non-synchronized systems of non-intersecting bridges. Let $\varepsilon > 0$. As in Sections 4.3 and 4.4 we introduce the following times:

$$T_1(\mathbf{S}) = \min_{k\geq 0} \{k \geq 0, \mathsf{Gap}(\mathbf{S}_k) > n^{\varepsilon}\}$$

and

$$T_2(\mathbf{S}) = \max_{k\geq 0} \{k \geq 0, \mathsf{Gap}(\mathbf{S}_k) > n^{\varepsilon}\}.$$

Lemma 4.5.19. *There exists $\varepsilon > 0$ sufficiently small such that there exists a positive constant $c > 0$ such that:*

$$\mathbf{P}_{(0,x)}\left[T_1(\mathbf{S}) > n^{1-\varepsilon}, T_2(\mathbf{S}) < n - n^{1-\varepsilon} | \mathbf{S} \in \mathsf{Hit}_{(n,y)}, \mathsf{NonIntDiam}(\mathbf{S})\right] < \frac{1}{c}\exp\left(-cn^{\varepsilon}\right),$$

where $\mathbf{P}_{(0,x)}$ is the distribution of a non-synchronized system of directed random walks.

Proof. Recall that $\check{\mathbf{S}}$ denotes the synchronized system of random walks embedded in $\mathbf{S}$. Then,

$$\begin{aligned}\mathbf{P}_{(0,x)}\big[T_1(\mathbf{S}) > n^{1-\varepsilon}, &T_2(\mathbf{S}) < n - n^{1-\varepsilon} | \mathbf{S} \in \mathsf{Hit}_{(n,y)}, \mathsf{NonIntDiam}(\mathbf{S})\big] \\ &\leq \mathbf{P}_{(0,x)}\left[T_1(\check{\mathbf{S}}) > n^{1-\varepsilon}, T_2(\check{\mathbf{S}}) < n - n^{1-\varepsilon} | \mathbf{S} \in \mathsf{Hit}_{(n,y)}, \mathsf{NonIntDiam}(\mathbf{S})\right] \\ &\leq 2\frac{\mathbf{P}_{(0,x)}\left[T_1(\check{\mathbf{S}}) > n^{1-\varepsilon}\right]}{\mathbf{P}_{(0,x)}\left[\mathbf{S} \in \mathsf{Hit}_{(n,y)}, \mathsf{NonIntDiam}(\mathbf{S})\right]} \\ &\leq \frac{2}{c}\exp(-cn^{\varepsilon})n^{\frac{r^2}{2}}(V(x)V(y))^{-1},\end{aligned}$$

which proves the lemma for another constant $c' < c$ provided that n is large enough. The last inequality comes from Lemma 4.5.8 for upper bounding the numerator, and from Lemma 4.5.16 for lower bounding the denominator. □

We are now ready to prove Theorem 4.5.3. The technique is very similar to the proof of Theorem 4.1.11. Indeed, we shall wait for a sublinear time that the walks attain a gap of order n^{ε}. After this time, we know that - looking at the process as a system of synchronized decorated random walks - the diamonds are very likely not to intersect so that the convergence of the synchronized embedded system towards the Brownian watermelon can be transmitted to the whole system.

Proof of Theorem 4.5.3. Let $\mathbf{S}$ be sampled according to the measure

$$\mathbf{P}_{(0,x)}\left[\,.\,\middle|\mathbf{S} \in \mathsf{NonIntDiam}(\mathbf{S}), \mathsf{Hit}_{(n,y)}\right].$$

Again, since we are going to work between the random times T_1 and T_2, we need to implement the strategy given by Lemma 4.3.10. Let $\delta > 0$ and $f^{\delta} : \mathcal{C}([\delta, 1-\delta], \mathbb{R}^r) \to \mathbb{R}$, continuous and bounded. Introduce $\mathbf{S}_n(t)$ the scaled version of $\mathbf{S}$:

$$\mathbf{S}_n(t) = \frac{1}{\sqrt{n}}\mathbf{S}(nt).$$

Our goal is to show that (we keep implicit the restrictions of $\mathbf{S}_n$ and $\mathsf{BW}^{(r)}$ to the interval $[\delta, 1-\delta]$):

$$\mathbb{E}\left[f^{\delta}(\mathbf{S}_n)|\mathbf{S} \in \mathsf{NonIntDiam}(\mathbf{S}), \mathsf{Hit}_{(n,y)}\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[f^{\delta}(\sigma\mathsf{BW}^{(r)})\right].$$

We claim that - thanks to Lemma 4.5.19 and the usual deviation argument for random walks - with probability $1 + o(1)$, there exist $T_1 > 0$ and $T_2 < n$ two random times such that T_1 and T_2 are synchronization times for $\mathbf{S}$, and such that

$$T_1 < 2n^{1-\varepsilon} \text{ and } T_2 > n - 2n^{1-\varepsilon}, \tag{4.56}$$

and

$$\begin{cases} \|\mathbf{S}(T_1)\|_2, \|\mathbf{S}(T_2)\|_2 = o(\sqrt{n}), \\ \min\{\mathsf{Gap}(\mathbf{S}(T_1)), \mathsf{Gap}(\mathbf{S}(T_2))\} > \frac{1}{2}n^{\varepsilon}. \end{cases} \tag{4.57}$$

In the rest of the proof, we then condition on the values of $T_1, T_2, \mathbf{S}(T_1)$ and $\mathbf{S}(T_2)$ satisfying (4.38) and (4.39). Moreover for sake of simplicity in the proof let us call $u = (T_1, \mathbf{S}(T_1))$ and $v = (T_2, \mathbf{S}(T_2))$. As soon as n is large enough so that $n\delta > T_1$ and $n(1-\delta) < T_2$, the Markov property for random walks ensures that:

$$\mathbb{E}\Big[f^{\delta}(\mathbf{S}_n | u, v, \mathbf{S} \in \mathsf{NonIntDiam}(\mathbf{S}), \mathsf{Hit}_{(n,y)}\Big] = \mathbb{E}_u\Big[f^{\delta}(\mathbf{S}_n | \mathbf{S} \in \mathsf{NonIntDiam}(\mathbf{S}), \mathsf{Hit}_v\Big],$$

where $\mathbb{E}_u$ denotes the expectation under the measure $\mathbf{P}_u$.

Let us consider $\check{\mathbf{S}}$ to be the synchronized system embedded into $\mathbf{S}$, and $\check{\mathbf{S}}(t)$ be its linear interpolation. By standard estimates on the max of a linear number of independent random variables with exponential tails, one gets:

$$\begin{aligned} \mathbf{P}_{(0,x)}\Big[\sup_{0\le t\le n}\Big|\mathbf{S}(t) - \check{\mathbf{S}}(t)\Big| > \log^2 n | \mathbf{S} \in \mathsf{NonIntDiam}(\mathbf{S}), \mathsf{Hit}_{(n,y)}\Big] \\ \le \frac{\mathbf{P}_{(0,x)}\left[\sup_{0\le t\le n}\left|\mathbf{S}(t) - \check{\mathbf{S}}(t)\right| > \log^2 n\right]}{\mathbf{P}_{(0,x)}\left[\mathbf{S} \in \mathsf{NonIntDiam}(\mathbf{S}), \mathsf{Hit}_{(n,y)}\right]} \\ \le \frac{1}{c}\exp\left(-c(\log^2 n)\right)(V(x)V(y))^{-1} n^{\frac{r^2}{2}}, \end{aligned}$$

where we used Lemma 4.5.16 for the last step. We now work under the event that

$$\sup_{0\le t\le n}\left|\mathbf{S}(t) - \check{\mathbf{S}}(t)\right| > \log^2 n.$$

Hence, for our purpose it is sufficient to show that:

$$\mathbb{E}_u\big[f^{\delta}(\check{\mathbf{S}}_n(t)) | \mathbf{S} \in \mathsf{Hit}_v, \mathsf{NonIntDiam}(\mathbf{S})\big] \xrightarrow[n\to\infty]{} \mathbb{E}\left[f^{\delta}(\sigma \mathsf{BW}^{(r)})\right].$$

The next step is to replace the conditioning over $\mathbf{S}$ belonging to the non-intersection of diamonds and connection event by a conditioning over $\check{\mathbf{S}}$ belonging to the non-intersection and connection event. Indeed, assuming that we managed to show that this change of conditioning was justified, the result would follow by Theorem 4.5.7. Our target estimate is then:

$$\mathbf{P}_u\left[\{\mathsf{NonIntDiam}(\mathbf{S})\}\,\Delta\left\{\check{\mathbf{S}} \in \mathcal{W}_{T_2-T_1}\right\} | \check{\mathbf{S}} \in \mathcal{W}_{T_2-T_1}, \mathsf{Hit}_v\right] \xrightarrow[n\to\infty]{} 0.$$

Observe that because we work under the event $\{\sup_{0\le t\le n}\left|\mathbf{S}(t) - \check{\mathbf{S}}(t)\right| > \log^2 n\}$, then

$$\begin{aligned} \mathbf{P}_u\left[\{\mathbf{S} \in \mathsf{NonIntDiam}(\mathbf{S})\}\,\Delta\left\{\check{\mathbf{S}} \in \mathcal{W}_{T_2-T_1}\right\}\right] \\ \le \mathbf{P}_u\Big[\inf_{t\in[T_1,T_2]}\mathsf{Gap}(\check{\mathbf{S}}(t)) < 4\log^3 n | \check{\mathbf{S}} \in \mathcal{W}_{[T_1,T_2]}, \mathsf{Hit}_v\Big]. \end{aligned}$$

By Lemma 4.5.12 we know that this probability decays to 0 at least polynomially fast. Thus, we proved that

$$\left|\mathbb{E}_u\left[f^\delta(\check{\mathbf{S}}_n(t))|\mathsf{NonIntDiam}(\mathbf{S}),\mathsf{Hit}_v\right] - \mathbb{E}_u\left[f^\delta(\check{\mathbf{S}}_n(t))|\check{\mathbf{S}}\in\mathcal{W}_{[T_1,T_2]},\mathsf{Hit}_v\right]\right| \xrightarrow[n\to\infty]{} 0. \tag{4.58}$$

Now because of (4.38) and (4.39), Theorem 4.5.7 applies and we get that

$$\mathbb{E}_u\left[f^\delta(\check{\mathbf{S}}_n(t))|\check{\mathbf{S}}\in\mathcal{W}_{[T_1,T_2]},\mathsf{Hit}_v\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[f^\delta(\sigma\mathsf{BW}^{(r)})\right]$$

for some $\sigma > 0$. This concludes the proof of the theorem: as previously condition (i) of Lemma 4.3.10 is a simple consequence of basic large deviations estimates. □

Appendix

Non-confinement in small tubes for a single directed random walk

Lemma 4.5.20 (Non-confinement of single directed random walk)**.** *There exists* $\varepsilon_0 > 0$ *such that that the following holds. Fix* $\varepsilon < \varepsilon_0$*. Let* $(S_n)_{n\geq 0}$ *be a directed random walk, and remember that* $S(t)$ *denotes its linear interpolation. Let* $f:\mathbb{R}^+\to\mathbb{R}$ *be any deterministic function. Then, for any* $\alpha\in(0,1]$*, there exists* $c>0$ *such that for any* $x\in\mathbb{R}$*,*

$$\mathbb{P}_{(0,x)}\left[\#\left\{k\in\{0,\ldots,n^{1-\varepsilon}\},|S(k)-f(k)|<n^\varepsilon\right\}>\alpha n^{1-\varepsilon}\right] < \mathrm{e}^{-cn^{1-3\varepsilon}}. \tag{4.59}$$

Limiting the times considered in (4.41) to $k\leq n^{1-\varepsilon}$ rather than the more natural choice $k\leq n$ is done only for coherence with the uses of this statement in other parts of the paper.

Proof. We cut up the interval $\{0,\ldots,n^{1-\varepsilon}\}$ in intervals of alternating lengths $C\frac{1}{2}\alpha n^{2\varepsilon}$ and $C(1-\frac{1}{2}\alpha)n^{2\varepsilon}$ (where C is some fixed constant to be determined). Call these buffer and main intervals. The buffer intervals occupy a proportion $\alpha/2$ of the whole walk, so

$$\begin{aligned}\mathbb{P}_{(0,x)}&\left[\#\left\{k\in\{0,\ldots,n^{1-\varepsilon}\},|S(k)-f(k)|<n^\varepsilon\right\}>\alpha n^{1-\varepsilon}\right]\\ &\leq \mathbb{P}_{(0,x)}\left[\#\{k\in\text{main intervals },|S(k)-f(k)|<n^\varepsilon\}>\tfrac{1}{2}\alpha n^{1-\varepsilon}\right].\end{aligned}$$

Call the indices k considered above “close points”. Call a main interval *bad* if it has a proportion of close points larger than $\alpha/4$. Then, for the above to be realized, one needs a proportion of at least $\alpha/4$ bad main intervals (the good main intervals account for at most $\frac{1}{4}\alpha n^{1-\varepsilon}$ close points).
Condition now on the trajectory in each of main interval. The only randomness comes from the starting positions of these main intervals, which are dictated by the buffer intervals.

One can then check that due to the pigeonhole principle, for each main interval, there are at most $\frac{4}{\alpha}\times n^\varepsilon$ starting positions that render them bad. Thus, just because of the buffer interval preceding each main interval, due to the Central limit Theorem, the probability of a main interval to be bad may be rendered small (smaller than any given constant, by choosing C large enough). Thus choose C so that

$$\mathbb{P}[\text{ one main interval is bad} \mid \text{all the RW except the preceding buffer interval}] \leq \alpha/8.$$

In total we have have $\frac{1}{C}n^{1-3\varepsilon}$ pairs of buffer and main intervals. Each main interval has a probability at least $1-\alpha/8$ to be good, independently of all other. Thus, the probability of having a proportion $\alpha/4$ of bad intervals is a large deviation estimate, and thus has a probability of order $e^{-cn^{1-3\varepsilon}}$ for some constant c that depends on C (itself depending on α). □

A brief sketch of proof of Theorems 4.5.5, 4.5.6 and 4.5.7

As explained previously, Theorems 4.5.5, 4.5.6 and 4.5.7 have already been proved in the case of regular random walks in [43, 44, 54], under a weaker moment assumption and the assumption that the coordinates of the walk are exchangeable — which suits to our setting. It has already been explained in [84] how to transfer Local limit Theorems proved for regular random walks to the case of directed random walks, and the same method applies *mutatis mutandi* to our setting. Indeed, the proof consists in conditioning on the number of steps of the walk called N, an considering three different cases. Indeed, large deviation estimates allow to rule out the case $N \notin [(\mu \pm \varepsilon)n]$, with $\mu := \mathbb{E}[\theta_1]^{-1}$. Then, the contribution of the indexes $N \in [(\mu-\varepsilon)n, \mu n - A\sqrt{n}] \cup [\mu n + A\sqrt{n}, (\mu+\varepsilon)n]$ is shown to be of order $f(A)n^{-\frac{r^2}{2}}V(x)V(y)$, with $f(A) \to 0$ when $A \to \infty$. As explained in [84], the important idea is to perform an exponential tilt of the random walk by the length of its time increments, and to analyze this new measure by standard random walks estimates. The proof finally reduces to the case where N lies in the interval $[\mu n \pm A\sqrt{n}]$, where A is a large constant. The proofs of [44] can then be mimicked. The discussion of [84, Proof of Thm 5.1] — in particular the observation that the harmonic function V does not depend on the time reference — shows that the result is uniform in starting points satisfying $\|x\|, \|y\| = o(\sqrt{n})$. It then may be considered as folklore that Theorems 4.5.5, 4.5.6 and 4.5.7 do hold in the case of synchronized directed random walks.

Chapter 5

Near-critical Ornstein–Zernike theory

5.1 Introduction

Rigorous understanding of the correlation structure of random fields is one of the main objectives of modern statistical mechanics. In this article, we focus on one of the most studied such models : the so-called planar random-cluster measure (also known as FK percolation). The classical Ornstein–Zernike (OZ) formula describes the behaviour of the two-point correlation function $G(x) := \phi_{p,q}[0 \longleftrightarrow x]$ when x goes to infinity, where $\phi_{p,q}$ is the random-cluster measure of parameters p and q, when p is such that the model is in its subcritical (or disordered) phase. It quantifies the behaviour of G beyond its exponential decay and is sharp in the regime in which x tends to infinity.

The OZ formula (it would be fair to speak of OZ theory, as the OZ formula is only one the outputs of the articles to be cited in this introduction) has a very long and rich history. The above-mentioned asymptotic formula was first conjectured in two very influential works by Ornstein and Zernike in 1914 [101], and Zernike [114]. Trying to correct a formula describing the phenomenon of *opalescence* in a crystal, they provided a non-rigorous computation of what would later become the OZ formula. Later on, Abraham and Kunz [2] and Paes–Leme [104] independently were able to derive the first rigorous implementation of Ornstein and Zernike's reasoning, in the context of classical lattice gases theory, by means of a graphical representation of the partition function of the model. The Ornstein–Zernike result for the order of decay of the correlations has then been shown to be true for a number of models *in a perturbative regime*, see [17, 97].

The next breakthrough consisted in a rigorous derivation of the Ornstein–Zernike asymptotic result in the whole regime of exponential decay of the correlation functions, and was done in the case of the self-avoiding walk along a direction given by the axis in [34] and later on in any direction in [82]. For percolation models, the case of Bernoulli percolation was first treated for an on-axis direction in [21] and later on in any direction in [22]. The case of subcritical Ising models was treated in [24] via the random-line graphical representation of the two-point correlation function. Finally, the analysis was carried out for all the subcritical random-cluster measures in [25]. In recent developments, the theory has been extended to Ising models with long-range interactions [10], and a new direction of research has been studied regarding the *failure* of Ornstein–Zernike behaviour in some long-range Ising models, when the coupling constants decay too slowly [8, 9].

All those works are concerned with the understanding of the two-points correlations at a fixed temperature. In this article, we are interested in the way in which the OZ formula behaves when the temperature parameter (p in the random-cluster model) approaches its critical value, in the *planar* setting. In particular, its principal output is the derivation of the asymptotic behaviour of the function $x \mapsto G(x)$ *uniformly both in* p *and* x, in the regime $p < p_c$ and x going to infinity. When $p = p_c$ and $q \in [1, 4]$ in the random-cluster model, the phase transition is known to be continuous and the function G behaves like a polynomial in $\|x\|$. The analysis carried out in this article and the main result derived allow to understand how the two-points correlation function switches from a disordered behaviour (caracterised by its exponential decay in $\|x\|$) to a near-critical behaviour in which the correlations start to decay at a polynomial speed.

The interest of this work is twofold. First, the result is of interest, as it may allow to understand the structure of the correlations in continuous fields constructed as near-critical limits of two dimensional statistical mechanics models (in particular one may think about the so-called planar *Ising magnetisation* field, constructed in [19, 20]); this will be the subject of further investigation. It is also an opportunity to revisit the approach to Ornstein–Zernike theories instigated in the last decades by the above-mentioned authors. Indeed, while our approach roughly follows the lines of [25], several differences may be highlighted. We circumvent the use of the "skeleton calculus" developed by the authors by analysing the structure of the percolation cluster at some fixed length scale L of order of the correlation length. Moreover, and this is the key contribution of this work, we replace all the arguments relying on finite energy type properties at the scale of the correlation length (which are non-uniform when $p \nearrow p_c$) by the box-crossing property given by near-critical RSW theory, and show that the latter is sufficient to obtain the mixing property needed in order to extract the random walk-type behaviour of the subcritical cluster.

We close the introduction by mentioning that the closely related question of understanding the behaviour of the Wulff crystal of the Ising model when $\beta \searrow \beta_c$ was raised in [30]. In this work, the authors proved that the Wulff construction remains valid when both the size of the box goes to infinity and the inverse temperature goes to β_c. In the context of the random-cluster model, our work allows to improve their result all the way up to the critical exponent $\nu = 1$, as the planar Ising model is integrable and all its critical exponents have been computed explicitly.

5.1.1 Definition of the model

We briefly define the model together with the basic notions that shall be used extensively in this article. We refer to the monographs [70, 49] for further background on the model.

This work is concerned with the *planar* random-cluster model. From now on, the dimension of the lattice will be fixed and equal to 2. We slightly abuse notation by writing $\mathbb{Z}^2 = (\mathbb{Z}^2, E(\mathbb{Z}^2))$ for the square lattice. For $G = (V(G), E(G))$ a finite subgraph of $\mathbb{Z}^2$, the space of percolation configurations on G is $\Omega^G := \{0,1\}^{E(G)}$. For an edge $e \in E(G)$ and $\omega \in \Omega^G$, we say that e is *open* if $\omega(e) = 1$ and *closed* else. A percolation configuration will be identified both with the set of its open edges as well as with the sub-graph of G with vertex set $V(G)$ and edge-set formed of the open edges of ω. In particular, we say that $x, y \in V(G)$ are connected in ω if there exists a sequence of vertices $x = x_1, \dots, x_k, x_k = y$ such that for any $1 \leq j < k$, $\|x_{j+1} - x_j\| = 1$ and the edge $\{x_j, x_{j+1}\}$ is open in ω. The maximal connected components for this notion of connectivity shall be called *open clusters* of ω.

The boundary ∂G of G is the set of vertices of G incident to at least one edge of $E(\mathbb{Z}^2) \setminus E(G)$. A *boundary condition* η is a partition of ∂G; we say that the vertices of ∂G that belong to the same component of η are *wired* together. To a boundary condition η and a percolation configuration $\omega \in \{0,1\}^{E(G)}$, associate the percolation configuration ω^η which is obtained by identifying all the mutually wired vertices of ∂G.

A percolation configuration ξ on $\mathbb{Z}^2$ induces certain boundary conditions on ∂G: two vertices are wired together if the•y are connected in $\xi \setminus E(G)$. We shall make a slight notational abuse by identifying the percolation configuration ξ with the boundary condition it induces on ∂G, and keeping the notation ω^ξ when ξ is a percolation configuration on $\mathbb{Z}^2 \setminus G$. Two boundary conditions play a special role: the *free* boundary conditions, denoted by 0, are those where no boundary vertices are wired together; in the *wired* boundary conditions, denoted by 1, all boundary vertices are wired together.

The random cluster measure on a finite subgraph G with boundary conditions η and parameters $p \in [0,1]$ and $q \geq 1$ is defined as follows. For a percolation configuration ω on G write $o(\omega)$ for number of open edges of ω, and call $k(\omega^\eta)$ the number clusters of ω^η. For any percolation configuration ω on G set

$$\phi^\eta_{G,p,q}[\omega] = \frac{1}{Z^\eta_{G,p,q}} \left(\frac{p}{1-p}\right)^{o(\omega)} q^{k^\eta(\omega)},$$

where, $Z^\eta_{G,p,q}$ is called the *partition function* of the model, and is the only constant guaranteeing that $\phi^\eta_{G,p,q}$ is a probability measure.

Random-cluster measures may also be defined on the full graph $\mathbb{Z}^2$ either through the DLR formalism or by taking limits of measures on increasing finite subgraphs of $\mathbb{Z}^2$. Due to monotonicity properties, it is classical that the free and wired measures $\phi^0_{G,p,q}$ and $\phi^1_{G,p,q}$

admit limits as G increases to $\mathbb{Z}^2$. These will be denoted $\phi^0_{p,q}$ and $\phi^1_{p,q}$ and are instances of *infinite volume measures*.

It was proved in [13] that the model exhibits a phase transition at the self-dual parameter $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$. That is, for any $p < p_c(q)$ (in the so-called *subcritical regime*), there exists a unique infinite volume measure ($\phi^0_{p,q} = \phi^1_{p,q}$) and connected components are almost surely finite. When $p > p_c(q)$, the infinite volume measure is unique and contains almost surely a unique infinite connected component. The phase transition was shown to be continuous for $q \in [1, 4]$ [53], in the sense that there exists a unique infinite-volume measure also at $p = p_c(q)$. Furthermore, RSW-type estimates are known to hold with that specific choice of parameters. For $q > 4$, the phase transition is discontinuous [50], which is to say that $\phi^0_{p_c(q),q} \neq \phi^1_{p_c(q),q}$, with the former having a sub-critical behaviour and the latter a super-critical one.

For the exposition of our results, we need to introduce two classical quantities in the study of subcritical and near-critical random-cluster model. Fix $p < p_c$. The *correlation length* of the model is defined as the following limit, the existence of which is based on super-multiplicativity arguments. For $\vec{v} \in \mathbb{R}^2 \setminus \{0\}$, set[1]

$$\xi_p(\vec{v}) = \big(\lim_{n\to\infty} -\tfrac{1}{n} \log \phi_p[0 \longleftrightarrow \lfloor n\vec{v} \rfloor]\big)^{-1}.$$

Notice that

$$\|\vec{v}\|\xi_p(\vec{v}) = \xi_p(\vec{v}/\|\vec{v}\|),$$

so we will mostly consider the case $\vec{v} \in \mathbb{S}^1$. The results of [13] imply that whenever $p < p_c$, $\xi_p(\vec{v}) > 0$ for any $\vec{v} \in \mathbb{S}^1$.

We also introduce the so-called *critical one-arm probability*. For any $R \geq 0$, define $\Lambda_R := \{-R, \dots, R\}^2$. Introduce the function π as follows:

$$\pi_1(R) = \phi^0_{p_c(q),q}[0 \longleftrightarrow \partial\Lambda_R].$$

As opposed to the correlation length, this quantity is computed *at* p_c: it is reminiscent of the behaviour of the connection probabilities at criticality.

5.1.2 Results

We are now ready to introduce our main results. For positive quantities f, g we write $f \lesssim g$ to mean that there exists a constant $C > 0$ such that $f \leq Cg$. The constants C may be chosen uniformly in certain parameters in the definitions of f and g which will be explicitly stated – these will generally be $p < p_c$, n and t below. Write $f \asymp g$ when $f \lesssim g$ and $g \lesssim f$. We will often use the expression *uniformly in* $p < p_c$ to mean uniformly for $p \in [\varepsilon, p_c)$ for some

[1] $\lfloor nx \rfloor$ denotes the vertex of $\mathbb{Z}^2$ which is the closest to nx.

$\varepsilon > 0$ fixed throughout the paper. The choice of ε is arbitrary, but may affect the value of certain constants.

Theorem 5.1.1 (Ornstein–Zernike asymptotics at the scale of the correlation length)**.** *Fix $q \geq 1$. Then, uniformly in $\vec{v} \in \mathbb{S}^1$, $n \geq \xi_p(\vec{v})$ and $p < p_c$,*

$$\phi_p\big[0 \longleftrightarrow \lfloor n\vec{v} \rfloor\big] \asymp \pi_1(\lfloor \xi_p(\vec{v}) \rfloor)^2 \sqrt{\tfrac{\xi_p(\vec{v})}{n}}\, \mathrm{e}^{-\frac{n}{\xi_p(\vec{v})}}. \tag{5.1}$$

Compared to the classical OZ results of [25], the advantage of our result is that it is uniform in p. Indeed, for $q \in [1,4]$, $\xi_p(\vec{v}) \to \infty$ as $p \nearrow p_c$. Previous results often employ local surgeries based on finite energy at scales lower than $\xi_p(\vec{v})$. As such they do not apply when ξ_p diverges. Our approach blends the critical and sub-critical behaviour, each accounting for one of the terms in the right-hand side of (5.1): the term $\mathrm{e}^{-\frac{n}{\xi_p(\vec{v})}}(\frac{n}{\xi_p(\vec{v})})^{-1/2}$ is reminiscent of classical OZ-type formulas and is entirely a sub-critical phenomenon, while the term $\pi_1(\lfloor \xi_p(\vec{v}) \rfloor)^2$ appears due to the near-critical behaviour of the model. Finally, when $n \leq \xi_p(\vec{v})$, the right-hand side may be replaced by $\pi_1(n)^2$, as proved in [51].

Theorem 5.1.1 is a consequence of a more detailed description of the cluster of 0 when conditioned to be connected to a half-plane at a distance n in the direction $\vec{w}$, dual to $\vec{v}$; see Theorem 5.3.2 below. We would like to emphasise that Theorem 5.3.2 also has other consequences, such as:

- Theorem 5.4.1, stating that the probability for the cluster of 0 to reach the half-space $\{\langle x, \vec{w} \rangle \geq n\}$ is pure exponential in n.
- Theorem 5.4.8, stating that the inverse correlation length ξ_p^{-1} is strictly convex, so as its convex dual, known as the *Wulff shape*.
- Theorem 5.4.9, establishing an invariance principle for the cluster of 0 when conditioned to be connected to $\lfloor n\vec{v} \rfloor$ or to a half-plane.

Contrary to previous approaches, Theorem 5.3.2 provides all these consequences simultaneously.

5.1.3 Overview of the proof

The idea of the proof is to slice the plane with lines orthogonal to $\vec{w}$ for some $\vec{w} \in \mathbb{S}^1$, at regular intervals of length comparable to ξ_p. We call these lines *hyperplanes*, as they should have co-dimension 1 in the more general d-dimensional setting. For simplicity, imagine $\vec{w} = e_1$ is the horizontal unit vector.

The cluster of 0 is then explored from left to right in a Markovian way: write X_k for the highest point of the exploration of the cluster when intersected with the k^{th} hyperplane. When no such intersection exists, write $X_k = \dagger$ and we say that X_k dies. We will show that the process $(X_k)_k$ has a certain renewal structure, even when conditioned on surviving for n

steps. As such, the process is decomposed into irreducible pieces and behaves essentially as a random walk, with all consequences following by standard tools.

The ideas are similar to the previous works [25], but we believe are rephrased in a different way. The decomposition of the cluster in irreducible pieces, and ultimately seen as a random walk, appeared already in these works. However, we do not use the "diamond decomposition" and prove directly that renewal times appear often along the cluster.

5.2 Background on the random-cluster model

Here we recall a few well-known properties of the random-cluster measure, previously introduced in Section 5.1.1. We also review some recent results of [51] about the near-critical regime of FK percolation.

Monotonicity properties. The set of percolation configurations on $\mathbb{Z}^2$ can classically be equipped with the partial order defined as follows. If ω and ω' are two percolation configurations, we declare that $\omega \leq \omega'$ if for any $e \in E(\mathbb{Z}^2)$, $\omega(e) \leq \omega'(e)$. An event A is said to be *increasing* if for any two percolation configurations $\omega \leq \omega'$, $\omega \in A \Rightarrow \omega' \in A$. Set $q \geq 1$, $p \in [0,1]$ and G a subgraph of $\mathbb{Z}^2$. The *FKG inequality* asserts that for any A, B increasing events, and any boundary condition η on G,

$$\phi^{\eta}_{G,p,q}[\mathcal{A} \cap \mathcal{B}] \geq \phi^{\eta}_{G,p,q}[\mathcal{A}]\phi^{\eta}_{G,p,q}[\mathcal{B}].$$

The random-cluster measure also displays the following monotonicity property. If $\xi \leq \xi'$ are boundary conditions on G (the order is defined by the usual partial order on the set of partitions of a given subset), then for any increasing event A,

$$\phi^{\xi}_{G,p,q}[A] \leq \phi^{\xi'}_{G,p,q}[A].$$

Domain Markov property. Let G be some subgraph of $\mathbb{Z}^2$. Fix $q \geq 1$ and $p \in (0,1)$. Let $G' = (V', E')$ be a subgraph of G. Then for any boundary condition η on G, any percolation configuration $\xi \in \{0,1\}^{E \setminus G'}$,

$$\phi^{\eta}_{G,p,q}[\cdot_{G'} |\, \omega_{E \setminus E'} = \xi] = \phi^{\xi^{\eta}}_{G',p,q}[\cdot], \tag{DMP}$$

where ξ^{η} is the boundary condition induced on the complement of G' by ξ together with the boundary condition η.

Duality. Consider the dual graph $(\mathbb{Z}^2)^*$ with vertex set $V(\mathbb{Z}^2) + (1/2, 1/2)$ and edge set $\{i + (1/2), j + (1/2)\}$, for i, j such that $\{i, j\} \in E(\mathbb{Z}^2)$. To any percolation configuration on $\mathbb{Z}^2$ we associate its *dual configuration*, defined on the graph $(\mathbb{Z}^2)^*$ by setting $\omega^*(e) =$

$1-\omega(e^*)$, where e^* is the unique edge of $(\mathbb{Z}^2)^*$ that crosses e. It is classical that when $\omega \sim \phi^0_{p,q}$ then $\omega^* \sim \phi^1_{p^*,q}$, where p and p^* are linked by the following duality relation:

$$pp^* = q(1-p)(1-p^*),$$

Note that the value $p_{\mathsf{sd}} := \frac{\sqrt{q}}{1+\sqrt{q}}$ is the unique solution of $p = p^*$, and coincides with the critical parameter p_c as first proved in [13].

5.2.1 Near-critical theory

The near-critical regime of the random-cluster model is the set of parameters n and p for which n is sufficiently small or p sufficiently close to p_c so that the system behaves critically at scale n. It is expected, and is indeed the case for the two dimensional random-cluster model, that the system behaves critically in the near-critical regime, and sub- or super-critically outside of it.

The rigorous understanding of the near-critical regime of percolation models in two dimensions started with Kesten's seminal work on Bernoulli percolation [91]. Kesten's results were adapted to the random-cluster model on $\mathbb{Z}^2$ with $q \in [1,4]$ in [51]. Here we will mention only the consequences of these works that are relevant to us.

For $q \in [1,4]$ fixed and $p < p_c$, set

$$L(p) = \inf\{n \geq 0 : \phi_p[\mathrm{Cross}(\Lambda_n)] \notin [\delta, 1-\delta]\},$$

where $\mathrm{Cross}(\Lambda_n)$ is the event that Λ_n contains an open path crossing it from left to right and $\delta > 0$ is some small fixed quantity.

It was proved in [51] that, for any $\vec{v} \in \mathbb{S}^1$,

$$L(p) \asymp \xi_p(\vec{v})$$

uniformly in $p < p_c$, for $q \in [1,4]$. Additionally, the RSW property was extended to the near-critical regime.

Let $\mathrm{Circ}(r,R)$ be the event that Λ_R contains an open circuit surrounding Λ_r. Write $\mathrm{Circ}^*(r,R)$ for the event that the dual configuration contains such a circuit, which for the primal model translates to Λ_r not being connected to $\partial\Lambda_R$.

Proposition 5.2.1 (RSW in the critical window)**.** *Fix $q \in [1,4]$. There exists $c > 0$ such that for any $p < p_c$ and $n \leq L(p)$*

$$c \leq \phi^0_{\Lambda_{2n},p}[\mathrm{Circ}(n,2n)] \leq 1-c \quad \textit{and} \quad c \leq \phi^1_{\Lambda_{2n},p}[\mathrm{Circ}^*(n,2n)] \leq 1-c. \tag{5.2}$$

Finally, the most significant contribution of [51] was to prove the stability of the arm event probabilities in the near-critical regime.

Theorem 5.2.2. *Fix $q \in [1,4]$. Then*

$$\phi^{\xi}_{\Lambda_{2R},p}[0 \longleftrightarrow \partial\Lambda_R] \asymp \pi_1(R), \tag{5.3}$$

uniformly in $p < p_c$, $R \le L(p)$ and any boundary conditions ξ on $\partial\Lambda_{2R}$

Henceforth we will use $L(p)$ rather than $\xi_p(\vec{v})$ to designate a quantity of the order of the correlation length, for instance when referring to the interspacing of the hyperplanes used to define the process $(X_n)_n$. We do so to emphasise that its use is not related to the direction $\vec{v}$ and is only important up to a bounded multiplicative constant.

5.3 Coarse renewal structure of a long subcritical percolation cluster

The key idea of the article is to show that a subcritical cluster admits what we call a "killed renewal structure", even when conditioned to hit a far-away half-plane. We start by defining a class of processes which will have random-walk like behaviour.

Definition 5.3.1. A stochastic process $(X_t, Y_t)_{t\in\mathbb{N}} \in (\mathbb{R} \cup \{\dagger\} \times \{0,1\})^{\mathbb{N}}$ is called a *killed Markov renewal process* (KMRP in short) with respect to some filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$ if

- It is adapted to $(\mathcal{F}_t)_{t\in\mathbb{N}}$;
- If $X_t = \dagger$, then $Y_t = 0$ and $X_{t+1} = \dagger$;
- If we set $T_0 = 0$ and $T_{k+1} = \inf\{t < T_k : Y_t = 1\}$, then for any $k \ge 1$, the future of the process $(X_{t+T_k} - X_{T_k}, Y_{t+T_k})_{t\in\mathbb{N}}$ conditionally on $\mathcal{F}_{T_k}$ and $X_{T_k} \neq \dagger$ has a fixed law $\mathcal{L}$.

We say that the process has exponential tails with constants $c_T, c_X > 0$ if for all $k, t \ge 0$

$$\mathbb{P}[n < T_{k+1} - T_k < \infty \,|\, \mathcal{F}_{T_k},\, T_k < \infty] \le \exp(-c_T\, n) \qquad \text{and}$$
$$\mathbb{P}[X_{T_k+j} \neq \dagger \text{ and } |X_{T_k+j} - X_{T_k}| \ge n \,|\, \mathcal{F}_{T_k},\, T_k < \infty] \le \exp(-c_X\, n/j) \qquad \forall n \ge 1.$$

We say that the model exhibits a mass gap of at least $\varepsilon > 0$ if for any k and $0 \le t \le n$,

$$\mathbb{P}[T_{k+1} \ge T_k + t | \mathcal{F}_{T_k},\, X_{T_k+n} \neq \dagger] \le C \exp(-\varepsilon t). \tag{5.4}$$

for some $C > 0$.

For such a process, and for $k \ge 1$, call the laws of $X_{T_{k+1}} - X_{T_k}$ and $T_{k+1} - T_k$ conditionally on $T_k \neq \infty$ the X-step and the T-step (or vertical and horizontal steps, respectively). Notice that these only depend on the law $\mathcal{L}$ mentioned above, and therefore do not depend on k. Define the X-step mean and variance as

$$\mu_X := \mathbb{E}[X_{T_{k+1}} - X_{T_k} \,|\, T_{k+1} < \infty] \quad \text{and} \quad \sigma_X := \mathrm{Var}[X_{T_{k+1}} - X_{T_k} \,|\, T_{k+1} < \infty]$$

also define the T-step mean and the killing rate as

$$\mu_T := \mathbb{E}[T_{k+1} - X_{T_k} \,|\, T_{k+1} < \infty] \quad \text{and} \quad \kappa := \mathbb{P}[T_{k+1} = \infty \,|\, T_k < \infty].$$

We will now describe how such general processes are related to the cluster of 0 under different conditionings.

Fix $\vec{w} \in \mathbb{S}^1$, $q \in [1, 4]$ and $p < p_c$. For all practical purposes, think of $\vec{w}$ as the unit vector in the horizontal direction, pointing to the right; this is only so that the vocabulary and illustrations below make sense. Define the half-spaces

$$\mathcal{H}^{\vec{w}}_{\leq t} = \{x \in \mathbb{R}^2 : \langle x, w\rangle \leq t\, L(p)\} \quad \text{and} \quad \mathcal{H}^{\vec{w}}_{\geq t} = \{x \in \mathbb{R}^2 : \langle x, w\rangle \geq t\, L(p)\}.$$

Call $\partial\mathcal{H}^{\vec{w}}_{\leq t} = \partial\mathcal{H}^{\vec{w}}_{\geq t} = \{x \in \mathbb{R}^2 : \langle x, w\rangle = t\, L(p)\}$ a hyperplane. In the following we will use an arbitrary integer approximation of these objects, which we do not detail. We will mostly work with $\vec{w}$ fixed, and will omit it from the notation whenever no ambiguity is possible.

Write C for the cluster of 0 and $\mathsf{C}_{\leq t}$ for the cluster of 0 in $\omega \cap \mathcal{H}_{\leq t}$. Note that this is contained in, but not always equal to $\mathsf{C} \cap \mathcal{H}_{\leq t}$. For any $t \in \mathbb{Z}$, set

$$X_t := \max\big\{h \in \mathbb{R} : tL(p) \cdot \vec{w} + h \cdot \vec{w}^{\perp} \in \mathsf{C}_{\leq t}\big\},$$

where $\vec{w}^{\perp} \in \mathbb{S}^1$ is a unit vector orthogonal to $\vec{w}$ (the direction of $\vec{w}^{\perp}$ is irrelevant for now; think of it as pointing upwards). If the set above is empty, which is to say that C does not intersect $\mathcal{H}_{\geq t}$, set $X_t = \dagger$. Thus X_t is the "highest" coordinate of the intersection of $\mathsf{C}_{\leq t}$ with $\partial\mathcal{H}_{\leq t}$.

The main objective of this section is the following result.

Theorem 5.3.2. *Fix $q \in [1, 4]$. For any $p < p_c$ and $\vec{w} \in \mathbb{S}^1$ there exists an enlarged probability space supporting a random process Y_t such that $(X_t/L(p), Y_t)$ is a killed renewal Markov process with exponential tails and mass gap, all bounded away from 0 uniformly in p and $\vec{w}$.*

Moreover, the killing rate is bounded away from 0 and 1 and the X-step variance is bounded away from 0 uniformly in p and $\vec{w}$. Finally, the initial step survival rate satisfies

$$\phi_p[T_1 < \infty] \asymp \pi_1(L(p)) \tag{5.5}$$

uniformly in p and $\vec{w}$.

The above will suffice to prove Theorem 5.1.1, as well as a large number of other properties of sub-critical clusters. We chose to formulate it using the concept of KMRP so as to separate the model-dependent part of the argument from the generic analysis of a class of processes with random-walk behaviour. While the formalism is new in this context, we do not claim the theorem to be entirely original; indeed, the only fundamental novelty compared to [25] is the uniformity in $p < p_c$ when $q \in [1, 4]$.

We mentioned uniform lower bounds on σ_X, but no upper bounds. The uniform exponential tails induce uniform upper bounds on σ_X, μ_X and μ_T.

Note that we do not claim that $\mu_X = 0$. This is the case when $\vec{w}$ is aligned to the coordinate axis due to symmetry, but is not generally true. As such, under the conditioning $X_n \neq \dagger$, the cluster does not "aim" for the point $n\vec{w}$, but rather for a point $n\vec{v}$ for some $\vec{v}$ depending on $\vec{w}$ (with $\langle \vec{v}, \vec{w} \rangle = 1$). The relation between $\vec{w}$ and $\vec{v}$ will yield the strict convexity of the Wulff shape and ultimately will prove how $\vec{w}$ needs to be chosen to deduce Theorem 5.1.1 for some direction $\vec{v}$ – see Section 5.4.3 for details.

The proof of Theorem 5.3.2 relies on a geometric analysis of C under the survival event $X_n \neq \dagger$. The analysis is performed at a scale $L(p)$, in particular showing that there exists a density of points at which the cluster is confined to boxes of size $L(p)$. Those boxes play the role of "pre-renewal times"; indeed, it will be shown that at each pre-renewal time, there is uniformly positive probability that the "future" cluster is sampled independently from its past. As such, the pre-renewal times play a similar role to the cone points in [25].

We should mention that any KMRP with exponential tails and a mass-gap has a exponential rate of survival

$$\mathbb{P}[X_n \neq \dagger] \asymp \exp(-n/\zeta),$$

as will be proved in Section 5.4. In our context, $\zeta = \zeta(p, \vec{w})$ depends on p and $\vec{w}$ but is uniformly bounded away from 0 and ∞. The constants in $\asymp$ above are not uniform in p; they will be shown to be of order $\pi_1(L(p))$ and are due to the requirement of survival up to the first renewal time. Finally, the mass-gap states that the exponential rate of survival of a single step is strictly smaller than ζ.

Remark 5.3.3. For KMRP with exponential tails but no mass gap a *condensation* phenomenon can occur, in which the process survives by making a very large step of linear order rather than many small steps of constant order. This was discussed in the context of long range Ising models in [8, 9].

The rest of the section is dedicated to proving Theorem 5.3.2. Its consequences such as Theorem 5.1.1 will be proved in Section 5.4.

For the rest of the section we will assume $\vec{w}$ to be the horizontal vector $(1, 0)$. This is purely for convenience of notation and has no impact on the proof. In particular, we will never use any symmetries of the lattice with respect to $\vec{w}$. One may imagine that the lattice is rotated so that $\vec{w}$ is horizontal.

Define the filtration $(\mathcal{F}_t)_{t \geq 0}$ as generated by the variables $\mathsf{C}_{\leq t}$. Later on we will extend this filtration to also contain some independent information.

For the rest of the section we will work with $p < p_c$ and write $L = L(p)$ when no confusion is possible. All constants and equivalences will be uniform in p unless otherwise stated.

5.3.1 Cone-connections

For integers $t \geq 0, k \in \mathbb{Z}$, define $\mathscr{L}_{t,k} := \{tL\} \times [kL, (k+1)L)$, so that the hyperplane $\partial\mathcal{H}_{\leq t}$ is the disjoint union of the line segments $\mathscr{L}_{t,k}$ when k runs over $\mathbb{Z}$. Also, write $x_{t,k} = (tL, (k+1/2)L)$ for the midpoint of $\mathscr{L}_{t,k}$.

For some $\alpha > 0$ to be fixed right after, write

$$\mathcal{Y} = \{z \in \mathbb{R}^2 : |\langle z, \vec{w}^{\perp}\rangle| \leq \alpha\langle z, \vec{w}\rangle\}.$$

This is the cone in direction $\vec{w}$ with aperture $2\arctan\alpha$.

We now explain how to choose α. We claim that there exists α_c such that for any $\beta \in \mathbb{R}$ such that $|\beta| > \frac{\alpha}{\sqrt{1+\alpha^2}}$, there exists $\varepsilon > 0$ such that for any $n \geq 3$,

$$\phi_p[\Lambda_L(0) \longleftrightarrow \Lambda_L(n\vec{w} + \beta\vec{w}^{\perp})] \leq \exp(-\varepsilon n)\phi_p[\Lambda_L(0) \longleftrightarrow \Lambda_L(n\vec{w})]. \quad (5.6)$$

In other words, reaching a box on the hyperplane $\mathcal{H}^{\vec{w}}_{\geq n}$ outside of the cone of opening α_c is exponentially more difficult than reaching a box in the same hyperplane, but in the middle of the cone. The construction of α_c follows by the convexity and homogeneity of ξ^{-1}: those are classical properties and imply that ξ^{-1} is a norm. Consider α_c to be the opening such that any facet of the unit ball[2] is included in a cone of apex 0 and opening α_c. Then (5.3.1) will be satisfied with this choice of α_c.

In what follows, we fix α to be some real number satisfying $\alpha > 2\alpha_c$.

Proposition 5.3.4 (Connections in cones)**.** *There exists a constant $c > 0$ such that, for any domain $D \supset \mathcal{H}_{\geq 0}$, any boundary conditions ξ on D and any $k, n \geq 1$,*

$$\phi^{\xi}_{D,p}\big[\Lambda_L \longleftrightarrow (\mathcal{Y} - (kL, 0))^c \,\big|\, \Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}\big] \leq e^{-ck}. \quad (5.7)$$

The lemma states that even when conditioned to reach a far away half-plane, the cluster of $\mathscr{L}_{0,0}$ will be contained in a cone with high probability. The probability approaches 1 exponentially fast when the cone is widened (with the aperture angle remaining constant).

Proof. For this proof we will introduce the slightly thinner cone

$$\mathcal{Z} = \{z \in \mathbb{R}^2 : |\langle z, \vec{w}^{\perp}\rangle| \leq \tfrac{1}{2}\alpha\langle z, \vec{w}\rangle\}.$$

Fix D, ξ, n and k as in the statement. Write C for the cluster of Λ_L. Fix $k \geq 1$.

[2]It will be shown in Section 5.4.3 that the unit ball is actually *strictly convex*, meaning that it has no such facets.

If C exits $(\mathcal{Y}-(kL,0))^c$, it does so either below $\mathcal{Y}-(kL,0)$ or above $\mathcal{Y}-(kL,0)$. We will bound the probability of each event separately, conditionally on C intersecting $\mathcal{H}_{\geq n}$.

We start with the first case, and call $\partial^{\mathsf{B}}\mathcal{Y}$ (resp. $\partial^{\mathsf{B}}\mathcal{Z}$) the bottom boundary of the cone $\mathcal{Y}$ (resp. of the cone $\mathcal{Z}$). We distinguish two scenarios:

(a) There exists a connection between Λ_L and $\mathcal{H}_{\geq n}$ not intersecting the region below $\mathcal{Z}-(\frac{1}{2}kL,0)$ or

(b) Any connection between Λ_L and $\mathcal{H}_{\geq n}$ intersects the region below $\mathcal{Z}-(\frac{1}{2}kL,0)$.

In situation (a), explore the top-most connection Γ between Λ_L and $\mathcal{H}_{\geq n}$. Conditionally on Γ, the probability that a point $z=(t,y)$ on the bottom boundary of $\mathcal{Y}-(kL,0)$ is connected to Γ is bounded above as

$$\phi^{\xi}_{D,p}\big[z \longleftrightarrow \Gamma \,\big|\, \Gamma\big] \leq \exp(-c(t+k)),$$

for some $c=c(\alpha)>0$. Summing over z we find

$$\phi^{\xi}_{D,p}\big[\partial^{\mathsf{B}}(\mathcal{Y}-(kL,0)) \longleftrightarrow \Gamma \,\big|\, \Gamma\big] \leq \exp(-ck),$$

for a possibly altered value of the constant $c>0$.

We are now going to prove that situation (b) itself is exponentially unlikely. We distinguish two subcases.

(i) Λ_L is connected to a point of $\partial\mathcal{H}_{\geq n}\cap(\mathcal{Z}-(\frac{1}{2}kL,0))^c$.

(ii) the intersection of the cluster of Λ_L and $\mathcal{H}_{\leq n}$ is contained within the cone $(\mathcal{Z}-(\frac{1}{2}kL,0))$.

Case (b,(i)) is easily bounded by the choice of the angle α being larger than the greatest facet of the Wulff shape $\mathcal{W}$. Indeed, there exists $\varepsilon>0$ such that for any ℓ such that the box $\Lambda_{n,\ell}:=\Lambda_L(nL,\ell L)$ is not in the cone $(\mathcal{Z}-(\frac{1}{2}kL,0))$,

$$\phi^{\xi}_{D,p}[\Lambda_L \longleftrightarrow \Lambda_{\ell,n}] \leq \exp(-\varepsilon\ell)\phi^{\xi}_{D,p}[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}].$$

Summing over all such ℓ yields the announced exponential decay.

For case (b, (ii)), we proceed as follows. Explore the interfaces starting at vertices of $\partial\mathcal{H}_{\geq n}\cap(\mathcal{Z}-(\frac{1}{2}kL,0))$, inside the cone $\mathcal{Z}-(\frac{1}{2}kL,0)$, from top to bottom. If case (b, (ii)) occurs together with the event $\{\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}\}$, then this exploration needs to hit a vertex of $\partial^B(\mathcal{Z}-(\frac{1}{2}kL,0))$. We stop the exploration when this event occurs, and call E_x the event that the exploration procedure is stopped at $x\in\partial^B(\mathcal{Z}-(\frac{1}{2}kL,0))$. For such an x, call A_x the line segment running from x to $\partial\mathcal{H}_{\geq n}$ along the bottom boundary of the cone $\mathcal{Z}-(\frac{1}{2}kL,0)$. The key observation is then that if case (b, (ii)) occurs together with the events

E_x and $\{\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}\}$, then it is the case that A_x needs to be connected by a primal path to Λ_L. Thus,

$$\phi_D^\xi[\text{case (b, (ii))}, \mathcal{H}_{\geq n}] \leq \sum_{x \in \partial^B(\mathcal{Z}-(\frac{1}{2}kL,0))} \phi_D^\xi[E_x]\phi_D^\xi[A_x \longleftrightarrow \Lambda_L | E_x]. \tag{5.8}$$

Fix $x \in \partial^B(\mathcal{Z} - (\frac{1}{2}kL, 0))$. We are going to bound the two factors appearing on the sum on the right-hand side of (5.3.1) separately. Write $x = (t, y)$, and write $\tilde{x} = (t, 0)$.

We first write

$$\phi_D^\xi[E_x] \leq \phi_D^\xi[\Lambda_L(x) \longleftrightarrow \mathcal{H}_{\geq n}] \asymp \phi_D^\xi[\Lambda_L(\tilde{x}) \longleftrightarrow \mathcal{H}_{\geq n}].$$

For the second term, observe that pushing away the boundary conditions induced by the exploration procedure yields

$$\phi_D^\xi[A_x \longleftrightarrow \Lambda_L | E_x] \leq \phi_D^{\tilde{\xi}(x)}[A_x \longleftrightarrow \Lambda_L],$$

where $\tilde{\xi}(x)$ is the boundary condition defined as follows: it is induced by ξ on D, and is wired on the bottom of the arc A_x and free on its top, and wired to the right of $\partial\mathcal{H}_{\geq n} \cap (\mathcal{Z} - (\frac{1}{2}kL, 0))$ and free on its left. (see figure**??**). We first observe that

$$\phi_D^{\tilde{\xi}(x)}[A_x \longleftrightarrow \Lambda_L] \lesssim \phi_D^{\tilde{\xi}(x)}[\Lambda_L(x) \longleftrightarrow \Lambda_L].$$

This is proved by splitting A_x into boxes of size L, by summing the connection probabilities from Λ_L to those boxes, and by using the exponential decay of the connection probabilities.

Now observe that due to the choice of α, writing $x = (t, y)$,

$$\phi_D^{\tilde{\xi}(x)}[\Lambda_L(x) \longleftrightarrow \Lambda_L] \leq \exp(-\varepsilon(t + k))\phi_D^\xi[\Lambda_L \longleftrightarrow \Lambda_L(\tilde{x})].$$

Putting everything together we proved that

$$\begin{aligned}\phi_D^\xi[E_x]\phi_D^\xi[A_x \longleftrightarrow \Lambda_L | E_x] &\leq \exp(-\varepsilon(t + k))\phi_D^\xi[\Lambda_L \longleftrightarrow \Lambda_L(\tilde{x})]\phi_D^\xi[\Lambda_L(\tilde{x}) \longleftrightarrow \mathcal{H}_{\geq n}] \\ &\leq \exp(-\varepsilon(t + k))\phi_D^\xi[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}],\end{aligned}$$

where we used FKG inequality for the last line. We conclude by summing over the value of t. As the same proof holds if the connection leaves the cone on its top boundary, this concludes the proof of the statement.

□

The next proposition is an extension of the usual mixing property of the subcritical random-cluster model to some specific infinite domains.

Lemma 5.3.5. *Uniformly in $p < p_c$ and in $\vec{w} \in \mathbb{S}^1$, for any event $\mathcal{A}$ (resp. $\mathcal{B}$) depending on the edges of $\mathcal{H}_{\leq 0}$ (resp. $\mathcal{Y} \cap \mathcal{H}_{\geq 1}$),*

$$\phi_p[\mathcal{A} \cap \mathcal{B}] \asymp \phi_p[\mathcal{A}]\phi_p[\mathcal{B}].$$

Proof. Let $\tilde{\mathcal{Y}}$ be the cone of larger opening 2α (*i.e.*, $\tilde{\mathcal{Y}} = \{z \in \mathbb{R}^2 : |\langle z, \vec{w}^\perp\rangle| \leq 2\alpha\langle z, \vec{w}\rangle\}$). We start by arguing that there exists $\eta \in (0,1)$, independent of the value of p, such that

$$\phi_p[\partial(\tilde{\mathcal{Y}} \cap \mathcal{H}_{\geq 1/2}) \longleftrightarrow \partial\mathcal{H}_{\leq 0}] \leq \eta \quad \text{and} \quad \phi_p[\partial(\tilde{\mathcal{Y}} \cap \mathcal{H}_{\geq 1/2}) \longleftrightarrow \partial(\mathcal{Y} \cap \mathcal{H}_{\geq 1})] \leq \eta. \tag{5.9}$$

Let us prove the claim of the first part of the statement, as the second part will follow by the same argument.

For $k \geq 1$, set $a_k = (0, 2^k L\alpha)$ and $b_k = (2^{k-1}, 2^k L\alpha)$. For $k \leq -1$, set $a_k = (0, -2^{-k} L\alpha)$ and $b_k = (2^{-(k-1)}, -2^{-k} L\alpha)$. Denote by R_k^1 (resp. R_k^2) the rhombi enclosed by the vertices $a_k, a_{k+1}, b_{k+1}, b_k$ (resp. $a_k, a_{k+2}, b_{k+2}, b_k$). Finally for a rhombus R, denote by $\mathcal{V}(R)$ (resp. $\mathcal{H}(R)$) the event that there exists a dual open crossing from the bottom side (resp. left side) of the rhombus R to the top side (resp. right side) of the rhombus.

It follows by basic percolation arguments and the equivalence between the characteristic length and the correlation length that there exists some constant $c > 0$ independent of p such that for any $k \in \mathbb{Z}$,

$$\phi^1_{(\mathcal{H}_{\geq 0} \cup \tilde{\mathcal{Y}})^c}[\mathcal{H}(R_1^k)] > 1 - \mathrm{e}^{-c2^{|k|-1}} \quad \text{and} \quad \phi^1_{(\mathcal{H}_{\geq 0} \cup \tilde{\mathcal{Y}})^c}[\mathcal{V}(R_2^k)] > 1 - \mathrm{e}^{-c2^{|k|-1}}. \tag{5.10}$$

By the FKG inequality, we obtain that

$$\phi^1_{(\mathcal{H}_{\geq 0} \cup \tilde{\mathcal{Y}})^c}\big[\bigcap_{k \in \mathbb{Z}} \mathcal{H}(R_1^k) \cap \mathcal{V}(R_2^k)\big] \geq \prod_{k \in \mathbb{Z}} \big(1 - \mathrm{e}^{-c2^{|k|}}\big)^2 = \eta > 0,$$

due to (5.5).

Now one may check that the occurence of $\bigcap_{k \in \mathbb{Z}} \mathcal{H}(R_1^k) \cap \mathcal{V}(R_2^k)$ prevents the occurence of $\{\partial(\tilde{\mathcal{Y}} \cap \mathcal{H}_{\geq 1/2}) \longleftrightarrow \partial\mathcal{H}_{\leq 0}\}$. This establishes the first half of (5.4). The second part of the statement can be proved by a similar construction.

With (5.5) in hand, the proof is now very much inspired by that of [51, Prop. 2.9]. We are going to prove that there exists some constant η such that for any boundary conditions ξ, ψ on $\partial\mathcal{H}_{\leq 0}$,

$$\left|\phi^\xi_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}] - \phi^\psi_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}]\right| < \eta\phi^\xi_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}].$$

This will imply the result by usual arguments, exploring the boundary conditions induced on $\partial\mathcal{H}_{\leq 0}$ by the occurrence of $\mathcal{A}$ and using (DMP).

As in [51], we fix a domain $\tilde{\mathcal{Y}} \cap \mathcal{H}_{\geq 1} \subset \Omega \subset (\mathcal{H}_{\leq 0})^c$. We fix a boundary condition ψ on $\partial\Omega$. We start by comparing $\phi^0_\Omega[\mathcal{B}]$ and $\phi^\psi_\Omega[\mathcal{B}]$. Let us call Ψ the increasing coupling between those two measures.

$$\begin{aligned}
\phi^\psi_\Omega[\mathcal{B}] - \phi^0_\Omega[\mathcal{B}] &= \Psi[\omega \notin \mathcal{B} \text{ but } \omega' \in \mathcal{B}] \\
&\leq \phi^\psi_\Omega[\omega' \in \mathcal{B}, \partial(\mathcal{Y} \cap \mathcal{H}_{\geq 1}) \longleftrightarrow \partial\Omega] \\
&\leq \phi^1_{\Omega \setminus (\mathcal{Y} \cap \mathcal{H}_{\geq 1})}[\partial(\mathcal{Y} \cap \mathcal{H}_{\geq 1}) \longleftrightarrow \partial\Omega]\phi^\psi_\Omega[\mathcal{B}] \\
&\leq \eta\phi^\psi_\Omega[\mathcal{B}],
\end{aligned}$$

where we used (5.4) in the last inequality.

We obtained that for any Ω as above,

$$\phi^\psi_\Omega[\mathcal{B}] \leq \tfrac{1}{1-\eta}\phi^0_\Omega[\mathcal{B}]. \tag{5.11}$$

Observe that a consequence of this bound is that for any Ω as above, one has

$$\phi^0_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}] = \sum_{\psi \text{ b.c. on } \partial\Omega} \phi^\psi_\Omega[\mathcal{B}]\phi^0_{(\mathcal{H}_{\leq 0})^c}[\omega \text{ induces } \psi \text{ on } \partial\Omega] \leq \tfrac{1}{1-\eta}\phi^0_\Omega[\mathcal{B}]. \tag{5.12}$$

We follow the strategy of [51] for providing a converse bound to (5.6). For any configuration ω in $\mathcal{H}^c_{\geq 0}$, let $\Omega(\omega)$ be the set of vertices *not* connected to ∂D_2. Then, observe that

$$\begin{aligned}
\phi^\psi_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}] &\geq \phi^\psi_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}, \partial(\tilde{\mathcal{Y}} \cap \mathcal{H}_{\geq 1/2}) \not\longleftrightarrow \partial\mathcal{H}_{\leq 0}] \\
&= \sum_{\tilde{\mathcal{Y}} \cap \mathcal{H}_{\geq 1} \subset \Omega \subset (\mathcal{H}_{\leq 0})^c} \phi^0_\Omega[\mathcal{B}]\phi^\psi_{(\mathcal{H}_{\leq 0})^c}[\Omega(\omega) = \Omega] \\
&\geq (1-\eta)\phi^0_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}]\phi^\psi_{(\mathcal{H}_{\leq 0})^c}[\partial(\tilde{\mathcal{Y}} \cap \mathcal{H}_{\geq 1/2}) \not\longleftrightarrow \partial\mathcal{H}_{\leq 0}] \\
&\geq (1-\eta)^2\phi^0_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}].
\end{aligned}$$

We used (5.7) on the third line and (5.4) on the fourth line. Setting $\tilde{\eta} := \max\{\frac{1}{1-\eta} - 1, 1 - (1-\eta)^2\}$, inserting $\Omega = (\mathcal{H}_{\leq 0})^c$ in (5.6) and combining with latter input yields:

$$|\phi^\psi_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}] - \phi^0_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}]| < \tilde{\eta}\phi^0_{(\mathcal{H}_{\leq 0})^c}[\mathcal{B}].$$

By the triangular inequality, this yields the desired inequality up to doubling the constant $\tilde{\eta}$. Observe that $\tilde{\eta}$ is an explicit function of the parameter η only; it is then uniform in p and in $\vec{w}$.

□

Finally, we state a direct consequence of Proposition 5.3.4 and Lemma 5.3.5 that will be useful at several stages of the paper.

Corollary 5.3.6. *For any $n \geq 1$ and any boundary conditions ξ on $\mathcal{H}_{\geq 0}$*

$$\phi^{\xi}_{\mathcal{H}_{\geq 0}}\big[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}\big] \asymp \phi^{1}_{\mathcal{H}_{\geq 0}}\big[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}\big]. \tag{5.13}$$

Proof. We will prove that $\phi^{0}_{\mathcal{H}_{\geq 0}}\big[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}\big] \asymp \phi^{1}_{\mathcal{H}_{\geq 0}}\big[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}\big]$, which is sufficient due to the fact that $\{\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}\}$ is increasing. As the upper bound is obvious, we focus on the lower bound. By Proposition 5.3.4, it is the case that there exists some $c > 0$ such that

$$\phi^0[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n}] \geq c\phi^0[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n} | \Lambda_L \not\longleftrightarrow \mathcal{Y}^c].$$

Now, Lemma 5.3.5 shows that

$$\phi^0[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n} | \Lambda_L \not\longleftrightarrow \mathcal{Y}^c] \asymp \phi^1[\Lambda_L \longleftrightarrow \mathcal{H}_{\geq n} | \Lambda_L \not\longleftrightarrow \mathcal{Y}^c],$$

which concludes the proof of (5.3.6) by a second application of Proposition 5.3.4. □

5.3.2 The number of active boxes is subcritical

A line segment $\mathscr{L}_{t,k}$ is said to be *active at time t* if $\mathsf{C}_{\leq t} \cap \mathscr{L}_{t,k} \neq \emptyset$. Write N_t for the number of active segments at time t. Some time t for which $N_t = 1$ will be called a *pre-renewal times*. The purpose of this section is the following.

Proposition 5.3.7 (Density of pre-renewal times)**.** *There exists $c > 0$ such that for any $p < p_c$ and any $t, r > 0$ and $n \geq t + r$,*

$$\phi_p[N_s > 1 \ \forall s \in \{t+1, \dots, t+r\} \,|\, \mathcal{F}_t,\, N_t = 1,\, X_n \neq \dagger] \leq \exp(-cr) \quad \text{and}$$
$$\phi_p[N_s > 1 \ \forall s \in \{1, \dots, r\} \,|\, X_n \neq \dagger] \leq \exp(-cr).$$

This tells us that pre-renewal times arrive quickly, even when conditioning on a survival event far in the future. Proposition 5.3.7 will follow from the result below.

Proposition 5.3.8. *For C large enough, there exist constants $\mu < 1$ and $K > 0$ such that for any $p < p_c$, any $t \geq 0$ and $n \geq t + C$,*

$$\phi_p[N_{t+C} \,|\, \mathcal{F}_t,\, X_n \neq \dagger] \leq \mu N_t + K. \tag{5.14}$$

The rest of this section is dedicated to the proofs of the two results above. As mentioned, Proposition 5.3.8 is the key step, and we start with it. First we explain how to fix the constant C.

Lemma 5.3.9. *There exists a constant $C > 0$ so that for any $p < p_c$ and $\vec{w} \in \mathbb{S}^1$,*

$$\sup_D \phi^1_{D,p}[\#j \text{ such that } \mathscr{L}_{0,0} \leftrightarrow \mathscr{L}_{C,j}] = \eta < 1, \tag{5.15}$$

where the supremum is taken over all domains D containing $\mathcal{H}_{\geq 0}$.

Proof of Lemma 5.3.9. Fix a domain D containing $\mathcal{H}_{\geq 0}$, and some $K \in \mathbb{N}^*$. Observe that if the number of indices j satisfying $\{\mathscr{L}_{0,0} \leftrightarrow \mathscr{L}_{C,j}\}$ is larger than K, then there exists at least one $j_0 \in \mathbb{N}$ such that $j_0 \geq \lfloor K/2 \rfloor$ and $\mathscr{L}_{0,0} \leftrightarrow \mathscr{L}_{C,j_0}$. For such a j_0 we may lower bound the Euclidean distance between $\mathscr{L}_{0,0}$ and $\mathscr{L}_{C,j_0}$ by $L\sqrt{C^2 + j_0^2}$. By the equivalence between the characteristic length and the correlation length and a basic union bound, we obtain:

$$\begin{aligned}\phi^1_{D,p}[\#j \text{ such that } \mathscr{L}_{0,0} \leftrightarrow \mathscr{L}_{C,j} > K] &\leq \sum_{j_0 \geq \lfloor K/2 \rfloor} \exp(-c\sqrt{C^2 + j_0^2}) \\ &\leq c^{-1} \exp(-c(C + K/2)).\end{aligned}$$

for some $c > 0$ an absolute constant independent of the value of p and of the choice of the domain D. Choosing C large enough so that $\sum_{K \geq 0} c^{-1} \exp(-c(C + K/2)) < 1$ concludes the proof. □

Proof of Proposition 5.3.8. Fix the parameters p, $\vec{w}$, n and t as in the statement. Also fix some realisation of $\mathsf{C}_{\leq t}$. We will always work conditionally on this realisation of the "past cluster", and write $\phi_{\mathcal{F}_t,p}$ for this conditional measure. All notions of connections and clusters below refer to the configuration in $\mathsf{C}^c_{\leq t}$ only. All constants and equivalences below are uniform in the choices of p, $\vec{w}$, n, t and $\mathsf{C}_{\leq t}$.

Fix C given by Lemma 5.3.9. Write $\mathbf{j}$ for the index of the top-most active box $\mathscr{L}_{t,j}$ such that $\mathscr{L}_{t,j} \cap \mathsf{C}_{\leq t}$ is connected to $\mathcal{H}_{\geq n}$. If no such connection exists, write $\mathbf{j} = \emptyset$. Also write K for the minimal value K such that the cluster of $\mathscr{L}_{t,\mathbf{j}}$ is contained in $\mathcal{Y}_K := \mathcal{Y} + x_{t,\mathbf{j}} - (KL, 0)$ (recall that $x_{t,j}$ denotes the middle point of $\mathscr{L}_{t,j}$).

Our goal is to bound

$$\phi_p[N_{t+C} \,|\, \mathbf{j} \neq \emptyset] = \sum_j \phi_p[\mathbf{j} = j \,|\, \mathbf{j} \neq \emptyset] \phi_{\mathcal{F}_t,p}[N_{t+C} \,|\, \mathbf{j} = j],$$

and we will do so by bounding each of the terms $\phi_{\mathcal{F}_t,p}[N_{t+C} \,|\, \mathbf{j} = j]$ individually.

We first argue that

$$\phi_{\mathcal{F}_t,p}[\mathsf{K} > k \,|\, \mathbf{j} = j] \leq \mathrm{e}^{-c_0 k} \tag{5.16}$$

for some constant $c_0 > 0$ and all $k \geq 3$.

We say $\mathscr{L}_{t,j}$ is a top-most seed if $\mathscr{L}_{t,j} \cap \mathsf{C}_{\leq t}$ is connected to $\partial\Lambda_{2L}(x_{t,j})$ but not to $\mathscr{L}_{t,j+1}$ in $\mathsf{C}^c_{\leq t} \cap \Lambda_{2L}(x_{t,j})$. We also write $s_t = \phi^1_{\mathcal{H}_{\leq 0},p}[\mathscr{L}_{0,0} \longleftrightarrow \mathcal{H}_{\geq t}]$. We start off by estimating the probability of $\mathbf{j} = j$.

Lemma 5.3.10 (Gluing lemma)**.** *Uniformly in $p < p_c$, $\vec{w}$ and n, t and r*

$$\phi_{\mathcal{F}_t,p}[\mathbf{j} = j] \asymp s_{n-t}\, \phi_{\mathcal{F}_t,p}[j \textit{ is top-most seed}\,]. \tag{5.17}$$

We defer the proof of the lemma to later in the section and finish that of Proposition 5.3.8. Due to (5.3.10) we have

$$\begin{aligned}
\phi_{\mathcal{F}_t,p}[\mathsf{K} > k \,|\, \mathbf{j} = j] &\lesssim \frac{\phi_{\mathcal{F}_t,p}[\mathsf{K} \geq k \text{ and } \mathbf{j} = j]}{s_{n-t}\, \phi_{\mathcal{F}_t,p}[j \text{ is top-most seed}]} \\
&\leq \frac{\phi_{\mathcal{F}_t,p}[j \text{ is top-most seed, } \Lambda_{2L}(x_{t,j}) \leftrightarrow \mathcal{H}_{\geq n} \text{ and } \Lambda_{2L}(x_{t,j}) \leftrightarrow \mathcal{Y}_k^c]}{s_{n-t}\, \phi_{\mathcal{F}_t,p}[j \text{ is top-most seed}]} \\
&\leq \frac{\sup_{D,\xi} \phi^{\xi}_{D,p}[\Lambda_{2L}(x_{t,j}) \leftrightarrow \mathcal{H}_{\geq n} \text{ and } \Lambda_{2L}(x_{t,j}) \leftrightarrow \mathcal{Y}_k^c]}{s_{n-t}} \\
&\lesssim \mathrm{e}^{-c_0 k}
\end{aligned}$$

where the supremum in the before-last line is over all the domains D and boundary conditions ξ given by $\mathsf{C}_{\leq t}$ and the configuration in $\Lambda_{2L}(x_{t,j})$. The last inequality is due to Proposition 5.3.4. This proves (5.3.2).

Now, for j fixed, by the almost sure finiteness of the cluster of 0,

$$\phi_{\mathcal{F}_t,p}[N_{t+C} \,|\, \mathbf{j} = j] = \sum_{k \geq 1} \phi_{\mathcal{F}_t,p}[N_{t+C} \,|\, \mathsf{K} = k,\, \mathbf{j} = j] \phi_{\mathcal{F}_t,p}[\mathsf{K} = k \,|\, \mathbf{j} = j]. \tag{5.18}$$

To bound $\phi_{\mathcal{F}_t,p}[N_{t+C} \,|\, \mathsf{K} = k,\, \mathbf{j} = j]$ explore first the connected component $\tilde{\mathsf{C}}$ of $\mathscr{L}_{t,j}$ and observe that it is connected to at most $2kC$ intervals $\mathscr{L}_{t+C,\ell}$, since it is contained in $\mathcal{Y}_k$.

Conditionally on $\tilde{\mathsf{C}}$, all other active intervals $\mathscr{L}_{t,\ell}$ are connected in $\tilde{\mathsf{C}}^c$ to a random number of intervals $\mathscr{L}_{t+C,\ell'}$ with an average at most η. Indeed, the conditioning on $\tilde{\mathsf{C}}$ induces free boundary conditions on $\tilde{\mathsf{C}}^c$, and (5.3.9) also applies in $(\mathsf{C}_{\leq t} \cup \tilde{\mathsf{C}})^c$ with the boundary conditions induced by the explored edges.

We conclude that $\phi_{\mathcal{F}_t,p}[N_{t+C} \,|\, \mathsf{K} = k,\, \mathbf{j} = j] \leq \eta N_t + 2Ck$. Inserting this into (5.3.2) and using (5.3.2) we conclude that

$$\phi_{\mathcal{F}_t,p}[N_{t+C} \,|\, \mathbf{j} = j] \leq \eta N_t + 2C \sum_k k \phi_{\mathcal{F}_t,p}[\mathsf{K} = k \,|\, \mathbf{j} = j] \leq \eta N_t + 2CC_0 \sum_k k\mathrm{e}^{-c_0 k}$$

which produces the desired result with $K = 2CC_0 \sum_k ke^{-ck} < \infty$. □

Recall that we still need to prove the "gluing estimate" (5.3.10). This is a relatively standard, but tedious use of the RSW property (5.2).

Proof of Lemma 5.3.10. Fix all parameters as in the lemma and Proposition 5.3.8. All constants appearing below will be independent of these parameters. The upper bound on $\phi_{\mathcal{F}_t,p}[\mathbf{j} = j]$ is immediate, since $\mathbf{j} = j$ requires j to be a top-most seed and requires that $\Lambda_{2L}(j)$ to be connected to $\mathcal{H}_{\geq n}$. These two events depend on the inside of $\Lambda_{2L}(j)$ and the outside $\Lambda_{2L}(j)$, respectively. The mixing property proved in (5.3.6) allows one to factorise the probabilities of these two events.

We now focus on the lower bound. The idea is to show that, conditionally on j being a top-most seed, the probability that it $\mathbf{j} = j$ is a uniformly positive multiple of s_{n-t}. In this proof we will use the notation $\mathcal{H}_{\geq s}$ also for s non-integer, as we will have to perform surgeries at scales lower than L.

Fix $R = L/10$ and write $x = x_{t,j}$ for the center of the interval $\mathscr{L}_{t,j}$. Write $\mathcal{A}$ for the points of the cluster $\mathsf{C}_{\leq t}$ on $\mathscr{L}_{t,j}$; we call these the active vertices. Also write $\mathcal{B}$ for of the points $\mathsf{C}_{\leq t}$ on $\mathscr{L}_{t,\ell}$ with $\ell > j$; these all lie on $\partial\mathcal{H}_{\geq t}$, above $\mathscr{L}_{t,\ell}$.

Notice first that, due to (5.3.4), (5.3.6) and a standard construction using (5.2)

$$\phi_{\mathcal{F}_t,p}\Big[\Lambda_R(x + (5R, -R)) \xleftrightarrow{\mathcal{H}_{\geq t+1/2}} \partial\mathcal{H}_{\geq n-t} \text{ and } \Lambda_R(x + (5R, R)) \xleftrightarrow{\omega^* \cap \mathcal{H}_{\geq t+1/2}} \infty\Big] \gtrsim s_{n-t}. \tag{5.19}$$

To ensure that $\mathbf{j} = j$, it suffices to link the cluster ensuring the first connection above to $\mathcal{A}$, while the dual path ensuring the second connection should be used to separate $\mathcal{A}$ from $\mathcal{B}$. One may be tempted to assume that this occurs with positive probability due to RSW estimates below the correlation length. Unfortunately this is generally false, but the probability of this event will be shown to be proportional to $\phi_{\mathcal{F}_t,p}[j$ is top-most seed$]$.

Let x_+ and x_- be the top-most and bottom-most points of $\mathcal{A}$ respectively. Write y_- for the bottom most point of $\mathcal{B}$. Note that x_+ and y_- are linked by a free path of the boundary of $\mathsf{C}_{\geq t}$.

Write $\partial_\infty\Lambda_{2R}(x_+)$ for the arc of $\partial\Lambda_{2R}(x_+)$ separating x_+ from ∞ in $\mathsf{C}^c_{\leq t}$. This contains at least $\partial\Lambda_{2R}(x_+) \cap \mathcal{H}_{\geq t}$. Split it into the top and bottom part, which lie above and below the point $x_+ + (2R, 0)$, respectively.

Write Γ for the top boundary of the cluster of $\mathcal{A}$ in $\mathsf{C}^c_{\geq t}$. This is a path that will be indexed by $[0, 1]$ starting from the point x_+. Since the cluster of $\mathcal{A}$ is a.s. finite, Γ eventually ends at x_-.

When j is a top-most seed, Γ exists in $\Lambda_{2L}(x)$ without connecting $\mathcal{A}$ to $\mathcal{B}$. We distinguish three cases

(1) x_+ and y_- are at a distance larger than $2R$ of each-other;

(2) x_+ and y_- are at a distance smaller than $2R$ of each-other and Γ first touches $\partial_\infty\Lambda_{2R}(x_+)$ on its bottom part, or

(3) x_+ and y_- are at a distance smaller than $2R$ of each-other and Γ first touches $\partial_\infty \Lambda_{2R}(x_+)$ on its top part.

Each case will be treated differently.

In the first case, explore nothing and set χ_0 to be the part of the boundary of $\mathsf{C}_{\geq t}$ between x_+ and y_-. Also write $z = x_+$ and $\tilde{\mathsf{C}} = \mathsf{C}_{\leq t}$.

In the second and third case explore Γ up to the first time τ it touches $\partial_\infty \Lambda_{2R}(x_+)$. Write z for its endpoint and $\tilde{\mathsf{C}} = \mathsf{C}_{\leq t} \cup \Gamma([0,\tau])$ for the explored part. In the second case, write χ_0 for the top part of $\Gamma([0,\tau])$; while in the third case we denote by χ_1 the bottom part of $\Gamma([0,\tau])$.

In the first two cases, χ_0 is *accessible* by a dual path in $\tilde{\mathsf{C}}^c$, meaning that there exists a tube of width $R/4$ and length at most $100R$ connecting χ_0 to $\Lambda_R(x_{t,j} + (5R, R))$ in $\tilde{\mathsf{C}}^c$. By RSW estimates (5.2), this tube contains a dual path with positive probability, which may then be prolonged to infinity at constant probabilistic cost (see (5.3.2)) – such a path will separate $\mathcal{A}$ from $\mathcal{B}$. In case (3), χ_1 is *accessible* by a primal path in $\tilde{\mathsf{C}}^c$, meaning that there exists a tube connecting χ_1 to $\Lambda_R(x_{t,j} + (5R, -R))$ in $\tilde{\mathsf{C}}^c$. If this tube contains a path, this path may then be linked to $\mathcal{H}_{\geq n-t}$ at a cost of at most s_{n-t} (see again (5.3.2)), and therefore connect $\mathcal{A}$ to to $\mathcal{H}_{\geq n-t}$.

In any case, our present exploration only suffices to create the desired connection in the primal or dual model, but not both. We will perform one more exploration which will ensure that the second connection may also be created with positive probability. We start off with cases (1) and (2) which will be treated in the same way.

Cases (1) and (2): Define the region Λ' as $\Lambda' = \Lambda_R(z)$ in case (1) and, in case (2), as the union of $\Lambda_R(u)$ for u in the bottom part of $\partial_\infty \Lambda_{2R}(x_+)$. Write $\partial_\infty \Lambda'$ for the arc of $\partial \Lambda'$ separating z from ∞ in $\tilde{\mathsf{C}}^c$. Finally, split $\partial_\infty \Lambda'$ at $x_+ + (3R, 0)$ into its top and bottom sections – see Figure **??**. Consider now τ' as the first time after τ when Γ touches $\partial_\infty \Lambda'$. We distinguish two sub-cases

(a) $\Gamma_{\tau'}$ is in the top part of $\partial_\infty \Lambda_R(z)$ or

(b) $\Gamma_{\tau'}$ is in the bottom part of $\partial_\infty \Lambda_R(z)$.

In case (a), write χ_1 for the wired arc of $\Gamma_{[0,\tau']}$. Notice that χ_1 is accessible by a primal path and χ_0 is accessible by a dual path, both in $(\tilde{\mathsf{C}} \cup \Gamma_{[0,\tau']})^c$, and the tubes used for each of them may be taken disjoint.

In case (b) a more complicated construction is needed. Write $\tilde{\Gamma}$ for the exploration path starting at x_+, leaving vertices connected to $\mathcal{A}$ in $\mathsf{C}^c_{\leq t}$ on its left (including those of $\mathcal{A}$, but not other vertices of $\mathsf{C}_{\leq t}$), and all other vertices on the right. See Figure **??** for an illustration. Write τ'' for the first time this path touches $\partial_\infty \Lambda_R(z)$; Figure **??** shows why such a time exists when j is a top-most seed, and why $\tilde{\Gamma}_{\tau''}$ is "below" $\Gamma_{\tau'}$ on $\partial_\infty \Lambda_R(z)$. It follows that the wired arc χ_1 of $\tilde{\Gamma}_{[0,\tau'']}$ is accessible in $\tilde{\mathsf{C}}^c$.

Notice that we do not claim that χ_1 is accessible when $\Gamma_{[\tau,\tau']}$ has been explored. Thus, one should make a choice at the time stopping time τ whether to continue exploring Γ up to time τ' or whether to explore $\tilde{\Gamma}$ up to time τ''.

Notice however that either

$$\phi_p\big[\text{case (a)} \,\big|\, \text{case (1) or (2), } \Gamma_{[0,\tau]} \text{ and } j \text{ top-most seed}\big] \geq 1/2 \text{ or}$$
$$\phi_p\big[\text{case (b)} \,\big|\, \text{case (1) or (2), } \Gamma_{[0,\tau]} \text{ and } j \text{ top-most seed}\big] \geq 1/2.$$

If (a) is more probable, then explore Γ up to time τ'. If this corresponds to case (a), define χ_1 as above, otherwise define $\chi_1 = \emptyset$. If (b) is more probable, explore $\tilde{\Gamma}$ up to time τ''. If the wired part of $\tilde{\Gamma}_{[0,\tau'']}$ is indeed accessible, denote it by χ_1; otherwise define $\chi_1 = \emptyset$.

Our analysis proves that

$$\phi_p\big[\chi_0 \text{ and } \chi_1 \neq \emptyset \big| \text{ case (1) or (2) and } j \text{ top-most seed}\big] \geq 1/2,$$

and (5.2) ensures that

$$\phi_{\mathcal{F}_t,p}[\mathbf{j} = j \,\big|\, \text{case (1) or (2) and } j \text{ top-most seed}\big] \gtrsim s_{n-t}.$$

Cases (3): This case is treated similarly to case (2), except that the goal is now to define χ_0. Set Λ' to be the union of $\Lambda_R(u)$ for u in the top part of $\partial_\infty \Lambda_{2R}(x_+)$. Define $\partial_\infty \Lambda'$ and its top and bottom sections similarly to how this was done in case (2).

Define sub-cases (a) and (b) depending on where Γ exits Λ'; set τ' to be the exit time. If it is more likely to exit on the bottom part, define χ_0 as the free (top) side of $\Gamma_{[\tau,\tau']}$. If it is more likely to exit on the top part, start an exploration $\tilde{\Gamma}$ from x_+ leaving wired on the left and free on the right (this exploration runs along $\partial \mathcal{C}_{\leq t}$ up to y_-, and continues at least up to $\partial \Lambda_{2L}(x)$ when j is a top-most seed). Define then χ_0 as the free side of $\tilde{\Gamma}$ up to the first exit time $\partial_\infty \Lambda'$.

The same type of analysis as above shows that

$$\phi_{\mathcal{F}_t,p}[\mathbf{j} = j \,\big|\, \text{case (3) and } j \text{ top-most seed}\big] \gtrsim s_{n-t}.$$

□

We finally turn to the proof of Proposition 5.3.7, which follows by a standard martingale argument.

Proof of Proposition 5.3.7. Assume $C > 0$ is chosen large enough for Proposition 5.3.8 to hold. For the sake of simplicity, we argue as if $C = 1$. The general proof follows by considering the process N_{Ct} instead of N_t. Fix $p < p_c$ and $t < n$ two integers. Set $\ell = K/(1-\mu)$, and observe that $\ell > 0$ is the unique real number satisfying the relation $\ell = \mu\ell + K$. Also call $\tilde{\ell} = \lfloor \ell \rfloor + 1 \in \mathbb{N}$ and $\varepsilon = \tilde{\ell} - \ell > 0$. Our first target estimate is the following: if $N_t > \ell$ and $t + R \leq n$, then we are going to prove that

$$\phi_p[N_{t+s} > \ell, \forall s \in \{1, \dots, R\} | X_n \neq \dagger, N_t] \lesssim N_t \mu^R. \tag{5.20}$$

Set $M_s := N_{t+s} - \ell$, observe that it follows from (5.8) that

$$\phi_p[M_{s+1} | M_s, X_n \neq \dagger] \leq \mu M_s. \tag{5.21}$$

Iterating this relation we obtain that

$$\phi_p[M_{s+1} | M_0, X_n \neq \dagger] \leq \mu^s M_0.$$

Let $\tau := \min\{s \geq 0, M_s \leq 0\}$, and consider the process $\tilde{M}_s := M_{s\wedge\tau} \wedge 0$. An easy computation shows that (5.10) also holds for the process $\tilde{M}$, and that this process is non-negative. By Markov's inequality we get that

$$\begin{aligned}
\phi_p[N_{t+s} > \ell, \forall s \in \{1, \dots, R\} | X_n \neq \dagger, N_t] &= \phi_p[N_{t+s} \geq \tilde{\ell}, \forall s \in \{1, \dots, R\} | X_n \neq \dagger, N_t] \\
&= \phi_p[M_s > \varepsilon, \forall s \in \{1, \dots, R\} | X_n \neq \dagger, N_t] \\
&= \phi[\tilde{M}_R > \varepsilon | X_n \neq \dagger, \tilde{M}_0] \\
&\leq \varepsilon^{-1} \tilde{M}_0 \mu^R = \varepsilon^{-1} M_0 \mu^R
\end{aligned}$$

This establishes (5.9), in the case in which $N_t > \ell$.

We are going to deduce that the excursions of N outside of the set $\{t \in \{1, \dots, r\}, N_t \leq \ell\}$ are typically small. Indeed, write

$$\begin{aligned}
&\phi_{\mathcal{F}_t,p}[N_s > \ell, \forall s \in \{t+1, \dots, t+r\} | N_t \leq \ell, X_n \neq \dagger] \\
&\quad \leq \phi_{\mathcal{F}_t,p}[N_s > \ell, \forall s \in \{t+1, \dots, t+r\} | N_{t+1} \leq r\ell, N_t \leq \ell, X_n \neq \dagger] \\
&\qquad + \phi_{\mathcal{F}_t,p}[N_{t+1} > r\ell | N_t \leq \ell, X_n \neq \dagger].
\end{aligned}$$

Due to (5.9), the first summand is upper bounded by $r\ell\mu^r$.

For the second summand, observe that due to Proposition 5.3.4,

$$\phi_{\mathcal{F}_t,p}[N_{t+1} > r\ell | N_t \leq \ell, X_n \neq \dagger] \leq \exp(-cr).$$

We finally obtain that there exists a possibly different value of $c > 0$ such that:

$$\phi_p[N_{t+s} > \ell, \forall s \in \{1, \dots, r\} | X_n \neq \dagger, N_t] \leq \exp(-cr). \tag{5.22}$$

We then prove the following: “finite energy”-type property: there exists $\eta > 0$ such that for any $t \leq n$,

$$\phi_p[N_{t+1} = 1 | N_t \leq \ell, X_n \neq \dagger] \geq \eta. \tag{5.23}$$

The proof of this item is very similar to that of Proposition 5.3.8, and we shall be quite synthetic about it. Condition on the event $\{\mathbf{j} = j\}$, as in the proof of Proposition 5.3.8. On the event that $\mathscr{L}_{t,j}$ only has one child, it easy to conclude by exploring the cluster of $\mathscr{L}_{t,j}$, and observe that it induces free boundary conditions on the remainder of the space. As there are at most $\ell - 1$ active boxes which clusters need to be explored, the probability that each one dies before reaching $\mathcal{H}_{\geq t+1}$ can be lower bounded by $\varepsilon^{\ell-1}$, where ε is a constant independent of p.

Next, we prove that there exists an absolute η such that

$$\phi_{\mathcal{F}_t,p}[N(j) = 1 | \mathbf{j} = j] \geq \eta > 0,$$

where we denoted by $N(j)$ the number of line segments activated by $\mathscr{L}_{t,j}$.

We write

$$\begin{aligned}\phi_{\mathcal{F}_t,p}[N(j) = 1 | \mathbf{j} = j] &= \frac{\phi_{\mathcal{F}_t,p}[N(j) = 1, \mathbf{j} = j]}{\phi_{\mathcal{F}_t,p}[\mathbf{j} = j]} \\ &\gtrsim \frac{\phi_{\mathcal{F}_t,p}[N(j) = 1, \mathbf{j} = j]}{s_{n-t}\phi_{\mathcal{F}_t,p}[j \text{ is a top-most seed}]},\end{aligned}$$

where the upper bound for the denominator is just an inclusion of events and mixing. It remains to prove that:

$$\phi_{\mathcal{F}_t,p}[N(j) = 1, \mathbf{j} = j] \asymp s_{n-t}\phi_{\mathcal{F}_t,p}[j \text{ is a top-most seed}].$$

Looking back to the proof of Lemma 5.3.10, it is easy to modify the RSW construction to ensure that the connection from $\mathscr{L}_{t,j}$ intersects the hyperplane $\partial\mathcal{H}_{\leq t+1}$ only along a segment of length L. We refer to the proof of Lemma 5.3.10 for details.

Thanks to this input together with (5.11), we are able to conclude. Indeed, we start with the first bit of Proposition 5.3.7. Remember that (5.11) shows that the durations of the excursions outside of the level set $\{s \in \{1, \dots, r\}, N_{t+s} \leq \ell\}$ are stochastically dominated by a family of independent and identically distributed exponential random variables of parameter e^{-c}. Fix $\alpha > \frac{1}{1+\mathrm{e}^{-c}}$. We first write, thanks to (5.12):

$$\begin{aligned}&\phi_{\mathcal{F}_t,p}[N_s > 1 \ \forall s \in \{t+1, \dots, t+r\} \,|\, N_t = 1, \ X_n \neq \dagger] \\ &\qquad \leq (1-\eta)^{\alpha r} + \phi_{\mathcal{F}_t,p}[|\{s \in \{1, \dots, r\}, N_{t+s} > \ell\}| > \alpha r | N_t = 1, X_n \neq \dagger] \\ &\qquad \leq (1-\eta)^{\alpha r} + \exp(-c(\alpha) r),\end{aligned}$$

where $c(\alpha)$ is an absolute constant coming from the law of large numbers for the family of exponential random variables of parameter e^{-c}.

This proves the first item of Proposition 5.3.7.

The proof of the second item is very similar to the one of the first. It is sufficient to prove that $\phi_p[N_0 > k | X_n \neq \dagger] \leq \mathrm{e}^{-ck}$ for some constant $c > 0$, and the same proof follows *mutatis mutandi*. We use a reasoning similar to that of the proof of Proposition 5.3.8. Indeed, we claim that a standard RSW construction implies that

$$\phi_p[0 \longleftrightarrow \mathcal{H}^{\vec{w}}_{\geq n}] \asymp \phi_p[0 \longleftrightarrow \partial\Lambda_L] s_{n-t}.$$

Together with this "gluing lemma", we can repeat the same proof as the one of Proposition 5.3.8 to get that

$$\phi_p[\mathsf{K} > k | 0 \longleftrightarrow \mathcal{H}^{\vec{w}}_{\geq n}] \leq \mathrm{e}^{-ck}.$$

In particular, we obtain that $\phi_p[N_0 > k | X_n \neq \dagger] \leq \mathrm{e}^{-ck}$, which concludes the proof. □

5.3.3 Uniform mixing when $N_k = 1$

We now prove that the process X_k satisfies a mixing property when $N_k = 1$.

Write $\mathsf{C}_{\geq t} = \mathsf{C} \setminus \mathsf{C}_{\leq t}$ and $\mathsf{C}_{s \leq \cdot \leq t} = \mathsf{C}_{\leq t} \setminus \mathsf{C}_{\leq s}$ for $0 \leq s \leq t$. For $n \geq 1$ let $\mathcal{L}_n$ be the law of the cluster of $\mathscr{L}_{0,0}$ under the measure $\phi^1_{\mathcal{H}_{\geq 0},p}[\cdot | \mathscr{L}_{0,0} \longleftrightarrow \mathcal{H}_{\geq n}]$. When writing C under the measure $\mathcal{L}_n$, we refer to the cluster of $\mathscr{L}_{0,0}$.

We are now ready to prove the main decoupling estimate.

Proposition 5.3.11. *There exists $\eta > 0$ such that for any $p \leq p_c$, $1 \leq t \leq n$ and any realisation χ of $\mathsf{C}_{\geq 1/2}$ contained in $\mathcal{Y} - (L, 0)$ (where $\mathsf{C}_{\geq 1/2}$ is sampled according to $\mathcal{L}_{n-t}$),*

$$\phi_p\big[\mathsf{C}_{\geq t+1/2} = \chi + X_t \,\big|\, \mathcal{F}_t, N_t = 1, X_n \neq \dagger\big] \geq \eta\, \mathcal{L}_{n-t}[\mathsf{C}_{\geq 1/2} = \chi], \tag{5.24}$$

where $\chi + X_t$ is the translation of χ by X_t.

Moreover, for any $1 \leq s < t \leq n$ and any possible realisation ζ of $\mathsf{C}_{s \leq \cdot \leq t}$ such that $N_t = 1$, if χ is contained in $\mathcal{Y} - (\frac{t-s}{2} L, 0)$

$$\sum_{\chi} \Big| \phi_p[\mathsf{C}_{\geq t} = \chi + X_s \,|\, \mathcal{F}_t, N_s = 1, \mathsf{C}_{s \leq \cdot \leq t} = \zeta + X_s, X_n \neq \dagger] - \mathcal{L}_{n-s}[\mathsf{C}_{\geq t-s} = \chi \,|\, \mathsf{C}_{\leq t-s} = \zeta] \Big| \leq e^{-\eta(t-s)}. \tag{5.25}$$

The first inequality, together with Proposition 5.3.4 states that at every pre-renewal time, there is a positive probability that the future is sampled independently of the past, therefore creating a true renewal. It is tempting to think that this implies that a positive proportion of pre-renewals are actual renewals, thus proving Theorem 5.3.2. Unfortunately this is not the case, hence the need for the second part of the proposition, which states that if the clusters sampled according to $\phi_p[.\,|\,\mathcal{F}_s, N_s = 1, X_n \neq \dagger]$ and $\mathcal{L}_{n-s}$ are coupled for $t-s$ steps, then the probability that they couple the rest of the process is exponentially close to 1. Crucially, the value of $\eta > 0$ may be chosen uniformly in p, the direction $\vec{w}$ as well as in t, s and n and the realisation of the cluster up to t.

Remark 5.3.12. Note that (5.13) does not make a statement about the whole "future" of C, but rather about the future after a buffer zone of width $L/2$. We call $\mathsf{C}_{t\leq.\leq t+1/2}$ the "link". Thus (5.13) states that for cone-contained futures, their probabilities under $\phi_p\big[.\,\big|\,\mathcal{F}_t, N_t = 1, X_n \neq \dagger\big]$ are comparable to their probabilities under $\mathcal{L}_{n-t}$, and therefore only have a limited dependence on $\mathcal{F}_t$. This statement does not apply to the link, which does depend strongly on $\mathcal{F}_t$.

This is a crucial difference with [25], where the link is trivial due to the use of cone points.

The rest of the section is dedicated to proving Proposition 5.3.11.

Proof of Proposition 5.3.11. We start with the proof of (5.13), which is the most complicated property. Recall that we write $\mathcal{L}_{n-t}$ for the law of the cluster C of $\mathscr{L}_{0,0}$ under $\phi^1_{\mathcal{H}_{\geq 0}}[.\,|\,\mathscr{L}_{0,0} \longleftrightarrow \mathcal{H}_{\geq n-t}]$.

For a possible realisation χ of $\mathsf{C}_{\geq 1/2}$, write $\partial_0\chi$ as the set of vertices not contained in χ but adjacent to it and which are connected in $\mathcal{H}_{\geq 1/2}$. Write $\partial_1\chi$ for the vertices of $\chi \cap \partial\mathcal{H}_{\geq 1/2}$ Then

$$\mathcal{L}_{n-t}[\mathsf{C}_{\geq 1/2} = \chi] = \tfrac{1}{Z(\mathcal{L})}\phi^1_{\mathcal{H}_{\geq 0}}[\omega(e) = 1 \text{ for } e \in \chi \text{ and } \omega(e) = 0 \text{ for } e \in \partial\chi \cap \mathcal{H}_{\geq 1/2}] \cdot \phi^\sigma_{\mathcal{H}_{\geq 0}\setminus\chi}[\mathscr{L}_{0,0} \Leftrightarrow \chi],$$

where σ is the boundary condition which is free on $\partial_0\chi$, wired on $\partial\mathcal{H}_{\geq 0}$ and induced by the connection in χ on $\partial_1\chi$; $Z(\mathcal{L})$ is a normalising constant ensuring that $\mathcal{L}_n$ is a probability measure. We used the notation $\Leftrightarrow$ to indicate that all connected components of χ should be connected together and to $\mathscr{L}_{0,0}$, but should not connect to any other point of $\mathcal{H}_{\geq 1/2}$.

Similarly, if we set $D = ((\mathsf{C}_{\leq t} \cup (\chi + X_t))^c$

$$\begin{aligned}\phi_p\big[\mathsf{C}_{\geq t+1/2} =& \chi + X_t\,\big|\,\mathcal{F}_t, N_t = 1, X_n \neq \dagger\big]\\ &= \tfrac{1}{Z(\mathcal{F}_t)}\phi_p\big[\omega(e) = 1 \text{ for } e \in \chi + X_t \text{ and } \omega(e) = 0 \text{ for } e \in \partial(\chi + X_t) \cap \mathcal{H}_{\geq t+1/2}\big]\\ &\quad\cdot\phi^\sigma_D[\mathsf{C}_{\leq t} \Leftrightarrow \chi],\end{aligned}$$

where σ is the boundary condition on D defined as above on $\partial_0\chi + X_t$ and $\partial_1\chi + X_t$ and are wired on all vertices of $\mathsf{C}_{\leq t} \cap \partial\mathcal{H}_{\leq t}$ and free on $\partial\mathsf{C}_{\leq t} \cap \mathcal{H}_{\leq t}$. The normalising constant $Z(\mathcal{F}_t)$ ensured that the above is a probability measure.

We now claim that, when $\chi \subset \mathcal{Y} - (L, 0)$

$$\begin{aligned}
&\phi_p\big[\omega(e) = 1 \text{ for } e \in \chi + X_t \text{ and } \omega(e) = 0 \text{ for } e \in \partial(\chi + X_t) \cap \mathcal{H}_{\geq t+1/2}\big] \\
&\qquad \asymp \phi^1_{\mathcal{H}_{\geq 0}}[\omega(e) = 1 \text{ for } e \in \chi \text{ and } \omega(e) = 0 \text{ for } e \in \partial\chi \cap \mathcal{H}_{\geq 1/2}] \qquad \text{and (5.26)} \\
&\phi^\sigma_D[\mathsf{C}_{\leq t} \Leftrightarrow \chi] \\
&\qquad \asymp \phi^\sigma_D[\mathsf{C}_{\leq t} \longleftrightarrow \mathcal{H}_{\geq t+1/4}]\phi^\sigma_{\mathcal{H}_{\geq 0}\setminus\chi}[\mathscr{L}_{0,0} \Leftrightarrow \chi]. \qquad (5.27)
\end{aligned}$$

Indeed, (5.3.3) is a direct consequence of Lemma 5.3.5. The second relation (5.3.3) is more complicated. It should be understood as stating that connecting $\mathsf{C}_{\leq t}$ to χ is comparable to the product of the probabilities for $\mathsf{C}_{\leq t}$ and χ to be connected to distance $L/4$, with χ being connected in the sense of $\Leftrightarrow$. Indeed, once these connections are established, one may prove that $\mathsf{C}_{\leq t}$ and χ connect with positive probability. To do this, one should perform a particular exploration very similar to that used in the proof of Proposition 5.3.8. Notice that here the connection to $\mathsf{C}_{\leq t}$ is easier to establish, as it does not need to avoid any active vertices. The connection to χ is more delicate, as it requires all connected components of χ to be connected, but to not connect to any other point of $\mathcal{H}_{\geq 1/2}$. This last requirement complicates the construction slightly, but not in any fundamental way. We will not give further details of this argument here.

Finally, the same reasoning implies that to be connected to distance $L/4$ may be shown to be comparable to

$$\phi^\sigma_{\mathcal{H}_{\geq 0}\setminus\chi}[\partial\mathcal{H}_{\leq t+1/4} \Leftrightarrow \chi] \asymp \phi^\sigma_{\mathcal{H}_{\geq 0}\setminus\chi}[\mathscr{L}_{0,0} \Leftrightarrow \chi].$$

In conclusion, for χ contained in $\mathcal{Y} - (L, 0)$ we find

$$\frac{\phi_p\big[\mathsf{C}_{\geq t+1/2} = \chi + X_t \,\big|\, \mathcal{F}_t, N_t = 1, X_n \neq \dagger\big]}{\mathcal{L}_{n-t}[\mathsf{C}_{\geq 1/2} = \chi]} \asymp \frac{Z(\mathcal{L})}{Z(\mathcal{F}_t)}\phi^\sigma_D[\mathsf{C}_{\leq t} \longleftrightarrow \mathcal{H}_{\geq t+1/4}] \qquad (5.28)$$

Since both measures on the left-hand side assign positive mass to $\chi \in \mathcal{Y} - (L, 0)$ (see Lemma 5.3.4) we conclude that

$$\tfrac{Z(\mathcal{L})}{Z(\mathcal{F}_t)}\phi^\sigma_D[\mathsf{C}_{\leq t} \longleftrightarrow \mathcal{H}_{\geq t+1/4}] \asymp 1,$$

which, when combined with (5.3.3) yields (5.13).

The second property (5.3.11) is a direct consequence of Lemma 5.3.5 and the fact that under $\mathcal{L}_{n-s}[\cdot|\mathsf{C}_{\leq t-s} = \zeta]$, the event $\mathsf{C}_{\geq t-s} = \chi$ is entirely determined by the state of the edges in $\mathcal{Y} - (\frac{t-s}{2}L, 0)$. □

5.3.4 Proof of Theorem 5.3.2

Proof of Theorem 5.3.2. Fix a direction $\vec{w}$. For C, write $S_0 = 0$ and for $k \geq 0$ set

$$S_{k+1} = \inf\{t \geq S_k + 2 : \ N_t = 1\}$$

Note that we impose that $S_{k+1} - S_k \geq 2$; this is purely for technical reasons. Let K be the first index for which $S_K = \infty$.

Let $\mathsf{D}_k = \mathsf{C}_{\leq S_k} \setminus \mathsf{C}_{\leq S_{k-1}}$ for $k = 1, \ldots, K-1$. We also set $\mathsf{D}_K = \mathsf{C} \setminus \mathsf{C}_{\leq S_{K-1}}$; this is the only piece D_k which does not end with a time for which $N_t = 1$.

First we will describe how to sample the pieces D_k sequentially, which in turn also constructs the sequence S_k. For this proof, write $\mathbb{P}$ for the probability measure used to sample C according to the procedure described below. We will then argue that the sequence D_k has the properties necessary for Theorem 5.3.2.

Sequential sampling of C. Fix ζ a possible realisation of $\mathsf{C}_{\leq S_k}$. We will describe how to sample D_{k+1} conditionally on the event that $\mathsf{C}_{\leq S_k} = \zeta$. For simplicity write $s = S_k$ – the value of S_k is determined by the conditioning, so may be treated as a constant.

For $1 \leq j < s$ and χ a potential realisation of $\mathsf{C}_{\geq S_k} - X_k$ contained in $\mathcal{Y} + j/2$ set

$$q_j(\zeta, \chi) = \min_{\xi} \phi^{\xi}_{\mathcal{H}_{\geq 0}}\big[\mathsf{C}_{\geq j} = \chi + X_j \,\big|\, \mathsf{C}_{\leq j} = \zeta_{s-j \leq \cdot \leq s}\big],$$

For $j \geq s$, simply set $q_j(\zeta, \chi) = \phi_p[\mathsf{C}_{\geq j} = \chi + X_s \,\big|\, \mathsf{C}_{\leq s} = \zeta]$. Finally, for χ a potential realisation of $\mathsf{C}_{\geq S_k + 1/2} - X_k$ contained in $\mathcal{Y} - (L, 0)$, set

$$q_0(\zeta, \chi) = \min_{\xi} \phi^{\xi}_{\mathcal{H}_{\geq 0}}\big[\mathsf{C}_{\geq j+1/2} = \chi + X_j \,\big|\, \mathsf{C}_{\leq j} = \zeta_{s-j \leq \cdot \leq s}\big],$$

$$q_0(\zeta, \chi) = \min_{\xi} \phi^{\xi}_{\mathcal{H}_{\geq 0}}\big[\mathsf{C}_{\geq 1/2} = \chi + X_s\big].$$

Then define

$$Z_j(\zeta) := \sum_{\chi} q_j(\zeta, \chi).$$

By definition, the quantities $q_j(\zeta, \chi)$ and $Z_j(\zeta)$ are increasing in j. Define a random variable $\tilde{M}$ taking values in $\mathbb{N}$ with

$$\mathbb{P}[\tilde{M}_k \leq j \,|\, \mathsf{C}_{\leq s} = \zeta] = Z_j(\zeta).$$

Sample $\tilde{M}$, then, conditionally on $\tilde{M}$, set $\tilde{C}_{\geq t} = \chi$ with probability

$$\mathbb{P}[\tilde{C}_{\geq t} = \chi \,|\, \mathcal{F}_t, \, \tilde{M}_k = j] = \tfrac{1}{Z_j(\zeta)}\big(q_j(\zeta, \chi) - q_{j-1}(\zeta, \chi)\big).$$

where we ignore the last term when $j = 0$. With this definition, we find

$$\mathbb{P}[\{\tilde{C}_{\geq t} = \chi\} \cap \{\tilde{M}_k \leq j\} \,|\, \mathsf{C}_{\leq s} = \zeta] = q_j(\zeta, \chi),$$

and therefore

$$\mathbb{P}[\tilde{C}_{\geq t} = \chi \,|\, \mathsf{C}_{\leq s} = \zeta] = \phi_p[\mathsf{C}_{\geq j} = \chi + X_s \,\big|\, \mathsf{C}_{\leq s} = \zeta] \tag{5.29}$$

Finally, set $\mathsf{D}_{k+1} = \mathsf{D}_1(\tilde{C}_{\geq t})$, which is to say the piece of $\tilde{C}_{\geq t}$ up to its first pre-renewal.

Due to (5.3.4), we have

$$\mathbb{P}[\mathsf{D}_{k+1} = D \,|\, \mathsf{C}_{\leq s} = \zeta] = \phi_p[\mathsf{D}_{k+1} = D \,|\, \mathsf{C}_{\leq s} = \zeta],$$

for any realisation D of D_{k+1}. This implies that when sampling C according to this procedure, we indeed obtain the law of the cluster of 0 under the measure ϕ_p.

Memory variables and renewal structure. Observe that, due to Proposition 5.3.11,

$$\mathbb{P}[\tilde{M}_k > j | \mathcal{F}_t] \leq \eta \exp(-\eta j),$$

for all $j \geq 0$. As such, we may bound $\tilde{M}_k$ from above by a geometric variable M_k starting from 0, with some parameter $\eta' > 0$ depending only on η, which is independent of $\mathcal{F}_t$. It follows that the variables M_k are i.i.d geometric variables with a uniform parameter.

Observe now that, for $j \geq 1$, if $M_k \leq j$, then the choice of D_{k+1} is independent of $\mathcal{F}_{S_k - j}$, and in particular of $\mathcal{F}_{S_{k-j}+1}$. Moreover, when $M_k = 0$, it is only the link between D_{k+1} that depends on $\mathcal{F}_{S_k}$, the rest of D_{k+1} is independent of $\mathcal{F}_{S_k}$ – see also Remark 5.3.12.

If follows that if k is such that $M_{k+j} \leq j$ for all $j \geq 0$, then $\mathsf{C}_{\geq S_k}$ is independent of $\mathcal{F}_{S_k}$, except for the link. In particular, this implies that S_k is a renewal time for the process (X_t). Fix therefore the consecutive times $(T_\ell)_{\ell \geq 1}$ which correspond to such renewal times S_k.

Exponential tails. For $k \geq 1$, set

$$I_k = \inf\{\ell \geq k : M_\ell \geq \ell - k\}.$$

Note that $I_k = \infty$ implies that S_k is a renewal time.

Define $Y_t = 1$ if $t = S_k$ and $I_k = \infty$. Let us now prove that the inter-renewal times T_k of Definition 5.3.1 have exponential tails. It is clear that

$$\begin{aligned}\mathbb{P}[n < T_{k+1} - T_k < +\infty | \mathcal{F}_{T_k}, T_k < \infty] &= \mathbb{P}[\exists \ell > n, M_{k+\ell} \geq \ell] \\ &\lesssim \sum_{\ell > n} (1-\eta)^\ell \lesssim (1-\eta)^n.\end{aligned}$$

Thus the inter-renewal times have exponential tiails.

Finally, observe that, by construction, each piece $\mathsf{C}_{T_k \leq \cdot \leq T_{k+1}}$ is contained in the cone $\mathcal{Y} + (T_k, X_{T_k})$, which, together with the exponential decay of the inter-renewal times, implies the exponential decay of the steps of (X_t).

Mass gap. To prove the mass gap, we will first argue that the inter-renewal times also have exponential decay when conditioned on a survival event. Indeed, if we apply the above procedure conditionally on C hitting $\mathcal{H}_{\geq n}$, then at every time S_k, $\tilde{\mathsf{C}}_{\geq S_k}$ will be sampled according to $\phi_p[.|\mathcal{F}_{S_k}, X_n \neq \dagger]$. Due to (5.13) and (5.3.11), the variables $\tilde{M}_k$ may still be bounded from above by i.i.d. modified geometric variables with uniform parameter, even under this conditioning. We conclude in the same way as above that

$$\mathbb{P}[\ell < T_{k+1} - T_k < \infty \,|\, \mathcal{F}_{T_k},\, X_n \neq \dagger] \leq \exp(-c_T\,\ell), \tag{5.30}$$

for some constant $c_T > 0$ independent of p, $\vec{w}$, k and n.

This implies that

$$\begin{aligned}&\mathbb{E}\big[\mathbb{1}_{\{\}}\{\ell < T_{k+1} - T_k < \infty\}\mathbb{P}[X_n \neq \dagger \,|\, \mathcal{F}_{T_{k+1}}] \,|\, \mathcal{F}_{T_k}]\\ &\quad = \mathbb{P}[\ell < T_{k+1} - T_k < \infty \text{ and } X_n \neq \dagger \,|\, \mathcal{F}_{T_k}] \leq \exp(-c_T\,\ell)\mathbb{P}[X_n \neq \dagger \,|\, \mathcal{F}_{T_k}].\end{aligned}$$

Writing $t = T_{k+1} - T_k$, we have that

$$\mathbb{P}[X_n \neq \dagger \,|\, \mathcal{F}_t]/\mathbb{P}[X_n \neq \dagger \,|\, \mathcal{F}_{T_{k+1}}] \leq \mathbb{P}[X_t \dagger],$$

for any realisations of $\mathcal{F}_{T_k}$ and $\mathcal{F}_{T_{k+1}}$ as above. Im Check whether this is true

It follows that our process (X_t, Y_t) defined as above does indeed have a mass gap of $c_T > 0$, uniform in p and $\vec{w}$.

Initial step survival rate

It remains to prove (5.3.2). Observe that $\mathbb{P}[T_1 < \infty] \leq \phi[0 \longleftrightarrow \Lambda_L] \asymp \pi_1(L(p))$, where the last asymptotic equivalence is due to Theorem 5.2.2. For the lower bound, observe that standard RSW considerations imply that

$$\mathbb{P}[T_1 < \infty | 0 \longleftrightarrow \Lambda_L] \asymp \mathbb{P}[T_1 < \infty | \mathscr{L}_{0,0} \longleftrightarrow \mathcal{H}_{\geq n}] \geq c > 0,$$

due to the above analysis. □

Im edit this remark

Remark 5.3.13. Observe that this construction provides a rigorous construction of "the infinite cluster conditioned to survive in the direction $\vec{e_1}$". Indeed, call $\mathscr{L}$ the limit of the joint distribution of T_1 and Expl_{T_1} under $\mathscr{L}_n$ [3], and call $\mathscr{L}_\infty$ the concatenation of independent samples of $\mathscr{L}$. $\mathscr{L}_\infty$ satisfies the equation $\mathscr{L} \circ \mathscr{L}_\infty$, and is natural law for the infinite cluster of 0 conditioned to survive in the direction $\vec{e_1}$.

5.4 Local limit theorem for the Markov renewal process

5.4.1 Probability of hitting a half-space

Equipped with the renewal structure of the process (X_k), we start by proving the following.

Theorem 5.4.1. *For any $p < p_c$ and $\vec{w} \in \mathbb{S}^1$, there exists $\zeta(p, \vec{w})$ such that, for the chain (X_n) constructed with these parameters*

$$\phi_p[X_n \neq \dagger] \asymp \pi_1(L(p))\, \mathrm{e}^{-\frac{n}{\zeta(p,\vec{w})}}. \tag{5.31}$$

uniformly in p, $\vec{w}$ and $n \geq 1$. Moreover, ζ is uniformly bounded away from 0 and ∞.

Corollary 5.4.2. *For any $\vec{w} \in \mathbb{S}^1$ and $p < p_c$,*

$$\phi_p[0 \longleftrightarrow \mathcal{H}^{\vec{w}}_{\geq N}] \asymp \pi_1(L(p))\, \mathrm{e}^{-c\frac{N}{\zeta(p,\vec{w})L(p)}}.$$

uniformly in $N \geq L(p)$, p and $\vec{w}$.

The corollary states that the probability to hit a far-away half-space in the direction $\vec{w}$ decreases exponentially in the distance to the line with a parameter $\zeta(p, \vec{w})L(p)$. This does not imply that $\xi_p(\vec{w}) = \zeta(p, \vec{w})L(p)$, since $\xi_p(\vec{w})$ is not defined in terms of hitting a half-space, but rather a specific point. We will see how to deduce $\xi_p(\vec{v})$ in the next section.

Proof of Theorem 5.4.1. The proof consists in two distinct parts. The first one consists in establishing the pure exponential decay of $\phi_p[X_{n+T_1} \neq \dagger | \mathcal{F}_{T_1}, T_1 < \infty]$. The second one consists in a classical gluing construction to obtain the factor $\pi_1(L(p))$ corresponding to the probability of survival until the first renewal time T_1. We start with the first part. For sake of simplicity, we consider in the first part of the proof that $T_1 = 0$. We also use the notation $\mathbb{P}$ corresponding to the probability measure introduced in the last section. Let also τ be a random variable having the law of the inter-renewal time, *i.e.* $\tau \overset{(\mathrm{d})}{=} T_2 - T_1$.

Let us introduce $p_n := \mathbb{P}[\exists k \geq 0, T_k = n \text{ and } T_{k+1} = \dagger]$. We also set $a_n := \mathbb{P}[\tau = n]$. Remember that we note $\kappa := \mathbb{P}[\tau = \dagger]$. Finally we denote by P (resp. A) the generating series of the sequence (p_n) (resp. (a_n)). By conditioning on the value of the first step of the

[3] The uniform exponential decay of the tails of T_1 under $\mathscr{L}_n$ ensures that this limit is well-defined and is a probability measure when $n \to \infty$.

walk, observe that for any $n \geq 0$, we have $p_n = \sum_{k=1}^{n} a_k p_{n-k} + \kappa \mathbb{1}_{n=0}$, which yields the "killed renewal equation"

$$P(z) = \frac{\kappa}{1 - A(z)}. \tag{5.32}$$

The equality holds in the whole domain of convergence of P, which is a power series with positive coefficients.

Observe that A is a power series with positive coefficients that satisfies $A(1) = 1 - \kappa < 1$. By injectivity of A on the positive half-line, there exists a unique $t_0 > 1$ such that $A(t_0) = 1$.

We now argue that the radius of convergence of A is strictly larger than t_0, that is that there exists $\varepsilon > 0$ such that

$$t_0^k \mathbb{P}[\tau = k] \leq \mathrm{e}^{-\varepsilon k}.$$

This is due to the mass gap property. Indeed, observe that by definition of the radius of convergence, it is the case that $p_k = t_0^{-k(1+o(1))}$, which means that $\mathbb{P}[\exists r \geq 0, T_r \geq k] = t_0^{-k(1+o(1))}$. Now, due to (5.3.4), it is the case that

$$\mathbb{P}[\tau = k | \exists r \geq 0, T_r \geq k] = t_0^{k(1+o(1))} \mathbb{P}[\tau = k] \leq \mathrm{e}^{-c_T k}.$$

This shows that the radius of convergence of A is strictly larger than t_0.

For convenience, write $\tilde{P}$ (resp. $\tilde{A}$) for the power series $P(t_0 z)$ (resp. $A(t_0 z)$), and $\bar{\mathbb{D}}$ for the closed unit disc. We now are going to argue that 1 is the only 0 in $\bar{\mathbb{D}}$ of the series $\tilde{A} - 1$. Indeed, it easy to see that if $|z| < 1$, then

$$\Big| \sum_{n \geq 1} a_n (t_0 z)^n \Big| \leq \sum_{n \geq 1} a_n (t_0 |z|)^n < 1.$$

Now we claim that the aperiodicity of the distribution of τ — which itself follows from (5.12) — implies that $\tilde{A} - 1$ does not have an additional 0 on $\partial \bar{\mathbb{D}} \setminus \{1\}$. Indeed, if it were the case, then there would exist some $\theta \in (0, 2\pi)$ such that $\sum_{k=0}^{+\infty} a_k t_0^k \mathrm{e}^{\mathrm{i}k\theta} = 1$. By the equality case in the triangular inequality, one would then have that for all the $k \in \mathrm{Supp}(\tau)$, the $\mathrm{e}^{\mathrm{i}k\theta}$ are aligned; this contradicts the aperiodicity of τ. Finally, we claim that 1 is a simple zero of the function $\tilde{A} - 1$. Indeed, due to the fact that the radius of convergence of A is strictly greater than t_0, one has that $\tilde{A}$ has a positive derivative at 1. Summarizing the previous reasoning, we proved that the function $g(z) := \frac{1 - A(z)}{1 - z}$ does not vanish on $\bar{\mathbb{D}}$. We wish to apply Wiener's $1/f$ theorem (see [115, Theorem 5.2]) to the function g. To that end, we need to check if this series is summable at 1. It is a simple observation that

$$g(z) = \sum_{n=0}^{+\infty} z^n \left(\sum_{k=n+1}^{+\infty} t_0^k a_k \right).$$

But
$$\sum_{n=0}^{+\infty}\sum_{k=n+1}^{+\infty} t_0^k a_k = \sum_{k=1}^{+\infty} t_0^k k a_k = t_0 \tilde{A}'(1),$$
which is finite. By Wiener's $1/f$ theorem [115, Theorem 5.2], it is the case that $1/g$ can be expanded as a power series $\sum_n b_n z^n$ in $\bar{\mathbb{D}}$ and that $\sum_n |b_n| < \infty$.

Now observe that by (5.15), for any $z \in \bar{\mathbb{D}}$,
$$\frac{1}{g(z)} = \frac{1}{\kappa}(1-z)\tilde{P}(z) = \frac{1}{\kappa}\sum_{n\geq 0} z^n [t_0^n p_n - t_0^{n-1} p_{n-1}].$$

As Wiener's Theorem asserts that the sequence $(t_0^n p_n - t_0^{n-1} p_{n-1})_n$ is summable, this demonstrates thanks to Abel's Theorem that
$$\lim_n t_0^n p_n = p \lim_{z\to 1^-} \frac{1-z}{1-\tilde{A}(z)} = \frac{\kappa}{t_0 \tilde{A}'(t_0)} > 0. \tag{5.33}$$

Now observe that by definition of p_n, it is the case that
$$\mathbb{P}[\exists k \geq 0, T_k \geq n] = \sum_{k\geq n} p_k.$$

By summing the asymptotic expansion given by (5.16) (this is licit as we compare the remainders of convergent series with positive terms), we obtain that – as n tends to infinity,
$$\mathbb{P}[\exists k \geq 0, T_k \geq n] = (1+o(1)) \frac{\kappa}{t_0 \tilde{A}'(t_0)(t_0 - 1)} t_0^{-n}.$$

This concludes the first part of the proof. Indeed, setting $\zeta(p, \vec{w}) := (\log t_0)^{-1}$, we proved that
$$\mathbb{P}[\exists k \geq 0, T_k \geq n] \asymp \mathrm{e}^{-\frac{n}{\zeta(p,\vec{w})}}.$$

From this it is easy to conclude that
$$\phi_p[X_{n+T_1} \neq \dagger | \mathcal{F}_{T_1}, T_1 < \infty] \asymp \mathrm{e}^{-\frac{n}{\zeta(p,\vec{w})}}. \tag{5.34}$$

Let us now turn to the second part of the proof and analyze the behaviour of the term $\phi_p[T_1 < \infty]$. All constants below will be uniform in $\vec{w}$, p and $n \geq 1$. We have
$$\begin{aligned}\phi_p[X_n \neq \dagger] &= \phi_p[X_n \neq \dagger \text{ and } T_1 \geq n] + \sum_{k=1}^{n} \phi_p[T_1 = k]\mathbb{P}[X_n \neq \dagger | T_1 = k] \quad (5.35)\\ &\asymp \phi_p[X_n \neq \dagger \text{ and } T_1 \geq n] + \sum_{k=1}^{n} \phi_p[T_1 = k] e^{-(n-k)/\zeta(p,\vec{w})}\end{aligned}$$

where the second relation is ensured by (5.17). Finally, recall from Theorem 5.3.2 that

$$\mathbb{P}[T_1 \geq k \text{ and } X_n \neq \dagger] \leq \mathrm{e}^{-\varepsilon k}\,\phi_p[X_n \neq \dagger] \qquad \text{and}$$
$$\mathbb{P}[k \leq T_1 < \infty] \leq \pi_1(L(p))e^{-k(\zeta(p,\vec{w})^{-1}+\varepsilon)},$$

for some $\varepsilon > 0$. Inserting this into (5.4.1) we find (5.14).

□

Proof of Corollary 5.4.2. Fix $\vec{w}$ and $p < p_c$. All constants below will be uniform in $\vec{w}$, p and $n \geq 1$. We have

$$\phi_p\big[X_{\lfloor N/L(p)+1\rfloor} \neq \dagger\big] \leq \phi_p\big[0 \longleftrightarrow \mathcal{H}^{\vec{w}}_{\geq N}\big] \leq \phi_p\big[X_{\lfloor N/L(p)\rfloor} \neq \dagger\big]$$

Apply Theorem 5.4.1 to conclude. □

5.4.2 Endpoint concentration when conditioned on survival

It remains to study the behaviour of the process (X_k) when conditioned on the event $\{X_n \neq \dagger\}$. We prove that it satisfies a local limit theorem. Let $g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}$ be the Gaussian density with variance σ^2.

Proposition 5.4.3. *Fix $\vec{w} \in \mathbb{S}^1$ and $p < p_c$. There exists $\mu = \mu(p,\vec{w})$ and $\sigma = \sigma(p,\vec{w})$ such that, for any $k \in \mathbb{Z}$*

$$\left|\sqrt{n}\phi_p\Big[\lfloor X_n/L(p)\rfloor = \lfloor n\cdot\mu\rfloor + k \,\Big|\, X_n \neq \dagger\Big] - g_\sigma\left(\tfrac{k}{\sqrt{n}}\right)\right| \to 0. \qquad (5.36)$$

with the asymptotics being uniform in p and $\vec{w}$. Furthermore $|\mu(p,\vec{w})|$ is uniformly bounded away from ∞, $\sigma(p,\vec{w})$ uniformly bounded away from 0 and ∞.

Remark 5.4.4. Other properties typical of random walks may be extended to the process $(X_n)_n$ such as the existence of a uniform constant $C > 0$ such that

$$\phi_p\Big[\lfloor X_n/L(p)\rfloor = \lfloor n\cdot\mu\rfloor + k \,\Big|\, X_n \neq \dagger\Big] \leq C\phi_p\Big[\lfloor X_n/L(p)\rfloor = \lfloor n\cdot\mu\rfloor \,\Big|\, X_n \neq \dagger\Big] \qquad (5.37)$$

and a large deviation estimate

$$\phi_p\Big[|X_n/L(p) - \lfloor n\cdot\mu| \geq \alpha n \,\Big|\, X_n \neq \dagger\Big] = \mathrm{e}^{-I(\alpha)n+o(n)} \qquad (5.38)$$

where I is a continuous function on an interval $(-\varepsilon,\varepsilon)$ for some $\varepsilon > 0$ with $I(0) = 0$ and $I(\alpha) > 0$ for $\alpha \neq 0$. All constants appearing in the above are uniform in n, $\vec{w}$ and $p < p_c$. The rate function I does depend on the direction $\vec{w}$. For proofs of such statements, see [16].

Proof. Local limit-type theorems for compound Markov processes are classical, and can be found in a number of references. For instance, this result directly follows from [16, Theorem 2.1.2] □

We are finally ready to deduce some form of the OZ-formula. Recall the quantity $\zeta(p,\vec{w})$ defined in Theorem 5.4.1.

Corollary 5.4.5. *Fix $\vec{w}\in\mathbb{S}^1$ and $p<p_c$. There exists $\mu=\mu(p,\vec{w})$ and $\sigma=\sigma(p,\vec{w})$ such that*

$$\phi_p\big[0\longleftrightarrow x\big]\asymp\frac{\pi_1(L(p))^2}{\sqrt{n}}\mathrm{e}^{-\frac{n}{\zeta(p,\vec{w})}}, \tag{5.39}$$

uniformly in p, $\vec{w}$, $n\geq 1$ and any $x\in\mathbb{Z}^2$ with $\|x-nL(p)(\vec{w}+\mu\cdot\vec{w}^\perp)\|\leq L(p)/2$.

Remark 5.4.6. The same proof as below allow one to deduce from (5.4.4) a large deviation estimate for the hitting position

$$\phi_p\Big[0\longleftrightarrow\lfloor nL(p)(\vec{w}+(\mu(p,\vec{w})+\alpha)\cdot\vec{w}^\perp)\rfloor\,\Big|\,0\longleftrightarrow\mathcal{H}^{\vec{w}}_{\geq n}\Big]=\pi_1(L(p))\cdot\mathrm{e}^{-I(\alpha)n+o(n)} \tag{5.40}$$

for any α close enough to 0 and some continuous rate function I with $I(0)=0$ and $I(\alpha)>0$ for $\alpha\neq 0$.

Proof of Corollary 5.4.5. Fix p, $\vec{w}\in\mathbb{S}^1$ and $x\in\Lambda_{L(p)/2}(nL(p)(\vec{w}+\mu(p,\vec{w})\cdot\vec{w}^\perp))$ for some $n\geq 1$. For $n<3$ it was proved in [51] that $\phi_p[0\longleftrightarrow x]\asymp\pi_1(L(p))^2$. We will henceforth assume that $n\geq 3$. The upper and lower bound on $\phi_p[0\longleftrightarrow x]$ will be treated differently.

Write Γ for the exploration top-most open path of C between $\mathcal{H}_{\leq 0}$ and $\mathcal{H}_{\geq n-1}$. On the event that there exists a unique cluster of C crossing between these to half-spaces, X_{n-1} is the endpoint of Γ on $\mathcal{H}_{\geq n-1}$. Conditioning on a realisation $\Gamma=\gamma$ with $\lfloor X_{n-1}/L(p)\rfloor=\lfloor(n-1)\cdot\mu(p,\vec{w})\rfloor$, by an applying RSW-type construction in $\Lambda_{2L(p)}(x)$ (see Figure **??** Im TO DO) we find that

$$\phi_p[0\longleftrightarrow x\,|\,\Gamma=\gamma]\geq c\,\phi^0_{\Lambda_{L(p)},p}[x\longleftrightarrow\partial\Lambda_{L(p)/2}]\asymp\pi_1(L(p)),$$

where (5.2) and (5.3) state that all constants may be chosen uniform in $\vec{w}$ and p. Summing over γ as above, and keeping in mind that the probability that X_{n-1} is measurable in terms of Γ is exponentially high under the conditioning $X_{n-1}\neq\dagger$, we conclude that

$$\phi_p[0\longleftrightarrow x]\geq c\pi_1(L(p))\phi_p\big[\lfloor X_{n-1}/L(p)\rfloor=\lfloor(n-1)\cdot\mu_{\vec{w}}\rfloor\big]\asymp\frac{\pi_1(L(p))^2}{\sqrt{n}}\mathrm{e}^{-\frac{n}{\zeta(p,\vec{w})}},$$

where we used (5.4.5) and (5.14) in the last equivalence.

We turn to the upper bound, for which we decompose

$$\begin{aligned}\phi_p[0\longleftrightarrow x]&\leq\phi_p[0\longleftrightarrow\Lambda_{L(p)}(x)\text{ and }x\longleftrightarrow\partial\Lambda_{L(p)}(x)]\\&\leq\phi_p[0\longleftrightarrow\Lambda_{L(p)}(x)]\,\phi^1_{\Lambda_{L(p)}p}[0\longleftrightarrow\partial\Lambda_{L(p)}]\end{aligned} \tag{5.41}$$

where the second inequality is due to the fact that the two events are measurable in terms of what happens outside and inside of $\Lambda_{L(p)}(x)$, respectively. The second term is bounded by a constant multiple of $\pi_1(L(p))$ by (5.3). Write T for the last renewal time of before time n. Then the first term may be bounded by

$$\phi_p[0 \longleftrightarrow \Lambda_{L(p)}(x)] \leq \sum_{j\geq 0} \phi_p\big[T = n-j \text{ and } |X_T - n\mu_{\vec{w}}| \leq j\big]$$
$$\leq \sum_{j\geq 0} \phi_p\big[|X_{n-j} - n\mu_{\vec{w}}| \leq j\big]\, \phi_p[\text{no renewal in } [n-j,n] \text{ but } X_n \neq \dagger \,\big|\, n-j \text{ is renewal}\big],$$

where the decomposition in the last inequality is due to the fact that T is a renewal, and therefore the probability of having no renewal in the next j steps is independent of the position of the walk at time T. Due to (5.3.1) and (5.4.4)

$$\phi_p\big[0 \longleftrightarrow \Lambda_{L(p)}(x)\big] \leq C\pi_1(L(p)) \sum_{j\geq 0} j \tfrac{1}{\sqrt{n-j}} \mathrm{e}^{-\frac{n-j}{\zeta(p,\vec{w})}} \cdot \mathrm{e}^{-j(\zeta(p,\vec{w})^{-1}+\varepsilon)}$$
$$\leq C' \frac{\pi_1(L(p))}{\sqrt{n}} \mathrm{e}^{-n/\zeta(p,\vec{w})}$$

In the last line, we simply used the summability of $j\mathrm{e}^{-\varepsilon j}$. Inserting this together with the estimate on $\phi^1_{\Lambda_{L(p)}p}[0 \longleftrightarrow \partial\Lambda_{L(p)}]$ into (5.4.2), we deduce the matching upper bound. □

In light of the above, write

$$\vec{v}(\vec{w}) := \frac{\vec{w} + \mu(p,\vec{w}) \cdot \vec{w}^{\perp}}{\|\vec{w} + \mu(p,\vec{w}) \cdot \vec{w}^{\perp}\|}.$$

Taking $n \to \infty$ in (5.4.5), we conclude that

$$\xi_p(\vec{v}(\vec{w})) = \frac{L(p)\zeta(p,\vec{w})}{\|\vec{w} + \mu(p,\vec{w}) \cdot \vec{w}^{\perp}\|}$$

Lemma 5.4.7. *For any $\vec{v} \in \mathbb{S}^1$, there exists at least one $\vec{w} \in \mathbb{S}^1$ so that*

$$\vec{v}(\vec{w}) = \vec{v}.$$

Proof. We will prove that the function $\vec{w} \mapsto \vec{v}(\vec{w})$ is continuous. This is a simple geometric construction; see Figure **??** for an illustration.

Fix some $\vec{w} \in \mathbb{S}_1$ and assume for simplicity that $\mu(p,\vec{w}) \geq 0$. Write $\rho_\delta(.)$ be the rotation by an angle δ.

For $\delta > 0$, a connection between 0 and $x = nL(p)(\vec{w} + \mu \cdot \vec{w}^{\perp})$ intersects $\mathcal{H}^{\rho_\delta(\vec{w})}_{\geq n\cdot\cos\delta}$ Together with (5.18) and (5.14) this implies that

$$\zeta(p,\rho_\delta(\vec{w})) \geq \cos\delta \cdot \zeta(p,\vec{w})$$

For $\delta < 0$, write $\theta = \arctan(\mu(p, \vec{w}))$ – note that θ is uniformly bounded away from ∞ by Proposition 5.4.3 Then, a connection between 0 and $x = nL(p)(\vec{w} + \mu \cdot \vec{w}^{\perp})$ ensures that the cluster of 0 intersects $\mathcal{H}^{\rho_\delta(\vec{w})}_{\geq n\cos(\theta+\delta)/\cos(\theta)}$.

$$\zeta(p, \rho_\delta(\vec{w})) \geq \tfrac{\cos(\theta+\delta)}{\cos(\theta)} \cdot \zeta(p, \vec{w}).$$

Using the two displays above, and their versions with the roles of $\vec{w}$ and $\rho_\delta(\vec{w})$ inverted, we conclude that $\vec{w} \mapsto \zeta(p, \vec{w})$ is continuous.

The same reasoning as above proves that, for any $\varepsilon > 0$ and δ small enough

$$\phi_p\Big[0 \longleftrightarrow \tfrac{n}{\langle \vec{v}(\vec{w}), \rho_\delta(\vec{w})\rangle}\vec{v}(\vec{w}) \,\Big|\, 0 \longleftrightarrow \mathcal{H}^{\rho(\delta\vec{w})}_{\geq n}\Big] \geq \exp(-\varepsilon\, n).$$

Together with the large deviation estimate (5.4.6) and the continuity of the rate function, this implies that $\vec{v}(\vec{w})$ is close to $\vec{v}(\rho_\delta(\vec{w}))$. Thus we conclude the proof of the continuity of $\vec{w} \mapsto \vec{v}(\vec{w})$.

To finish the proof of the Lemma, observe that for $\vec{w}$ a coordinate vector $\mu(p, \vec{w}) = 0$ due to the symmetry of the lattice. It follows that $\vec{v}(\vec{w}) = \vec{w}$ in this case. By the intermediate value theorem, we conclude that the image of the function $\vec{w} \mapsto \vec{v}(\vec{w})$ covers the whole unit circle, as claimed. □

We can finally prove Theorem 5.1.1.

Proof of Theorem 5.1.1. Fix $q \geq 1$, $\vec{v} \in \mathbb{S}^1$ $p < p_c$ and $n \geq \xi_p(\vec{v})$. Define $\vec{w}$ as a vector such that $\vec{v}(\vec{w}) = \vec{v}$. Then (5.4.5) implies that

$$\phi_p\big[0 \longleftrightarrow \lfloor n\vec{v} \rfloor\big] \asymp \frac{\pi_1(L(p))^2}{\sqrt{m}} \mathrm{e}^{-\frac{m}{\zeta(p,\vec{w})}}, \tag{5.42}$$

where $m = \frac{n}{\langle \vec{v}, \vec{w}\rangle L(p)}$. Recall that the drift $\mu(p, \vec{w})$ is bounded uniformly, and therefore $\langle \vec{v}, \vec{w}\rangle$ is uniformly positive. Thus $m \asymp n/L(p)$. Moreover, the above implies that $\xi_p(\vec{v}) = \langle \vec{v}, \vec{w}\rangle L(p)\zeta(p, \vec{w}) \asymp L(p)$.

Inserting the above in (5.4.2) we conclude that

$$\phi_p\big[0 \longleftrightarrow \lfloor n\vec{v} \rfloor\big] \asymp \pi_1(\lfloor \xi_p(\vec{v}) \rfloor)^2 \sqrt{\tfrac{\xi_p(\vec{v})}{n}}\, \mathrm{e}^{-\frac{n}{\xi_p(\vec{v})}},$$

as claimed. □

5.4.3 Strict convexity of Wulff shape

Fix $p < p_c$. Define $\xi_p^*(\vec{w})$ by

$$(\xi_p^*(\vec{w}))^{-1} = -\lim_{n\to\infty} \frac{1}{n} \log \phi_p[0 \longleftrightarrow \mathcal{H}_{\geq n}^{\vec{w}}].$$

Notice that this definition, as well as the one for ξ_p are valid for any vector in $\mathbb{R}^2 \setminus \{0\}$. For coherence, we set $\xi_p(0) = \xi_p^*(0) = \infty$.

It is classical [25] that both $(\xi_p(.))^{-1}$ and $(\xi_p^*(.))^{-1}$ define norms on $\mathbb{R}^2$. Indeed they are both positive homogeneous, and FKG inequalities imply that they are convex. Define their unit balls

$$\mathcal{U}_p = \{\vec{v} \in \mathbb{R}^2 : \xi_p(\vec{v}) \geq 1\} \quad \text{and} \quad \mathcal{W}_p = \{\vec{w} \in \mathbb{R}^2 : \xi_p^*(\vec{w}) \geq 1\}.$$

The latter is called the *Wulff shape* (see [15] for an extensive review on the subject).

The goal of this section is to explain how our approach allows to re-prove some known results about $\mathcal{U}_p$ and $\mathcal{W}_p$. The method is different of [25], as it does not rely on the analysis of the regularity properties of the associated Ruelle–Perron–Frobenius operator.

Theorem 5.4.8. *For any $p < p_c$, $\mathcal{U}_p$ and $\mathcal{W}_p$ are strictly convex bounded sets of $\mathbb{R}^2$, symmetric with respect to the coordinate axis with differentiable boundaries. Moreover, they are convex dual to each other,* i.e. *they are linked by the following relation:*

$$\mathcal{U}_p = \{\vec{v} \in \mathbb{R}^2, \langle \vec{v}, \vec{n}\rangle \leq \xi_p^*(\vec{n}), \forall \vec{n} \in \mathbb{R}^2\} \quad \textit{and} \quad \mathcal{W}_p = \{\vec{w} \in \mathbb{R}^2, \langle \vec{w}, \vec{n}\rangle \leq \xi_p(\vec{n}), \forall \vec{n} \in \mathbb{R}^2\}.$$

Proof. We start by proving the relation between $\mathcal{U}_p$ and $\mathcal{W}_p$, and start with the expression for $\mathcal{U}_p$. Let $\vec{v}$ be a direction. Let $\vec{w}$ be a unit vector. Observe that the hyperplane with normal vector $\vec{w}$ containing the vertex $n\vec{v}$ is given by $\partial\mathcal{H}_{\geq n\langle \vec{v},\vec{w}\rangle}^{\perp\vec{w}}$. Thus, it is clear that

$$\phi_p[0 \longleftrightarrow n\vec{v}] \leq \phi_p[0 \longleftrightarrow \mathcal{H}_{\geq n\langle \vec{v},\vec{w}\rangle}^{\perp\vec{w}}].$$

Identifying the constants leading the exponential decay of these two quantities, we obtain the following equation

$$\xi_p(\vec{v})\langle \vec{v}, \vec{w}\rangle \leq \xi_p^*(\vec{w}). \tag{5.43}$$

This inequality being valid for any choice of directions $\vec{w}, \vec{v}$, we observe that if $\vec{v}$ satisfies $\xi_p(\vec{v}) \geq 1$, then for any $\vec{w} \in \mathbb{S}^1$, $\xi_p^*(\vec{w}) \leq \langle \vec{v}, \vec{w}\rangle$. This gives

$$\mathcal{U}_p \subset \{\vec{v} \in \mathbb{R}^2, \langle \vec{v}, \vec{n}\rangle \leq \xi_p^*(\vec{n}), \forall \vec{n} \in \mathbb{R}^2\}.$$

For the converse inclusion, let $\vec{v}$ be a unit vector such that for any $\vec{n}$, $\langle \vec{v}, \vec{n}\rangle \leq \xi_p^*(\vec{n})$. Observe that Lemma 5.4.7 yields the existence of $\vec{w} \in \mathbb{S}^1$ such that $v(\vec{w}) = \vec{v}$. Observe that the proof of the OZ formula and in particular (5.4.2) implies that

$$\xi_p(\vec{v}) = \zeta(p, \vec{w})L(p)\langle \vec{v}, \vec{w}\rangle.$$

By definition, it is the case that $\zeta(p,\vec{w})L(p) = \xi_p^*(\vec{w})$. Thus, $\vec{v} = v(\vec{w})$ achieves the equality case in (5.19). Thus we conclude by writing that

$$\xi_p(\vec{v}) = \frac{\xi_p^*(\vec{w})}{\langle \vec{v}, \vec{w} \rangle} \geq 1,$$

where the last inequality is a consequence of our assumption on $\vec{v}$. Thus,

$$\mathcal{U}_p = \{\vec{v} \in \mathbb{R}^2, \langle \vec{v}, \vec{n} \rangle \leq \xi_p^*(\vec{n}), \forall \vec{n} \in \mathbb{R}^2\}. \tag{5.44}$$

In the language of convex bodies, $\mathcal{U}_p$ is the convex body with *gauge function* given by ξ_p. Now, (5.20) gives that its *support function* is ξ_p^*. Classical results on convex duality [112, polar duality theorem, p. 238] then imply that the support function of $\mathcal{U}_p^*$ is ξ_p, and that its gauge function is ξ_p^*, which immediately proves that $\mathcal{W}_p = \mathcal{U}_p^*$ and

$$\mathcal{W}_p = \{\vec{w} \in \mathbb{R}^2, \langle \vec{w}, \vec{n} \rangle \leq \xi_p(\vec{n}), \forall \vec{n} \in \mathbb{R}^2\}.$$

We notice *en passant* that as respective gauge and support functions of two dual convex bodies, ξ and ξ^* are the generalized Legendre transforms one of each other.

We now turn to strict convexity properties. Assume that $\mathcal{U}_p$ possesses a flat facet. This means that there exists $v_1 \neq v_2 \in \partial\mathcal{U}_p$ such that the line segment $[v_1, v_2]$ is contained in $\partial\mathcal{U}_p$. We are going to see that it contradicts the large deviation principle stated in (5.4.6). Indeed, call $\vec{w}$ the unit vector normal to the hyperplane containing v_1 and v_2. Call $\vec{v_1} = \frac{v_1}{\|v_1\|}$ and $\vec{v_2} = \frac{v_2}{\|v_2\|}$. Also call $\vec{v} = \vec{v}(\vec{w})$, and v the point in the direction $\vec{v}$ lying on $\partial\mathcal{U}_p$. Due to the convexity of $\mathcal{U}_p$, it is the case that v, v_2 and v_1 must lie on the same line, and we assume that $v \in [v_1, v_2]$. Also, let $\alpha > 0$ be such that $v_1 = v + \alpha w^\perp$. Then on the one hand, due to (5.4.6),

$$\phi_p[0 \longleftrightarrow nv_1 | 0 \longleftrightarrow \mathcal{H}_{\geq n}^{w^\perp}] = \pi_1(L(p))\mathrm{e}^{-I(\alpha)n + o(n)}$$

with $I(\alpha) > 0$. On the other hand however, by definition of v and the fact that $v \in [v_1, v_2]$,

$$\lim_n -\tfrac{1}{n} \log \phi_p[0 \longleftrightarrow \mathcal{H}_{\geq n}^{w^\perp}] = \lim_n -\tfrac{1}{n} \log \phi_p[0 \longleftrightarrow nv] = \lim_n -\tfrac{1}{n} \log \phi_p[0 \longleftrightarrow nv_1].$$

This is the required contradiction.

We proved that $\mathcal{U}_p$ is strictly convex. This implies that ξ_p is strictly convex. Further classical considerations on Legendre transforms imply that the boundary of $\mathcal{W}_p$ is everywhere differentiable, and so is ξ_p^*.

A similar reasoning establishes the strict convexity of $\mathcal{W}_p$, and the differentiability of the boundary of $\mathcal{U}_p$ and of ξ_p. □

5.4.4 Invariance principle

Fix $p < p_c$ and $\vec{w} \in \mathbb{S}^1$. Define the linear interpolation $X(t)_{t\geq 0}$ of the discrete process $(X_n)_{n\geq 0}$. The representation of the cluster in terms of a random-walk like object allows to prove several invariance principles, similar to Donsker's Theorem for random walks. For instance, non-uniform invariance principles similar to Theorem 5.4.9 were derived in [25] and [38]. Our representation allows to prove the two following invariance principles.

Theorem 5.4.9. *Under the family of measures $\phi_p[\ \cdot\ |\ 0 \longleftrightarrow \mathcal{H}^{\vec{w}}_{\geq n}]$, uniformly in p and $\vec{w}$, when $n \to \infty$,*

$$\big(\frac{1}{\sqrt{n}} X(nt)\big)_{t\in(0,1)} \Rightarrow \Big(\mathsf{B}_t^{\mu(p,\vec{w}),\sigma(p,\vec{w})}\Big)_{t\in(0,1)},$$

where $B_t^{\mu(p),\sigma(p)}$ denotes the one-dimensional Brownian motion started at 0 with drift $\mu(p,\vec{w})$ and variance $\sigma(p,\vec{w})$.

Moreover, under the family of measures $\phi_p[\ \cdot\ |\ 0 \longleftrightarrow n\vec{v}]$, uniformly in p and $\vec{v}$, when $n \to \infty$,

$$\big(\frac{1}{\sqrt{n}} X(nt)\big)_{t\in(0,1)} \Rightarrow \Big(\mathsf{BB}_t^{\sigma(p,\vec{w})}\Big)_{t\in(0,1)},$$

where $\vec{w}$ is the unique unit vector such that $\vec{v}(\vec{w}) = \vec{v}$ and $\mathsf{BB}_t^{\sigma(p,\vec{w})}$ is the one-dimensional Brownian bridge started at 0 and ending at $\vec{v}$ with variance $\sigma(p,\vec{w})$.

In both statements, the convergence occurs in the space of continuous functions from $[0,1]$ to $\mathbb{R}$, endowed with the topology of uniform convergence.

Proof. As previously, this follows directly from Theorem 5.3.2 together with classical considerations on Markov renewal processes. The first invariance principle directly follows from [16, Theorem 1.5.3]. The second invariance principle is a classical consequence of the first one together with the local limit estimate given by Proposition 5.4.3. As this is classical, we do not give further details, and refer to [38] for the formal reasoning leading to the invariance principle. □

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