Exact cube-root fluctuations in an area-constrained random walk model

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Abstract

This article is devoted to the study of the behaviour of a (1+1)-dimensional model of random walk conditioned to enclose an area of order N^2. Such a conditioning enforces a globally concave trajectory. We study the local deviations of the walk from its convex hull. To this end, we introduce two quantities — the mean facet length MeanFL and the mean local roughness MeanLR — measuring the typical longitudinal and transversal fluctuations around the boundary of the convex hull of the random walk. Our main result is that MeanFL is of order $N^{2/3}$ and MeanLR is of order $N^{1/3}$. Moreover, following the strategy of Hammond (Ann. Prob., 2012), we identify the polylogarithmic corrections in the scaling of the *maximal* facet length and of the *maximal* local roughness, showing that the former one scales as $N^{2/3}(\log N)^{1/3}$, while the latter scales as $N^{1/3}(\log N)^{2/3}$. The object of study is intended to be a toy model for the interface of a two-dimensional statistical mechanics model (such as the Ising model) in the phase separation regime — we discuss this issue at the end of this work.

Contents

1 Introduction

The phase separation problem is concerned with the study of the boundary appearing between two different phases of a statistical mechanics model, in a regime where those two phases can coexist. In his seminal work, Wulff [Wul01] proposed that such a boundary should macroscopically adopt a deterministic limit shape given by the solution of a variational problem involving thermodynamic quantities such as the surface tension. This prediction has been an object of intense study and has by now been made rigorous in a very wide variety of settings, see for instance the monographs [DKS92, BIV00, CP00, Cer06].

While this macroscopic shape is dependent on the model, the *fluctuations* of the random phase boundary are widely believed to behave in an universal way. A first natural candidate to measure these fluctuations is the deviations from the limit shape. These have been shown to be Gaussian in [DH96] in the context of area-constrained random walks, and later on in [DH97] for the 2D Ising phase boundary at low temperatures. Two other quantities of interest are given by the *maximal facet length* and the *maximal local roughness* of the interface, that are respectively the length of the largest segment of the convex hull of the interface and the maximal deviation of the interface from this convex hull. In a seminal paper [Ale01], Alexander conjectured that the exponent governing the scaling of the maximal local roughness should be $1/3$. In the context of percolation models in the phase separation setting, he derived upper bounds for an averaged version of the local roughness. Alexander and Usun [UA03] then complemented this work by providing lower bounds for the local roughness in the setup of Bernoulli percolation. Later on, in a remarkable series of papers, Hammond [Ham11, Ham12a, Ham12b] was able to identify the exact scale of the maximal facet length and the maximal local roughness of a droplet of volume N^2 in the planar subcritical random-cluster model. Indeed, he proved that the former is of order $N^{2/3}(\log N)^{1/3}$, while the latter is of order $N^{1/3}(\log N)^{2/3}$, validating the exponent derived in [Ale01]. These results are built on the identification of $N^{2/3}$ as the scale at which the curvature effect enforced by the conditioning has the same order of magnitude as the Gaussian fluctuations of the interface. Finally, let us mention that in an earlier paper, Hammond and Peres [HP08] introduced a continuous and Brownian version of the phase separation problem, studying a two-dimensional Brownian loop conditioned to enclose a large area. They proved results in favour of the appearance of the cube-root fluctuations in this setting.

In this paper, we study a model of random walks, with geometrically randomised length, conditioned to enclose an area at least equal to N^2. This model was suggested by Hammond in [Ham12a, Section 1.0.4] and later in [Ham19]. We prove that, as predicted by the author, his techniques can successfully be applied to this setting, enabling us to establish the above-mentioned polylogarithmic corrections. However, the main innovation of this paper is the identification of the scaling of the *typical* facet length and local roughness (rather than their maximal values). Call MeanFL (resp. MeanLR) the length (resp. the local roughness) of the facet intersecting a given line. We prove that $\mathsf{MeanFL} \asymp N^{2/3}$ and $\mathsf{MeanLR} \asymp N^{1/3}$, see Theorem 1.3.

The scaling exponents $1/3$ and $2/3$ have been shown to arise in various related contexts in statistical mechanics. An important example is the *critical prewetting* in the Ising model, which has been extensively studied in a beautiful series of papers [HV04, Vel04, ISV15, GG21, IOSV22] to mention a few of them. A remarkable aspect of these works is that the results are derived without the help of any integrable feature. Let us also note that [CLM$^+$16, CKL23] provided strong evidence for a similar behaviour in the context of the SOS model above a wall in $(2+1)$ dimensions.

To the best of our knowledge, it is the first time that those exponents are identified in a context such as ours. Indeed, in the above-mentioned works, the interface lies above a facet of length much longer than $N^{2/3}$ — this facet being deterministic and artificially created by

looking at the interface along a side of a large box for instance. However, in our work, the facets are not deterministic, and themselves reflect the competition between the randomness and the curvature induced by the conditioning.

Finally, we strongly believe (supported by [Ham12a]) that our approach is robust and should allow to derive the same result for a large variety of models, including the fluctuations of the outermost circuit in a subcritical random-cluster model conditioned to enclose a large area, which itself is a good toy model for the boundary of a droplet in a supercritical Potts model. Note that this suggests that at scale $N^{2/3}$, our model and more general phase boundaries models should lie in the same universality class. This is discussed in Section 5.

1.1 Definition of the model and statement of the main results

Let Λ be the set of finite paths in the first quadrant of $\mathbb{Z}^2$ which start on the y-axis and end on the x-axis, and which are oriented in the sense that they only take rightward and downward steps, see Figure 1. For $\gamma \in \Lambda$, we define $|\gamma|$ to be its length, i.e. its number of steps (which is also the sum of the y-coordinate of its starting point and the x-coordinate of its ending point). It will be convenient to identify an element $\gamma \in \Lambda$ with the set of points of $\mathbb{N}^2$ it passes by, i.e $\gamma = (\gamma(k))_{0 \leq k \leq |\gamma|}$. We set Λ_n to be the subset of Λ of such oriented paths of length n. It is clear that $|\Lambda_n| = 2^n$. If $a, b \in \mathbb{N}^2$, we denote by $\Lambda^{a \to b}$ the set of downright paths from a to b.

Let $0 < \lambda < \frac{1}{2}$. We define a probability measure $\mathbb{P}_\lambda$ on Λ by requiring that, for $\gamma \in \Lambda$,

$$\mathbb{P}_\lambda[\gamma] = \frac{1}{Z_\lambda} \lambda^{|\gamma|}, \tag{1.1}$$

where Z_λ is the normalisation constant given by $Z_\lambda = (1 - 2\lambda)^{-1}$.
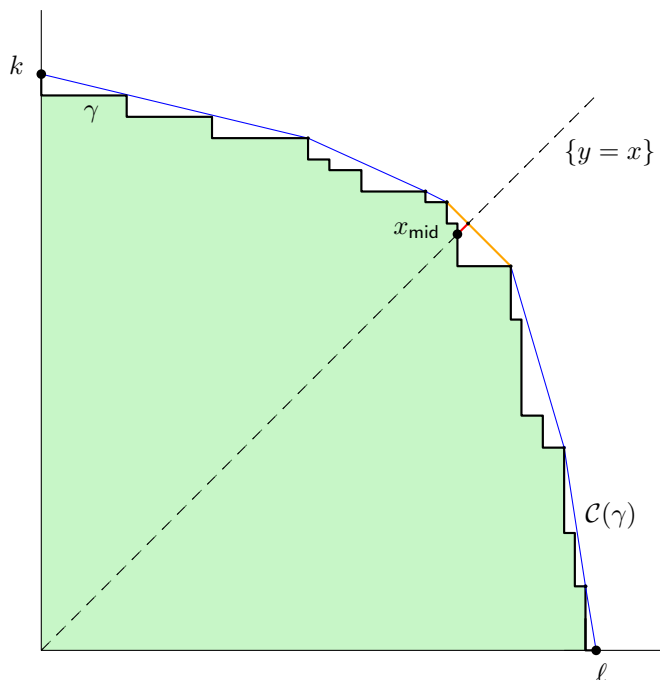
Figure 1: An example of path γ (in bold black) with $|\gamma| = k + \ell$, and its least concave majorant $\mathcal{C}(\gamma)$ (in blue). The mean facet of γ is coloured in orange, while the mean local roughness is the length of the red segment. The area $\mathcal{A}(\gamma)$ is the area of the light green shaded region.

The probability measure $\mathbb{P}_\lambda$ is to be seen as a *background* measure which imposes an exponential decay of the probability of sampling a long path. We now enforce a competing constraint of enclosing a large area. For $\gamma \in \Lambda$, we define the area of γ, denoted by $\mathcal{A}(\gamma)$, to be the area

enclosed by the graph of γ and the two coordinate axes (see Figure 1). Let $N \in \mathbb{N}$ and let Γ be a sample of $\mathbb{P}_\lambda$. We define a new probability measure $\mathbb{P}_\lambda^{N^2}$ on Λ by

$$\mathbb{P}_\lambda^{N^2}[\,\cdot\,] := \mathbb{P}_\lambda[\,\cdot\,\mid \mathcal{A}(\Gamma) \geq N^2]. \tag{1.2}$$

We also introduce $\Lambda^{N^2} \subset \Lambda$, the set of all paths of Λ that capture an area greater than N^2. Samples of $\mathbb{P}_\lambda^{N^2}$ typically experience a competition between capturing a large area and minimizing the length of the paths. As explained before, it suggests that for large values of N, a typical sample of $\mathbb{P}_\lambda^{N^2}$ will have a global curvature imposed by the area condition and Gaussian fluctuations imposed by the background measure (once the length is fixed, the background measure is the uniform law). The object of interest in our work, which tracks the competition between these two phenomena, is defined below.

Definition 1.1 (Least concave majorant, facets)**.** Let $\gamma \in \Lambda$. We set $\mathsf{Ext}(\gamma)$ to be the set of non-zero extremal points of the convex hull of $\gamma \cup \{(0,0)\}$. Let $\mathcal{C}(\gamma)$ be the *least concave majorant* of γ, that is, the graph of the piecewise affine function passing through all the points of $\mathsf{Ext}(\gamma)$. Observe that with this definition, $\mathcal{C}(\gamma)$ becomes a finite union of line segments of $\mathbb{R}^2$ which are called *facets* (see Figure 1), the endpoints of these facets precisely being the elements of $\mathsf{Ext}(\gamma)$. For $x \in \gamma$, we define its *local roughness* to be its Euclidean distance to $\mathcal{C}(\gamma)$,

$$\mathsf{LR}(x) := \mathsf{d}(x, \mathcal{C}(\gamma)). \tag{1.3}$$

Our main result concerns the tightness of the length of a typical facet intercepting a given ray emanating from the origin (resp. the fluctuations of Γ along the facet) at scale $N^{2/3}$ (resp. $N^{1/3}$). Let us first define the quantities of interest.

Definition 1.2 (MeanFL, MeanLR)**.** Let $\gamma \in \Lambda$. We define $\mathsf{MeanFac}(\gamma)$ to be the facet intersecting the line $\{y = x\}$ (if we are in the case where the ray $\{y = x\}$ intercepts the endpoint of two consecutive facets, we arbitrarily choose the leftmost one for $\mathsf{MeanFac}(\gamma)$). We call $x_{\mathsf{mid}}(\gamma)$ the point of $\mathbb{N}^2 \cap \gamma$ that is the closest to the line $\{y = x\}$ (as previously, in case of conflict, we arbitrarily choose the leftmost one). Assume that $\mathsf{MeanFac} = [a, b] (= \{at + (1-t)b,\ t \in [0,1]\})$, with $a, b \in \mathbb{N}^2$. Define the following random variables,

$$\mathsf{MeanFL}(\gamma) := \mathsf{d}(a, b), \qquad \mathsf{MeanLR}(\gamma) := \mathsf{LR}(x_{\mathsf{mid}}). \tag{1.4}$$

In words, MeanFL is the length of $\mathsf{MeanFac}$ and MeanLR is the local roughness of the "mean vertex" of γ.

The following result is the main contribution of the paper.

Theorem 1.3 (Tightness of MeanFL and MeanLR at scales $N^{2/3}$ and $N^{1/3}$)**.** *Let $0 < \lambda < \frac{1}{2}$. For any $\varepsilon > 0$, there exist $c, C > 0$ and $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$,*

$$\mathbb{P}_\lambda^{N^2}\big[cN^{\frac{2}{3}} < \mathsf{MeanFL}(\Gamma) < CN^{\frac{2}{3}}\big] > 1 - \varepsilon, \tag{1.5}$$

and

$$\mathbb{P}_\lambda^{N^2}\big[cN^{\frac{1}{3}} < \mathsf{MeanLR}(\Gamma) < CN^{\frac{1}{3}}\big] > 1 - \varepsilon. \tag{1.6}$$

We chose for convenience to formulate Theorem 1.3 in terms of the statistics of $\mathsf{MeanFac}$. However, the proof never uses the symmetry of the model around the line $\{y = x\}$ so the result is not specific to the choice of the line $\{y = x\}$: for any $\alpha > 0$, the result holds[1] for $\mathsf{MeanFac}_\alpha$ which is defined to be the unique facet intercepting the line $\{y = \alpha x\}$.

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[1]Note however that the constants c, C might now depend on α.

Remark 1.4. In the proof of Theorem 1.3, we actually derive a slightly stronger statement. Indeed, we obtain stretch-exponential upper tails for MeanFL and MeanLR (see Propositions 3.1 and 3.2)

Following [Ham12a] and [Ham12b], our second result identifies the logarithmic corrections to the *maximal* facet length and to the *maximal* local roughness along $\mathcal{C}(\gamma)$. We call these quantities MaxFL(γ) (resp. MaxLR(γ)).

Theorem 1.5. *Let* $0 < \lambda < \frac{1}{2}$. *There exist* $c, C > 0$ *such that,*

$$\mathbb{P}_\lambda^{N^2} \big[cN^{\frac{2}{3}} (\log N)^{\frac{1}{3}} < \mathsf{MaxFL}(\Gamma) < CN^{\frac{2}{3}} (\log N)^{\frac{1}{3}} \big] \xrightarrow[N\to\infty]{} 1, \tag{1.7}$$

and

$$\mathbb{P}_\lambda^{N^2} [cN^{\frac{1}{3}} (\log N)^{\frac{2}{3}} < \mathsf{MaxLR}(\Gamma) < CN^{\frac{1}{3}} (\log N)^{\frac{2}{3}}] \xrightarrow[N\to\infty]{} 1. \tag{1.8}$$

1.2 Related works and known results

As pointed out in the introduction, similar models have been studied quite extensively in the literature, especially in [DH96], where they are introduced as toy models for the study of a low-temperature interface of a (1+1)-dimensional SOS model. In this work, the authors investigate the behaviour of the model at the macroscopic scale N and at the mesoscopic scale $N^{1/2}$. It is possible to extend their result to our setup. To properly state it, we introduce the following parametrisation of Γ: let $\Gamma(t)$ be the linear interpolation between the points $\Gamma(k)$ for $0 \le k \le |\Gamma|$. Using the discussion of [DH96, Section 1], together with the basic estimates given by Lemma 2.1 and Proposition 2.4, one can obtain the following result.

Theorem 1.6. *Let* $0 < \lambda < \frac{1}{2}$. *There exists a deterministic, concave and continuous function* $f_\lambda : [0,1] \to \mathbb{R}^+$ *such that for any* $\varepsilon > 0$,

$$\mathbb{P}_\lambda^{N^2} \Big[\sup_{t\in[0,1]} |N^{-1}\Gamma_N(|\Gamma_N|t) - f_\lambda(t)| > \varepsilon \Big] \xrightarrow[N\to\infty]{} 0. \tag{1.9}$$

Remark 1.7. Using Lemma 2.1, the time scaling in (1.9) is linear in N.

Such a phenomenon is by now very well known under the name of a *limit shape phenomenon*, and is known to arise in a large variety of situations (see for instance [Oko16, Ken18]). In (1+1) dimensions such as in our setting, two different points of view can be adopted to prove a statement such as Theorem 1.6: the first one is the classical theory of sample path large deviations culminating with the celebrated Mogulski'i Theorem (see [DZ10, Section 5]). The second possible point of view has a more statistical mechanics flavour and is known as *Wulff theory* (see the reference monograph [DKS92]). In the work [DH96], it is shown that both approaches can be implemented and yield the same result. In both cases, the function f_λ is identified as the minimiser of a deterministic variational problem. Let us conclude this discussion by noticing that actually much stronger statements than (1.9) can be obtained, and in particular large deviations principles for the sample path Γ, though we will not focus on results of this type.

The main result of [DH96, Theorem 2.1] focuses on the fluctuations of Γ_N around f_λ. Again, minor modifications of their proof lead to the following result.

Theorem 1.8. *Let* $0 < \lambda < \frac{1}{2}$. *There exists a (centered) Gaussian process* ξ_λ *on the space* $\mathcal{C}([0,1])$ *(equipped with the topology of the uniform convergence) such that under the measure* $\mathbb{P}_\lambda^{N^2}$,

$$\Big(\frac{1}{\sqrt{N}} (\Gamma_N(|\Gamma_N|t) - Nf_\lambda(t)) \Big)_{t\in[0,1]} \xrightarrow[N\to\infty]{(d)} \xi_\lambda, \tag{1.10}$$

where the convergence holds in distribution.

These two results can be stated heuristically as follows: at the macroscopic scale N, the conditioning on the event $\{\mathcal{A}(\Gamma) \geq N^2\}$ enforces a deterministic and global curvature, whereas at the mesoscopic scale $N^{1/2}$ the conditioning has no effect on $\mathbb{P}_\lambda$ and the Gaussian nature of the measure $\mathbb{P}_\lambda$ is unchanged. As explained above, Theorem 1.3 identifies $N^{2/3}$ as being the scale at which those two competing effects are of the same order.

1.3   A resampling strategy

The proofs below heavily relie on a particularly simple (yet crucial) feature of the model: the so-called *Brownian Gibbs property* (see [CH14]). It can be stated as follows: start from a sample Γ of $\mathbb{P}_\lambda^{N^2}$, choose two points $a, b \in \Gamma$ with any random procedure "explorable from the exterior[2] of the path" (by this we mean that the event $\{a = x, b = y\}$ is measurable with respect to $\Gamma \setminus \Gamma_{x,y}$). Then, conditionally on (a, b) and $\Gamma \setminus \Gamma_{a,b}$, the distribution of the random variable $\Gamma_{a,b}$ is the uniform distribution on $\Lambda^{a \to b}$ **conditionally on the fact that the resulting path of Λ encloses an area greater than** N^2. This apparently naive observation allows one to implement a strategy of *resampling*. Indeed, one can construct several Markovian dynamics on Λ leaving the distribution $\mathbb{P}_\lambda^{N^2}$ invariant in a quite general fashion: start from a sample Γ of $\mathbb{P}_\lambda^{N^2}$, choose two points $a, b \in \Gamma$ according to the above procedure and replace $\Gamma_{a,b}$ by a sample of the uniform distribution of $\Gamma_{a,b}$ subject to the above-mentioned conditioning. Then, it is clear, thanks to the preceding observation, that the distribution of the output is $\mathbb{P}_\lambda^{N^2}$. In what follows, we shall call such a dynamic on Λ a *resampling procedure*.

This point of view will be used several times in the proofs to analyse marginals of $\mathbb{P}_\lambda^{N^2}$ in well-chosen regions of the first quadrant. A simple illustration is given in the proof of Proposition 2.10.

1.4   Open problems

In light of the preceding discussions, two natural questions arise. We now describe them.

As explained above, we expect the methods developed in this article to help the analysis of the droplet of a subcritical planar random-cluster model. We strongly believe that our strategy could complement the results obtained by Hammond in [Ham11, Ham12a, Ham12b]. We provide strong heuristics towards this result in Section 5.

Open Problem A (Droplet boundary in the subcritical planar random-cluster model)**.** *Extend Theorem 1.3 to the study of a typical facet in the setup of the droplet in the subcritical planar random-cluster model.*

Once tightness is obtained in Theorem 1.3, it is quite natural to try to identify the candidate for the scaling limit of the excursion below a typical facet. A similar question has been answered in [IOSV22] where the authors obtained convergence (in the bulk phase) of the object of study to the so-called *Ferrari–Spohn diffusion*. In our setup, the excursion below the mean facet should correspond to the excursion of a Ferrari-Spohn process. We intend to study this question in the future.

Open Problem B (Scaling limit of the excursion below a facet)**.** *Construct the Ferrari–Spohn excursion* (FSE)*. Prove that the piece of path lying below the mean facet, after a proper rescaling, converges to the* FSE *under* $\mathbb{P}_\lambda^{N^2}$ *in the limit* $N \to \infty$.

Compared to [IOSV22], the study of Open Problem B requires two additional ingredients. Indeed, one needs a fine understanding of the way that the $O(1)N^{1/3}$ excursions of size $N^{2/3}$ interact together along the convex hull of the droplet. Moreover, the construction of FSE and the proof of the convergence of the mean excursion towards it are non-trivial as the pinning condition takes place *at the scale of the correlation length* of the system.

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[2]This terminology is directly inspired by the notion of explorable set in percolation theory.

1.5 Organisation of the paper

The paper is organised as follows: in Section 2, we gather some preliminary results that are going to be our toolbox for the proofs of the main results. Section 3 is then devoted to the proof of Theorem 1.3, while Section 4 consists in an adaptation of the arguments of [Ham12a, Ham12b] to prove Theorem 1.5. Finally, Section 5 is devoted to a discussion regarding the extension of the results to other statistical mechanics models in the Wulff setting.

Notations and conventions. We shall adopt Landau formalism for real valued-sequences. Namely, whenever (a_n) and (b_n) are two real-valued sequences, we will write $a_n = o(b_n)$ when $|a_n|/|b_n| \xrightarrow[n\to\infty]{} 0$. We will also use the notation $a_n = O(b_n)$ when there exists some constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all n large enough. If $a_n = O(b_n)$ and $b_n = O(a_n)$, we shall write that $a_n \asymp b_n$. Finally we shall write $a_n \sim b_n$ when $a_n/b_n \xrightarrow[n\to\infty]{} 1$.

If A is a set, we denote by $\mathcal{P}(A)$ the power set of A. For $x = (x_1, x_2) \in \mathbb{R}^2$, $\|x\| := \sqrt{x_1^2 + x_2^2}$ denotes the Euclidean norm of x. If $S \subset \mathbb{R}^2$ is a Borel set, we denote by $|S|$ its Lebesgue measure. Moreover, for $x \in \mathbb{Z}^2$ we will write $\arg(x) \in [0, 2\pi)$ to denote the complex argument of x seen as an element of $\mathbb{C}$. For $t \in \mathbb{R}$, $\lfloor t \rfloor$ will denote the integer part of t and $\lceil t \rceil := \inf\{k \in \mathbb{Z}, \ k \geq t\}$.

2 Preliminary results

In the rest of this work, we fix $0 < \lambda < \frac{1}{2}$.

2.1 Basic statistics of typical samples of $\mathbb{P}_\lambda^{N^2}$

In this subsection, we study some basic properties of a typical sample of $\mathbb{P}_\lambda^{N^2}$.

Lemma 2.1 (Tail estimates for the length of a sample of $\mathbb{P}_\lambda^{N^2}$)**.** *There exist $c, C > 0$ such that for any $N \geq 1$, any $t \geq 0$,*

$$\mathbb{P}_\lambda^{N^2} [|\Gamma| \geq tN] \leq C e^{-cN(t-2\sqrt{2})}. \tag{2.1}$$

Proof. It is clear that

$$\mathbb{P}_\lambda^{N^2} [|\Gamma| \geq tN] \leq \frac{\mathbb{P}_\lambda [|\Gamma| \geq tN]}{\mathbb{P}_\lambda [\mathcal{A}(\Gamma) \geq N^2]}. \tag{2.2}$$

An easy computation yields

$$\mathbb{P}_\lambda [|\Gamma| \geq tN] = \frac{1}{Z_\lambda} \sum_{k \geq \lfloor tN \rfloor} (2\lambda)^k = (2\lambda)^{\lfloor tN \rfloor}. \tag{2.3}$$

It remains to lower bound $\mathbb{P}_\lambda [\mathcal{A}(\Gamma) \geq N^2]$. Let $\mathsf{y}(\Gamma)$ (resp. $\mathsf{x}(\Gamma)$) be the y-coordinate (resp. x-coordinate) of the first (resp. last) vertex of Γ. The measure $\mathbb{P}_\lambda$ conditioned on $(\mathsf{y}(\Gamma), \mathsf{x}(\Gamma))$ is exactly the uniform measure over path starting at $(0, \mathsf{y}(\Gamma))$ and ending at $(\mathsf{x}(\Gamma), 0)$. We claim that

$$\mathbb{P}_\lambda [\mathcal{A}(\Gamma) \geq N^2 \mid (\mathsf{y}(\Gamma), \mathsf{x}(\Gamma)) = (\lceil \sqrt{2}N \rceil, \lceil \sqrt{2}N \rceil)] \geq \frac{1}{2}. \tag{2.4}$$

Indeed, the square formed by the vertices $(0, \lceil \sqrt{2}N \rceil), (\lceil \sqrt{2}N \rceil, \lceil \sqrt{2}N \rceil), (\lceil \sqrt{2}N \rceil, 0)$ and $(0,0)$ has an area at least equal to $2N^2$. Hence, a symmetry argument shows that the proportion

of oriented paths starting at $(0, \lceil \sqrt{2}N \rceil)$ and ending at $(\lceil \sqrt{2}N \rceil, 0)$ that fulfill the requirement $\{\mathcal{A}(\Gamma) \geq N^2\}$ is at least $1/2$. By a standard computation, we find $c_1 = c_1(\lambda) > 0$ such that, for all $N \geq 1$,

$$\mathbb{P}_\lambda \left[(\mathsf{y}(\Gamma), \mathsf{x}(\Gamma)) = (\lceil \sqrt{2}N \rceil, \lceil \sqrt{2}N \rceil) \right] = \frac{1}{Z_\lambda} \lambda^{2\lceil \sqrt{2}N \rceil} \binom{2\lceil \sqrt{2}N \rceil}{\lceil \sqrt{2}N \rceil} \geq c_1 (2\lambda)^{2\sqrt{2}N} N^{-1/2}. \qquad (2.5)$$

Putting all the pieces together

$$\mathbb{P}_\lambda^{N^2}[|\Gamma| \geq tN] \leq c_1^{-1} \sqrt{N} (2\lambda)^{tN} (2\lambda)^{-2\sqrt{2}N}. \qquad (2.6)$$

The proof follows readily. $\qquad \square$

This tail estimate allows us to argue that a typical sample of $\mathbb{P}_\lambda^{N^2}$ stays confined between two balls of linear radii with very high probability. For $K \geq 0$, let B_K be the Euclidean ball of radius K centered at 0.

Lemma 2.2 (Confinement lemma)**.** *There exist $K_1, K_2, c, C > 0$ such that for all $N \geq 1$,*

$$\mathbb{P}_\lambda^{N^2}[\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}] \geq 1 - C\mathrm{e}^{-cN}. \qquad (2.7)$$

Proof. Using Lemma 2.1 and a simple geometric observation, we get that for $K_1 \in \mathbb{N}_{>0}$,

$$\begin{aligned} \mathbb{P}_\lambda^{N^2} \left[\Gamma \cap (B_{K_1 N})^c \neq \emptyset \right] &\leq \mathbb{P}_\lambda^{N^2}[|\Gamma| \geq K_1 N] \\ &\leq C\mathrm{e}^{-c(K_1 - 2\sqrt{2})N}, \end{aligned}$$

which gives that $\Gamma \subset B_{K_1 N}$ with high probability for $K_1 > 2\sqrt{2}$.

For the second part of the statement, notice that the area of a path that is contained in $B_{K_1 N}$ and intersects B_{KN} (for $K < K_1$) is necessarily smaller than $2K_1 K N^2$. Choosing $K_2 < 1/(2K_1)$ so that the preceding quantity is smaller than N^2 yields that

$$\mathbb{P}_\lambda^{N^2} \left[\Gamma \cap B_{K_2 N} \neq \emptyset, \Gamma \cap (B_{K_1 N})^c = \emptyset \right] = 0. \qquad (2.8)$$

Hence,

$$\begin{aligned} \mathbb{P}_\lambda^{N^2}[\Gamma \cap B_{K_2 N} \neq \emptyset] &= \mathbb{P}_\lambda^{N^2} \left[\Gamma \cap B_{K_2 N} \neq \emptyset, \Gamma \cap (B_{K_1 N})^c \neq \emptyset \right] + 0 \\ &\leq \mathbb{P}_\lambda^{N^2} \left[\Gamma \cap (B_{K_1 N})^c \neq \emptyset \right], \end{aligned}$$

and the result follows. $\qquad \square$

In what follows, K_1, K_2 will always denote the constants given by Lemma 2.2.

The following quantity will be of particular interest for the rest of this work.

Definition 2.3 (Excess area)**.** The *excess area* of a path $\gamma \in \Lambda^{N^2}$ is the quantity $\mathsf{ExcessArea}(\gamma)$ defined by

$$\mathsf{ExcessArea}(\gamma) := \mathcal{A}(\gamma) - N^2. \qquad (2.9)$$

Since $\mathbb{P}_\lambda^{N^2}$ exponentially penalizes long paths, we may expect the typical area of a path to be close to N^2. The following result quantifies this observation and will be very useful later.

Proposition 2.4 (Tail behaviour of $\mathsf{ExcessArea}$)**.** *There exists $c > 0$ such that for all $0 \leq t \leq N$,*

$$\mathbb{P}_\lambda^{N^2}[\mathsf{ExcessArea}(\Gamma) \geq tN] \leq 2\mathrm{e}^{-ct}. \qquad (2.10)$$

The idea of the proof is simple. By Lemma 2.1, a sample Γ of $\mathbb{P}_\lambda^{N^2}$ has a length of order N. Assume that this path satisfies $\mathcal{A}(\Gamma) > N^2 + tN$. If we remove its first t steps we obtain a path of area (roughly) at least N^2 and which is exponentially (in t) favoured by $\mathbb{P}_\lambda^{N^2}$.

Proof. If A is any subset of Λ^{N^2}, we shall write $Z_\lambda^{N^2}[A] = \sum_{\gamma \in A} \lambda^{|\gamma|}$ for the partition function of A. We will also write $Z_\lambda^{N^2} := Z_\lambda^{N^2}[\Lambda^{N^2}]$. By definition,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{ExcessArea}(\Gamma) \geq tN, \ \Gamma \subset B_{K_1 N} \setminus B_{K_2 N}] = \frac{1}{Z_\lambda^{N^2}} \sum_{K_2 N \leq a,b \leq K_1 N} \lambda^{a+b} \sum_{\substack{\gamma:(0,a)\to(b,0) \\ \mathcal{A}(\gamma) \geq N^2 + tN}} 1. \quad (2.11)$$

If $\gamma : (0,a) \to (b,0)$ with $K_2 N \leq a,b \leq K_1 N$, call $\mathsf{c} = \mathsf{c}(\gamma)$ the y-coordinate of the point of γ of x-coordinate $\lfloor t/K_1 \rfloor$. Splitting the path γ at the point of coordinates $(\lfloor t/K_1 \rfloor, \mathsf{c})$ splits γ into a pair of elements of Λ (after translation) that we denote by (γ_1, γ_2), where γ_1 is a path from $(0, a - \mathsf{c})$ to $(\lfloor t/K_1 \rfloor, 0)$ and γ_2 is an element of Λ^{N^2} from $(0, \mathsf{c})$ to $(b - \lfloor t/K_1 \rfloor, 0)$. As a result,

$$\sum_{\substack{\gamma:(0,a)\to(b,0) \\ \mathcal{A}(\gamma)\geq N^2 + tN}} 1 \leq \sum_{c=0}^{a} \left| \Lambda^{(0,a-\mathsf{c})\to(\lfloor t/K_1 \rfloor, 0)} \right| \cdot \left| \Lambda_{N^2}^{(0,\mathsf{c})\to(b-\lfloor t/K_1 \rfloor),0)} \right|, \quad (2.12)$$

where $\Lambda^{x \to y}$ is the set of path of starting at x and ending at y, and the subscript N^2 accounts for the condition of enclosing an area of at least N^2. Now, notice that

$$\sum_{b=K_2 N}^{K_1 N} \lambda^{c+b-\lfloor t/K_1 \rfloor} \left| \Lambda_{N^2}^{(0,\mathsf{c})\to(b-\lfloor t/K_1 \rfloor, 0)} \right| \leq Z_\lambda^{N^2}[\mathsf{y}(\Gamma) = c], \quad (2.13)$$

where we recall that $\mathsf{y}(\gamma)$ is the y-coordinate of the starting point of γ. Recall also that $\mathsf{x}(\gamma)$ is the x-coordinate of the last point of γ. Hence,

$$\sum_{K_2 N \leq a,b \leq K_1 N} \lambda^{a+b} \sum_{\substack{\gamma:(0,a)\to(b,0) \\ \mathcal{A}(\gamma)\geq N^2 + tN}} 1$$

$$\leq \sum_{a=K_2 N}^{K_1 N} \sum_{c=0}^{a} \left| \Lambda^{(0,a-\mathsf{c})\to(\lfloor t/K_1 \rfloor, 0)} \right| \lambda^{a-\mathsf{c}+\lfloor t/K_1 \rfloor} Z_\lambda^{N^2}[\mathsf{y}(\Gamma) = c]$$

$$\leq \sum_{c=0}^{K_1 N} Z_\lambda^{N^2}[\mathsf{y}(\Gamma) = c] \sum_{a\geq c} \left| \Lambda^{(0,a-\mathsf{c})\to(\lfloor t/K_1 \rfloor, 0)} \right| \lambda^{a-\mathsf{c}+\lfloor t/K_1 \rfloor}$$

$$\leq \sum_{c=0}^{K_1 N} Z_\lambda^{N^2}[\mathsf{y}(\Gamma) = c] Z_\lambda[\mathsf{x}(\Gamma) = \lfloor t/K_1 \rfloor]$$

$$\leq Z_\lambda^{N^2} Z_\lambda[|\Gamma| \geq \lfloor t/K_1 \rfloor]$$

$$\leq Z_\lambda^{N^2} (2\lambda)^{\lfloor t/K_1 \rfloor}.$$

The proof follows readily. $\qquad\square$

2.2 Non-existence of large flat sections of a typical sample of $\mathbb{P}_\lambda^{N^2}$

In the analysis of samples of $\mathbb{P}_\lambda^{N^2}$ in given angular sectors, we will sometimes need to ensure that the marginal is not "degenerate" in the sense that it is not supported on "flat" paths (i.e almost horizontal / vertical paths). Because of the oriented feature of the model, this is an additional difficulty in comparison to the setup of subcritical statistical mechanics models, where it is often known thanks to the Ornstein–Zernike theory that the surface tension is analytic and bounded away from 0 and infinity (see for instance [CIV08, Theorem A]).

Let $\mathbf{A}$ be a cone of apex the origin in the first quadrant of angle $\Theta_\mathbf{A} \in (0, \pi/2]$. For $\gamma \in \Lambda^{N^2}$, let $x_\mathbf{A} = x_\mathbf{A}(\gamma)$ (resp $y_\mathbf{A} = y_\mathbf{A}(\gamma)$) be the left-most (resp. right-most) point of $\gamma \cap \mathbf{A}$. Also, recall that the set $\Lambda^{x \to y} \subset \Lambda$ consists of all the oriented path going from x to y. Observe that if γ is a path of Λ containing x and y, then there is a natural notion of restriction of γ between x

and y: it is the only element of $\Lambda^{x \to y}$ which coincides with γ between x and y. We also denote by $\theta(x_{\mathbf{A}}, y_{\mathbf{A}}) \in [0, \pi/2]$ the angle formed by the segment $[x_{\mathbf{A}}, y_{\mathbf{A}}]$ and the horizontal line going trough $x_{\mathbf{A}}$. For $\varepsilon > 0$, we define the following events

$$\mathsf{Bad}^+_{\varepsilon, \mathbf{A}} := \{\gamma \in \Lambda^{N^2} , \, \theta(x_{\mathbf{A}}, y_{\mathbf{A}}) \in [0, \varepsilon]\}, \tag{2.14}$$

and,

$$\mathsf{Bad}^-_{\varepsilon, \mathbf{A}} := \{\gamma \in \Lambda^{N^2} , \, \theta(x_{\mathbf{A}}, y_{\mathbf{A}}) \in [\tfrac{\pi}{2} - \varepsilon, \tfrac{\pi}{2}]\}. \tag{2.15}$$

Proposition 2.5. *There exist $\varepsilon > 0$ and $c = c(\varepsilon), C = C(\varepsilon) > 0$ such that, for any cone $\mathbf{A}$ as above,*

$$\mathbb{P}^{N^2}_\lambda \big[\Gamma \in \mathsf{Bad}^+_{\varepsilon, \mathbf{A}} \cup \mathsf{Bad}^-_{\varepsilon, \mathbf{A}}\big] \leq C\mathbb{E}^{N^2}_\lambda \big[e^{-c\|x_{\mathbf{A}}(\Gamma) - y_{\mathbf{A}}(\Gamma)\|}\big]. \tag{2.16}$$

For the proof of Proposition 2.5, we will use a probabilistic version of the multi-valued map principle, stated and proved in the appendix (Lemma B.1).

Proof of Proposition 2.5. We are going to create an appropriate multi-valued map T, transforming a path belonging to $\mathsf{Bad}^+_{\varepsilon, \mathbf{A}}$ to another path of $\Lambda^{N^2} \setminus \mathsf{Bad}^+_{\varepsilon, \mathbf{A}}$, in such a way that this map has a lot of possible images but very few pre-images, and such that the probability of one element of the image of a path $\gamma \in \mathsf{Bad}^+_{\varepsilon, \mathbf{A}}$ by T is not too small compared to $\mathbb{P}^{N^2}_\lambda[\gamma]$.
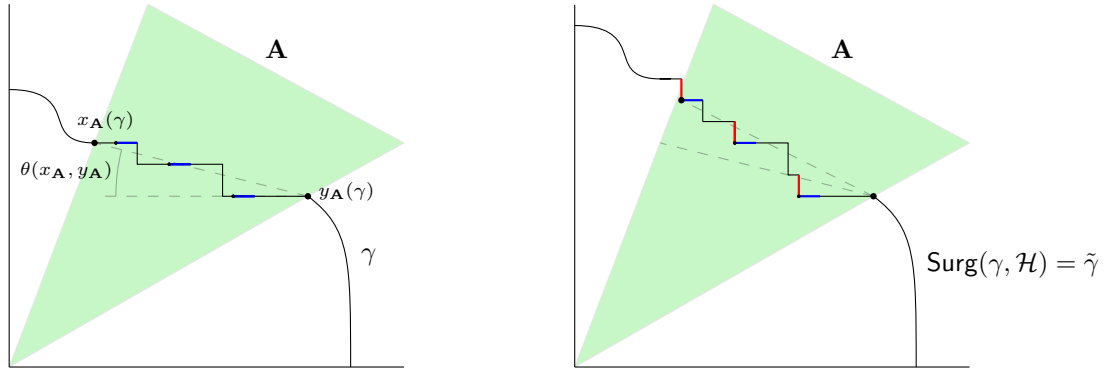

Figure 2: An illustration of the effect of the surgery map $\mathsf{Surg}(\gamma, \mathcal{H})$. The cone $\mathbf{A}$ is the light green region. The ηk edges of $\mathcal{H}$ that have been selected are the blue ones, and the vertical edges that have been added are depicted in red. We observe that the angle $\theta(x_{\mathbf{A}}, y_{\mathbf{A}})$ increases in $\mathsf{Surg}(\gamma, \mathcal{H})$.

Let $\eta > 0$ be a small parameter to be fixed later. For $\gamma \in \Lambda^{N^2}$, we denote by $\ell_{\mathbf{A}} = \ell_{\mathbf{A}}(\gamma)$ the length of the section of the path enclosed by $\mathbf{A}$, that is, $\ell_{\mathbf{A}}(\gamma) = |\gamma_{x_{\mathbf{A}}, y_{\mathbf{A}}}|$. We first define a surgical procedure modifying γ: if $\mathcal{H}$ is some subset of the set of the horizontal steps of γ inside $\mathbf{A}$, we set $\mathsf{Surg}(\gamma, \mathcal{H})$ to be the new path obtained by adding a vertical step just before each step belonging to $\mathcal{H}$ (see Figure 2). We are going to apply this procedure for $\mathcal{H}$ being a proportion η of the horizontal steps that γ makes in $\mathbf{A}$. Of course to be able to do so we need to ensure that the paths γ we apply the map to already contain a proportion at least η of horizontal steps in $\mathbf{A}$. Since we will apply the map to elements of $\mathsf{Bad}^+_{\varepsilon, \mathbf{A}}$, we expect the proportion of horizontal steps to be quite large. Indeed, if $\gamma \in \mathsf{Bad}^+_{\varepsilon, \mathbf{A}}$, and if $\alpha(\gamma)$ is the proportion of vertical steps in the piece of path of γ in $\mathbf{A}$, then one has that

$$\varepsilon \geq \arctan\left(\frac{\alpha(\gamma)}{1 - \alpha(\gamma)}\right), \tag{2.17}$$

so that $\alpha(\gamma) \leq \tan(\varepsilon) \leq 2\varepsilon$ (if ε is small enough). For technical reasons, it will also be convenient to work on a set of paths for which $\ell_{\mathbf{A}}$ is fixed. Introduce, for $k \geq 1$,

$$A_k := \mathsf{Bad}^+_{\varepsilon, \mathbf{A}} \cap \{\ell_{\mathbf{A}} = k\} \cap \Lambda^{N^2}, \tag{2.18}$$

and

$$B_k := \{k \leq \ell_{\mathbf{A}} \leq (1+\eta)k\} \cap \Lambda^{N^2}. \tag{2.19}$$

Define the multi-valued map $T_{\eta,k} : A_k \to \mathcal{P}(B_k)$ by setting for any $\gamma \in A_k$,

$$T_{\eta,k}(\gamma) = \{\mathsf{Surg}(\gamma, \mathcal{H}), |\mathcal{H}| = \eta k\}. \tag{2.20}$$

We notice that for $\gamma \in A_k$, $T_{\eta,k}(\gamma) \subset \Lambda^{N^2}$. Indeed, the area can only increase under the action of $\mathsf{Surg}(\gamma, \mathcal{H})$, for any $\mathcal{H}$ being a subset of the horizontal steps of γ. Moreover, after the surgery, the length of a path inside $\mathbf{A}$ has increased by at most ηk. Thus, the map $T_{\eta,k}$ is well-defined. We will consider the probability measures $\mathbb{P}_\lambda^{N^2}$ on A_k and B_k. This way, we are exactly in the setting of Lemma B.1. Introducing

$$\varphi(T_{\eta,k}) := \max_{a \in A_k} \max_{b \in T_{\eta,k}(a)} \frac{\mathbb{P}_\lambda^{N^2}[a]}{\mathbb{P}_\lambda^{N^2}[b]}, \qquad \psi(T_{\eta,k}) := \frac{\max_{b \in \tilde{B}_k} \left|T_{\eta,k}^{-1}(b)\right|}{\min_{a \in A_k} |T_{\eta,k}(a)|}, \tag{2.21}$$

where $\tilde{B}_k := \mathrm{Im}(T_{\eta,k})$. A direct application of Lemma B.1 yields that

$$\mathbb{P}_\lambda^{N^2}[A_k] \leq \varphi(T_{\eta,k})\psi(T_{\eta,k})\mathbb{P}_\lambda^{N^2}[B_k]. \tag{2.22}$$

We are now left with estimating $\varphi(T_{\eta,k})$ and $\psi(T_{\eta,k})$.

We start by observing that $\mathsf{Surg}(\gamma, \mathcal{H})$ increases the length of a path γ by $|\mathcal{H}|$ units, so that for any $\gamma \in A_k$ and $\tilde{\gamma} \in T_{\eta,k}(\gamma)$, the ratio defining φ is constant, and given by

$$\varphi(T_{\eta,k}) = \lambda^{-\eta k}. \tag{2.23}$$

Now, our task is to bound $\psi(T_{\eta,k})$. Let $\gamma \in A_k$. Then, it is clear that

$$\left|T_{\eta,k}(\gamma)\right| \geq \binom{(1-\alpha(\gamma))k}{\eta k} \geq \binom{(1-2\varepsilon)k}{\eta k}, \tag{2.24}$$

where $\alpha(\gamma)$ is the proportion of vertical steps in the piece of path of γ in $\mathbf{A}$ and where we used (2.17) to get that $\alpha(\gamma) \leq 2\varepsilon$ (if ε is small enough). Next, we have to bound $\left|T_{\eta,k}^{-1}(\tilde{\gamma})\right|$, for $\tilde{\gamma} \in \tilde{B}_k$. Some vertical steps may have been pushed outside the sector $\mathbf{A}$ by the effect of $T_{\eta,k}$, and we must explore them to reconstruct γ knowing $T_{\eta,k}(\gamma)$. Since at most ηk of these steps disappeared in $\mathbf{A}$, we obtain

$$\left|T_{\eta,k}^{-1}(\tilde{\gamma})\right| \leq \binom{(2\varepsilon+2\eta)k}{\eta k}. \tag{2.25}$$

Putting all the pieces together and using (2.22), we obtain

$$\mathbb{P}_\lambda^{N^2}[A_k] \leq \lambda^{-\eta k} \frac{\binom{(2\varepsilon+2\eta)k}{\eta k}}{\binom{(1-2\varepsilon)k}{\eta k}} \mathbb{P}_\lambda^{N^2}[B_k]. \tag{2.26}$$

Standard estimates[3] yield that, as $k \to \infty$,

$$\lambda^{-\eta k} \frac{\binom{(2\varepsilon+2\eta)k}{\eta k}}{\binom{(1-2\varepsilon)k}{\eta k}} \sim \exp(-k[(1-2\varepsilon)I\left(\frac{\eta}{1-2\varepsilon}\right) - (2\varepsilon+2\eta)I\left(\frac{\eta}{2\varepsilon+2\eta}\right) + \eta \log \lambda]), \tag{2.28}$$

where for $x \in (0,1)$, $I(x) := -x \log x - (1-x)\log(1-x)$.

---

[3]For any $c_1 < c_2$, as $k \to \infty$,

$$\binom{c_2 k}{c_1 k} \sim \mathrm{e}^{c_2 k I\left(\frac{c_1}{c_2}\right)}. \tag{2.27}$$

Set

$$c_{\varepsilon,\eta} = (1 - 2\varepsilon)I\left(\frac{\eta}{1 - 2\varepsilon}\right) - (2\varepsilon + 2\eta)I\left(\frac{\eta}{2\varepsilon + 2\eta}\right) + \eta\log\lambda. \tag{2.29}$$

It is easy to check that one can find a pair $\varepsilon, \eta(\varepsilon) > 0$ satisfying $c_{\varepsilon,\eta(\varepsilon)} > 0$. Let us fix such an $\varepsilon > 0$ and drop η from the notations. We obtained that there exists $C = C(\varepsilon) > 0$ such that for $k \geq 1$,

$$\lambda^{-\eta k}\frac{\binom{(2\varepsilon + 2\eta)k}{\eta k}}{\binom{(1 - 2\varepsilon)k}{\eta k}} \leq Ce^{-c_\varepsilon k}. \tag{2.30}$$

Hence,

$$\mathbb{P}_\lambda^{N^2}[\Gamma \in \mathsf{Bad}_{\varepsilon,\mathbf{A}}^+, \ell_A = k] \leq Ce^{-c_\varepsilon k}\mathbb{P}_\lambda^{N^2}[k \leq \ell_\mathbf{A} \leq (1 + \eta)k]. \tag{2.31}$$

Summing over k we finally obtain the existence of $c_1 = c_1(\varepsilon), C_1 = C_1(\varepsilon)$ such that

$$\mathbb{P}_\lambda^{N^2}[\Gamma \in \mathsf{Bad}_{\varepsilon,\mathbf{A}}^+] \leq C_1\mathbb{E}_\lambda^{N^2}[e^{-c_1\ell_\mathbf{A}}] \leq C_1\mathbb{E}_\lambda^{N^2}[e^{-c_1\|x_\mathbf{A}(\Gamma) - y_\mathbf{A}(\Gamma)\|}]. \tag{2.32}$$

A similar argument holds for $\mathsf{Bad}_{\varepsilon,\mathbf{A}}^-$. This concludes the proof. $\square$

2.3 Couplings between $\mathbb{P}_\lambda^{N^2}$ and a conditioned random walk

This subsection is devoted to the description of two couplings between the measure $\mathbb{P}_\lambda^{N^2}$ conditioned on a pinning event and the law of a random walk bridge. We first describe and prove the existence of these couplings and then explain what they will be useful for.

Let $a, b \in \mathbb{N}^2$ such that $\arg(a) > \arg(b)$. We introduce the event $\mathsf{Fac}(a, b)$ defined by

$$\mathsf{Fac}(a, b) := \{\gamma \in \Lambda, \text{ The segment } [a, b] \text{ is a facet of } \gamma\}. \tag{2.33}$$

Recall that the set $\Lambda^{a \to b} \subset \Lambda$ consists of all the oriented path going from a to b. We introduce the following partial order in the set $\Lambda^{a \to b}$: for $\gamma^1, \gamma^2 \in \Lambda^{a \to b}$, we shall say that $\gamma^1 \geq \gamma^2$ when

$$\forall(x_1, y_1) \in \gamma^1, (x_2, y_2) \in \gamma^2, x_1 = x_2 \Rightarrow y_1 \geq y_2. \tag{2.34}$$

Finally, we denote by $\mathbb{P}_{a,b}^-$ the uniform measure over all the paths of $\Lambda^{a \to b}$ that remain under the segment $[a, b]$.

With these definitions in hand, we are now able to describe the first coupling.

Proposition 2.6. *Let a and b be two vertices of $\mathbb{N}^2$ such that $\arg(a) > \arg(b)$. There exists a probability measure Ψ on the space $\Lambda^{a \to b} \times \Lambda^{a \to b}$ such that:*

(i) the marginal of Ψ on its first coordinate has the law of $\Gamma_{a,b}$ where Γ is sampled according to $\mathbb{P}_\lambda^{N^2}[\cdot \mid \mathsf{Fac}(a, b)]$,

(ii) the marginal of Ψ on its second coordinate has law $\mathbb{P}_{a,b}^-$,

(iii) $\Psi[\{(\gamma^1, \gamma^2) \in \Lambda^{a \to b} \times \Lambda^{a \to b}, \gamma^1 \geq \gamma^2\}] = 1$.

Proof. The proof relies on a dynamical Markov chain argument and is, for percolation *aficionados*, very similar to that of Holley's inequality (see for instance [Gri10]).

We shall start from the restriction of a sample of $\mathbb{P}_\lambda^{N^2}[\cdot \mid \mathsf{Fac}(a, b)]$ to $\Lambda^{a \to b}$ that we denote $\Gamma_{a,b}$, and then describe two Markovian dynamics $(\Gamma_k^1)_{k \geq 0}$ and $(\Gamma_k^2)_{k \geq 0}$ on the state space $\Lambda^{a \to b}$ with the following properties:

(P1) $\Gamma_0^1 = \Gamma_0^2 = \Gamma_{a,b}$,

(P2) $\forall k \geq 0, \Gamma_k^1 \geq \Gamma_k^2$,

(P3) the law of $\Gamma_{a,b}$ is the stationary measure of the Markov chain $(\Gamma_k^1)_{k \geq 0}$,

(P4) $\mathbb{P}_{a,b}^-$ is the stationary measure of the Markov chain $(\Gamma_k^2)_{k \geq 0}$.

Now, let us describe the Markovian dynamics. Let $p = \|a - b\|_1$ be the length of any path of $\Lambda^{a \to b}$. Let $(\ell_k)_{k \geq 0}$ be a family of independent random variables all having a uniform distribution on $\{1, \ldots, p - 1\}$, and $(X_k)_{k \geq 0}$ a family of independent random variables all having a uniform distribution on $\{0, 1\}$. At step k, the random variable ℓ_k is the position of the vertex on which we will proceed a random modification.

Looking at the two edges incident to the vertex $\Gamma_k^i(\ell_k)$ for $i \in \{1, 2\}$, three different configurations can arise:

1) If these two edges are co-linear, we do not modify anything: $\Gamma_{k+1}^i = \Gamma_k^i$ for $i \in \{1, 2\}$.

2) If these edges form a corner of the type $\llcorner$

    (*i*) For Γ^1 we flip the corner into the opposite corner $\urcorner$ if and only if we have $X_k = 1$ **and** the modified path does not go above the facet $[a, b]$. If any of these two conditions is not verified, we do not modify Γ_k^1.

    (*ii*) We apply the same procedure for Γ^2.

3) If these edges form a corner of the type $\urcorner$

    (*i*) For Γ^1, we flip this corner into the opposite corner $\llcorner$ if and only if $X_k = 1$ **and** the area enclosed by this new path is still greater than N^2. If any of these two conditions is not verified we do not modify Γ_k^1.

    (*ii*) For Γ^2, we flip the corner into the opposite one if and only if $X_k = 1$.

It is straightforward that these Markov dynamics are irreducible, aperiodic, and reversible. Then, Γ^1 (resp. Γ^2) admits a stationnary measure and converges towards it: it is straightforward to see that this stationary measure is the law of $\Gamma_{a,b}$ (resp $\mathbb{P}_{a,b}^-$). Moreover, by construction, one always has $\Gamma^1 \geq \Gamma^2$. This yields the desired stochastic domination. $\square$

This coupling will be very efficient when computing local statistics of a sample of $\mathbb{P}_\lambda^{N^2}$ along a facet (e.g. the local roughness along a facet) as it allows us to compare them to the ones of a random walk excursion on which we know much more information. More precisely, Proposition 2.6 has the following consequence: an upper bound on the length of a facet automatically converts into an upper bound on the local roughness under this facet.

Lemma 2.7. *There exist $c, C, N_0 > 0$ such that the following holds. For any $a, b \in \mathbb{N}^2$ with $\arg(a) > \arg(b)$ and $\|b - a\| \geq N_0$ and any $t \geq 0$,*

$$\mathbb{P}_\lambda^{N^2} \big[\max_{x \in \Gamma_{a,b}} \mathsf{LR}(x) > t \|b - a\|^{1/2} \,\big|\, \mathsf{Fac}(a, b) \big] \leq C \exp(-ct^2). \tag{2.35}$$

Proof. Observe that thanks to Proposition 2.6, the random variable $\max_{x \in \Gamma_{a,b}} \mathsf{LR}(x)$ where Γ is sampled according to $\mathbb{P}_\lambda^{N^2} [\,\cdot\,|\, \mathsf{Fac}(a, b)]$ is stochastically dominated by the random variable $\max_{x \in \omega_{a,b}} \mathsf{LR}(x)$ where $\omega_{a,b}$ is sampled according to $\mathbb{P}_{a,b}^-$.

The statement then follows by classical random walks arguments. Indeed, the results of [CC13] imply that a lattice random walk bridge conditioned to stay below a line segment with increments having exponential tails, after subtraction of the linear term corresponding to the equation of that segment, converges towards the standard Brownian excursion. As a consequence, knowing the fluctuations of the Brownian excursion (see [Chu75]) on an interval of size $\|b - a\|$ leads to the bound

$$\mathbb{P}_{a,b}^- \big[\max_{x \in \omega_{a,b}} \mathsf{LR}(x) \geq t \|b - a\|^{1/2} \big] \leq C \exp(-ct^2), \tag{2.36}$$

when $\|b - a\|$ is large enough. $\square$

Remark 2.8. In Section 4, we will use the preceding result in the regime $t = (\log \|b - a\|)^\alpha$ for some power $0 < \alpha < 1$. We claim that the result still holds true and is a consequence of classical moderate deviations estimates for random walks (see [dA92]).

The second coupling we shall need is closely related to the first one with the only exception that we do not assume that $[a, b]$ is a facet anymore. This time, we will build an "increasing coupling" (in the sense of the partial order of $\Lambda^{a \to b}$) between the law of $\Gamma_{a,b}$ where Γ is sampled according to $\mathbb{P}_\lambda^{N^2}[\cdot \mid a, b \in \Gamma]$ and the distribution of the random walk bridge between a and b. We denote by $\mathbb{P}_{a,b}$ the uniform distribution on $\Lambda^{a \to b}$.

Proposition 2.9. *Let a and b be two vertices of $\mathbb{N}^2$ such that $\arg(a) > \arg(b)$. There exists a probability measure $\overline{\Psi}$ on the space $\Lambda^{a \to b} \times \Lambda^{a \to b}$ such that:*

(i) the marginal of $\overline{\Psi}$ on its first coordinate has the law of $\Gamma_{a,b}$ where Γ is sampled according to $\mathbb{P}_\lambda^{N^2}[\cdot \mid a, b \in \Gamma]$,

(ii) the marginal of $\overline{\Psi}$ on its second coordinate has law $\mathbb{P}_{a,b}$,

(iii) $\overline{\Psi}\big[\{(\gamma^1, \gamma^2) \in \Lambda^{a \to b} \times \Lambda^{a \to b}, \gamma^1 \geq \gamma^2\}\big] = 1$.

Proof. The proof follows the strategy used to obtain Proposition 2.6 except that we do not have the condition of not going above the segment $[a, b]$ anymore. □

2.4 A rough upper bound on the length of a typical facet

The goal of this section is to give an illustration of the resampling strategy which will be used throughout the paper. The result proved in this section will also be useful for the proof of the upper bounds of Theorem 1.3.

Proposition 2.10. *Let $\epsilon > 0$. There exist $c = c(\epsilon), C = C(\epsilon) > 0$ such that, for all $1 \leq t \leq N^{2/3-\epsilon}$.*

$$\mathbb{P}_\lambda^{N^2}\big[\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{2}{3}+\epsilon}\big] \leq C e^{-ct^{3/2} N^{3\epsilon/2}}. \tag{2.37}$$

Moreover, there exists $c', C' > 0$ such that, for all $t \geq 1$

$$\mathbb{P}_\lambda^{N^2}\big[\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{4}{3}}\big] \leq C' e^{-c'tN^{4/3}}. \tag{2.38}$$

Before proving this result, we need to define an event which we will ask the resamplings to satisfy in order to ensure they capture a large enough area.

Let us first recall and introduce some notations. For $x, y \in \mathbb{N}^2$, let $\mathbf{A}_{x,y}$ be the cone of apex 0 bounded by x and y and let $\mathbf{T}_{0,x,y}$ be the triangle of apexes $0, x$ and y. For a cone $\mathbf{A}_{x,y}$ and $\gamma \in \Lambda$, we define $\mathrm{Enclose}(\gamma \cap \mathbf{A}_{x,y})$ to be the region delimited by γ and the two boundary axes of $\mathbf{A}_{x,y}$. The following event, illustrated in Figure 3, will be the one we will require our resampling to satisfy.

Definition 2.11 (Good area capture)**.** Let $\gamma \in \Lambda$ be a path, $\eta > 0$ and $x, y \in \mathbb{N}^2$. We say that γ realises the event $\mathsf{GAC}\,(x, y, \eta)$ (meaning "good area capture") if

(i) $\gamma \cap \mathbf{A}_{x,y} \in \Lambda^{x \to y}$, or in words, γ connects x and y by a oriented path,

(ii) $|\mathrm{Enclose}(\gamma \cap \mathbf{A}_{x,y})| - |\mathbf{T}_{0,x,y}| \geq \eta \|x - y\|^{\frac{3}{2}}$.

Proof of Proposition 2.10. Let $1 \leq t \leq N^{2/3-\epsilon}$. Introduce the event

$$\mathsf{BigMeanFL}(t) := \{\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{2}{3}+\epsilon}\}. \tag{2.39}$$
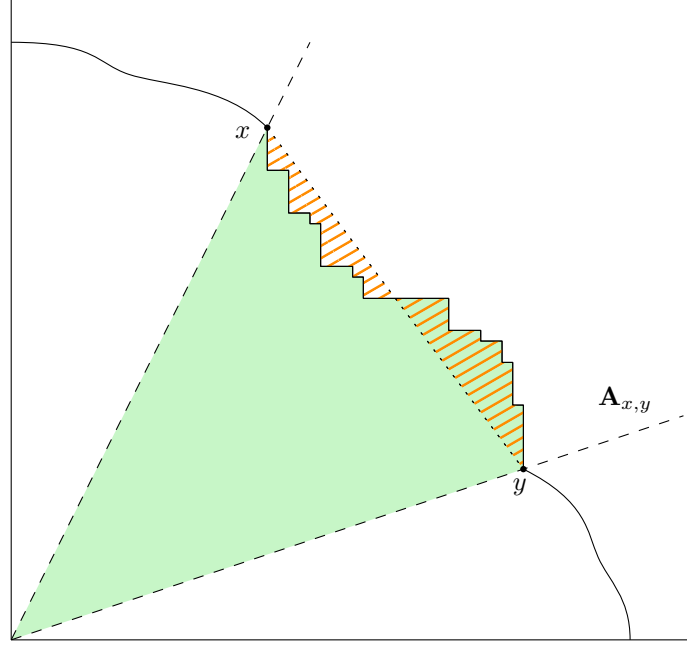
Figure 3: An illustration of the event $\mathsf{GAC}\,(x,y,\eta)$. The light green shaded region corresponds to $\mathsf{Enclose}(\gamma \cap \mathbf{A}_{x,y})$. The event that the path γ satisfies $\mathsf{GAC}\,(x,y,\eta)$ means that the (algebraic) area of the orange hatched region exceeds $\eta \|x - y\|^{3/2}$.

We pick uniformly at random, and independently of Γ, two points $\mathbf{x}, \mathbf{y}$ (with $\arg(\mathbf{x}) > \arg(\mathbf{y})$) in $B_{K_1 N}$ and call $\mathsf{GoodHit}$ the event that these two points coincide with the extremities of $\mathsf{MeanFac}$. The measure associated with this procedure is denoted $\mathbb{P}$.

Let us call $\mathsf{Bad}_\varepsilon^{+,-}$ the event that $\theta(\mathbf{x}, \mathbf{y}) \in [0, \varepsilon) \cup (\pi/2 - \varepsilon, \pi/2]$ for the $\varepsilon > 0$ given by Proposition 2.5. Let $\mathcal{E} := \mathsf{GoodHit} \cap (\mathsf{Bad}_\varepsilon^{+,-})^c \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}$. Resample Γ between $\mathbf{x}$ and $\mathbf{y}$ according to the uniform law among paths $\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}}$ such that $(\Gamma \setminus \Gamma_{\mathbf{x},\mathbf{y}}) \cup \gamma$ satisfies the area condition and call $\tilde{\Gamma}$ the resampling. Note that $\tilde{\Gamma}$ has law $\mathbb{P}_\lambda^{N^2}$.

Since we are resampling along a facet, if $(\Gamma, (\mathbf{x}, \mathbf{y}))$ realises $\mathcal{E}$, then $\Gamma_{\mathbf{x},\mathbf{y}}$ lies underneath the segment joining $\mathbf{x}$ and $\mathbf{y}$. Thus, if $(\Gamma, (\mathbf{x}, \mathbf{y})) \in \mathcal{E}$, replacing $\Gamma_{\mathbf{x},\mathbf{y}}$ by any path satisfying $\mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)$ still satisfies the area condition and[4]

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\tilde{\Gamma} \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta) \mid (\Gamma, (\mathbf{x}, \mathbf{y})) \in \mathsf{BigMeanFL}(t) \cap \mathcal{E}] \geq \mathbb{P}_{\mathbf{x},\mathbf{y}}[\mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)]. \qquad (2.40)$$

where we recall that $\mathbb{P}_{a,b}$ is the uniform law on $\Lambda^{a \to b}$. Indeed, conditioning on $(\Gamma, (\mathbf{x}, \mathbf{y})) \in \mathsf{BigMeanFL}(t) \cap \mathcal{E}$, the above inequality is equivalent to

$$\frac{|\{\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}}, \ \gamma \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)\}|}{|\{\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}}, \ (\Gamma \setminus \Gamma_{\mathbf{x},\mathbf{y}}) \cup \gamma \in \Lambda^{N^2}\}|} \geq \frac{|\{\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}}, \ \gamma \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)\}|}{|\Lambda^{\mathbf{x} \to \mathbf{y}}|}, \qquad (2.41)$$

which is trivially true. Now, if $\theta(\mathbf{x}, \mathbf{y}) \in [\varepsilon, \pi/2 - \varepsilon]$, standard computations on the simple random walk bridge (see Lemma 3.4 for more details) ensure that for $\eta > 0$ sufficiently small there exists $c_1 = c_1(\eta, \varepsilon) > 0$ such that,

$$\mathbb{P}_{\mathbf{x},\mathbf{y}}[\mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)] \geq c_1. \qquad (2.42)$$

Notice also that

$$\{\tilde{\Gamma} \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta), (\Gamma, (\mathbf{x}, \mathbf{y})) \in \mathsf{BigMeanFL}(t) \cap \mathcal{E}\} \subset \{\mathsf{ExcessArea}(\tilde{\Gamma}) \geq \eta t^{\frac{3}{2}} N^{1 + \frac{3}{2}\epsilon}\}. \qquad (2.43)$$

---

[4]We use the fact that $\mathsf{GAC}\,(\mathbf{x}, \mathbf{y}, \eta)$ can be seen as an event on $\Lambda^{\mathbf{x} \to \mathbf{y}}$ and for convenience we keep the same notation when looking at this event under the two measures of interest.

Putting all the pieces together, and using the fact that $\tilde{\Gamma}$ has law $\mathbb{P}_\lambda^{N^2}$,

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t) \cap \mathcal{E}] \leq c_1^{-1} \mathbb{P}_\lambda^{N^2}\left[\mathsf{ExcessArea}(\Gamma) \geq \eta t^{\frac{3}{2}} N^{1+\frac{3}{2}\epsilon}\right]. \tag{2.44}$$

Finally, notice that for some $c_2 > 0$,

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}\left[\mathsf{GoodHit} \mid \mathsf{BigMeanFL}(t) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}\right] \geq \frac{c_2}{N^4}. \tag{2.45}$$

Putting all the pieces together,

$$\begin{aligned}
\mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t)] &\leq \frac{N^4}{c_2} \cdot \mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t) \cap \mathcal{E}] \\
&+ \frac{N^4}{c_2} \cdot \mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t) \cap \mathsf{GoodHit} \cap (\mathsf{Bad}_\varepsilon^{+,-}) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}] \\
&+ \mathbb{P}_\lambda^{N^2}\left[\mathsf{BigMeanFL}(t) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}^c\right]. \tag{2.46}
\end{aligned}$$

To bound the second term we use Proposition 2.5 above. Notice that,

$$\begin{aligned}
\mathsf{BigMeanFL}(t) \cap \mathsf{GoodHit} \cap (\mathsf{Bad}_\varepsilon^{+,-}) \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\} \\
\subset \bigcup_{\substack{u,v \in B_{K_1 N} \setminus B_{K_2 N} \\ \|u-v\| \geq t N^{\frac{2}{3}+\epsilon}}} \{\Gamma \in \mathsf{Bad}_{\varepsilon, \mathbf{A}_{u,v}}^+ \cup \mathsf{Bad}_{\varepsilon, \mathbf{A}_{u,v}}^+\},
\end{aligned}$$

where $\mathbf{A}_{u,v}$ is the cone of apex 0 bounded by u and v. Finally, apply Lemma 2.2, Proposition 2.4, and Proposition 2.5 to get that for some $c = c(\varepsilon) > 0$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}(t)] \leq \frac{N^4}{c_2} \cdot c_1^{-1} \cdot \exp(-ct^{\frac{3}{2}} N^{\frac{3}{2}\epsilon}) + \frac{N^4}{c_2} \cdot (K_1 N)^4 \cdot 2 \exp(-ct N^{\frac{2}{3}+\epsilon}) + \exp(-cN), \tag{2.47}$$

and thus the first inequality (recall that $t \leq N^{2/3-\epsilon}$).

For $t \geq 1$[5] notice that

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanFL}(\Gamma) \geq t N^{\frac{4}{3}}\right] \leq \mathbb{P}_\lambda^{N^2}\left[|\Gamma| \geq (t N^{1/3}) \cdot N\right]. \tag{2.48}$$

Using Lemma 2.1 we get the result. $\square$

3   Analysis of the mean facet length and the mean local roughness

3.1   Upper bounds

In the following subsection, we actually prove a stronger statement than the upper bounds of Theorem 1.3. Recall that MeanFL and MeanLR where defined in Definition 1.2. The goal of this subsection is to prove upper tail estimates on MeanFL and MeanLR. Our first result is a refinement of Proposition 2.10.

Proposition 3.1 (Upper tail of MeanFL)**.** *There exist $\tilde{c}, c, C > 0$ such that, for any $\tilde{c} \leq t \leq N^{2/3}$,*

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanFL}(\Gamma) \geq t N^{\frac{2}{3}}\right] \leq C \mathsf{e}^{-ct^{3/2}}. \tag{3.1}$$

Proposition 3.2 (Upper tail of MeanLR)**.** *There exist $\tilde{c}, c, C > 0$ such that for any $\tilde{c} \leq t \leq N^{5/6}$,*

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MeanLR}(\Gamma) \geq t N^{\frac{1}{3}}\right] \leq C \mathsf{e}^{-ct^{6/5}}. \tag{3.2}$$

---

[5]In fact, one can take $t \geq 3\sqrt{2} N^{-1/3}$ here.

Remark 3.3. For larger values of t, we can use Lemma 2.1 and Proposition 2.10 to get explicit tails for MeanFL and MeanLR. In particular, we obtain that

$$\limsup_{N\to\infty} \mathbb{E}_\lambda^{N^2}\left[\frac{\mathsf{MeanFL}(\Gamma)}{N^{2/3}}\right] < \infty, \qquad \limsup_{N\to\infty} \mathbb{E}_\lambda^{N^2}\left[\frac{\mathsf{MeanLR}(\Gamma)}{N^{1/3}}\right] < \infty. \tag{3.3}$$

We keep the notations of Section 2.4. We will again use the event $\mathsf{GAC}\,(x,y,\eta)$ introduced in Definition 2.11. While it is difficult to estimate the probability of the event $\mathsf{GAC}\,(x,y,\eta)$ under $\mathbb{P}_\lambda^{N^2}$, it is much simpler to estimate it under $\mathbb{P}_{x,y}$ which is the uniform law over $\Lambda^{x\to y}$ (assuming that we chose $x,y\in\mathbb{N}^2$ such that $\Lambda^{x\to y}\neq\emptyset$). The following result, whose proof is postponed to the appendix, gives us a lower bound on $\mathbb{P}_{x,y}[\mathsf{GAC}\,(x,y,\eta)]$. Denote by $\theta(x,y)\in[0,\pi/2]$ the angle formed by the horizontal axis and the segment joining x and y.

Lemma 3.4. *Let $\varepsilon,\eta>0$. There exist $c=c(\varepsilon,\eta)>0$ and $N_0=N_0(\varepsilon,\eta)>0$ such that for any $x,y\in\mathbb{N}^2$ satisfying $\Lambda^{x\to y}\neq\emptyset$, $\varepsilon\le\theta(x,y)\le\pi/2-\varepsilon$ and $\|x-y\|\ge N_0$,*

$$\mathbb{P}_{x,y}[\mathsf{GAC}\,(x,y,\eta)] \ge c. \tag{3.4}$$

Remark 3.5.   1. The constant $c(\varepsilon,\eta)$ in the result above degenerates as $\varepsilon\to 0$. However, as we saw in Proposition 2.5, up to an event of small probability, we will be able to assume that θ stays bounded away from 0 and $\pi/2$.

   2. The constant $\tilde{c}$ in the statement of Propositions 3.1 and 3.2 will be chosen in terms of $N_0=N_0(\varepsilon,\eta)$ for ε given by Proposition 2.5 and a properly chosen $\eta>0$.

3.1.1   Proof of Proposition 3.1

As in the proof of Proposition 2.10, we would like to resample along the mean facet. However, as we saw, to pick the extremities of the mean facet without revealing any information about the path Γ comes with a cost of $O(N^4)$. This cost was previously handled by the requirement that $t\ge N^\epsilon$, but now we allow t to be of smaller order. To tackle this difficulty, we will not resample between the exact extremities of the mean facet but between points which approximate them (see Figure 4).

Proof of Proposition 3.1. Let $t>0$ be fixed, and let $\delta,\varepsilon>0$ be two small parameters that will be fixed later. Thanks to Proposition 2.10, we can additionally assume that $t\le N^\epsilon$ for some $\epsilon\in(0,1/3)$. Indeed, if $N^\epsilon\le t\le N^{2/3}$, using (2.37), we get

$$\mathbb{P}_\lambda^{N^2}\!\left[\mathsf{MeanFL}(\Gamma)\ge(tN^{-\epsilon})N^{2/3+\epsilon}\right]\le Ce^{-ct^{3/2}}. \tag{3.5}$$

We now assume $t\le N^\epsilon$. Let us introduce the following event

$$\mathsf{BigMeanFL}(t)=\left\{\mathsf{MeanFL}(\Gamma)\ge tN^{\frac{2}{3}}\right\}. \tag{3.6}$$

We also introduce, for any integer $k\ge 1$,

$$\mathsf{BigMeanFL}_k(t)=\left\{ktN^{\frac{2}{3}}\le\mathsf{MeanFL}(\Gamma)<(k+1)tN^{\frac{2}{3}}\right\}, \tag{3.7}$$

so that the family $(\mathsf{BigMeanFL}_k(t))_{k\ge 1}$ partitions the event $\mathsf{BigMeanFL}$.

Notice that for large values of k, we can use Proposition 2.10 again to conclude. If $k_N(\epsilon,t)$ is the smallest $k\ge 1$ such that $tk\ge N^\epsilon$, using (2.37) again,

$$\mathbb{P}_\lambda^{N^2}\!\left[\mathsf{MeanFL}(\Gamma)\ge tk_N(\epsilon,t)N^{\frac{2}{3}}\right]\le Ce^{-c(tk_N)^{3/2}}. \tag{3.8}$$

In particular it suffices to control the probabilities of $\mathsf{BigMeanFL}_k(t)$ for $1\le k\le k_N(\epsilon,t)$. In fact, it is sufficient to control the probabilities of $\mathsf{BigMeanFL}_k(t/3)$ for $3\le k\le k_N(\epsilon,t/3)=3k_N(\epsilon,t)$. We now let $s=t/3$ and $k_N=k_N(\epsilon,s)$.
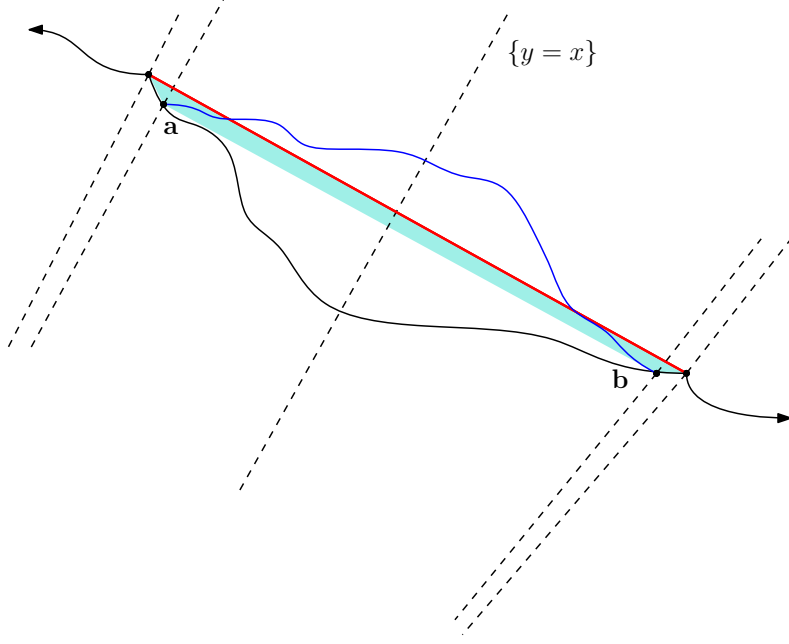
Figure 4: Illustration of the proof of Proposition 3.1. The original is path depicted in black. With large probability, the new resampled blue path captures an amount of area sufficient to compensate the area that could have been lost (in turquoise) due to the fact that **a** and **b** do not exactly coincide with the endpoints of the red facet. The event SD ensures that **a** and **b** are not atypically far from these two endpoints, allowing us to control the area of the turquoise region.

Fix $3 \leq k \leq k_N$. For $j \geq 0$, define the angles

$$\theta_j^+ = \theta_j^+(\delta, k) := \frac{\pi}{4} + \arctan\left(\frac{j\delta ks}{N^{1/3}}\right), \qquad \theta_j^- = \theta_j^-(\delta, k) := \frac{\pi}{4} - \arctan\left(\frac{j\delta ks}{N^{1/3}}\right), \qquad (3.9)$$

and denote $\ell_j^+ = \ell_j^+(\delta, k)$ (resp. $\ell_j^- = \ell_j^-(\delta, k)$) the half-line rooted at the origin and of argument θ_j^+ (resp. θ_j^-).

When $\mathsf{BigMeanFL}_k(s)$ occurs, there are at most $2/\delta$ indices j such that ℓ_j^+ (resp. ℓ_j^-) intersects the mean facet. For a sample Γ and $0 \leq j \leq 2/\delta$, we call $a_j = a_j(\Gamma, \delta, k)$ (resp. $b_j = b_j(\Gamma, \delta, k)$) the points of $\mathbb{N}^2$ that are the closest[6] to the intersection between Γ and ℓ_j^+ (resp. ℓ_j^-). Observe that these points are defined from Γ by a deterministic procedure which means that resampling a sample Γ between two of these points yields an output distributed according to $\mathbb{P}_\lambda^{N^2}$. In particular, we can define $\mathbf{a} = \mathbf{a}(\Gamma, \delta, k)$ (resp. $\mathbf{b} = \mathbf{b}(\Gamma, \delta, k)$) to be the point of Γ defined as follows: if j_0 (resp. j_1) is the largest j such that ℓ_j^+ (resp. ℓ_j^-) intersects the mean facet, then $\mathbf{a} = a_{j_0}$ (resp. $\mathbf{b} = b_{j_1}$).

In order to be able to pick **a** and **b** and resample between them without revealing the portion $\Gamma_{\mathbf{a},\mathbf{b}}$ of Γ between **a** and **b**, we introduce some extra randomness. We pick uniformly, a pair of points $(\mathbf{x}, \mathbf{y})$ in $\{a_j,\ 0 \leq j \leq 2/\delta\} \times \{b_j,\ 0 \leq j \leq 2/\delta\}$ and call $\mathbb{P}$ the measure associated with this random procedure. Let $\mathsf{GoodHit}_k$ be the event that $(\mathbf{x}, \mathbf{y}) = (\mathbf{a}, \mathbf{b})$. We are going to show that with uniformly positive probability, if we resample between **a** and **b**, the resampled path captures a linear excess of area, an event which is exponentially unlikely by Proposition 2.4.

As in the proof of Proposition 2.10, let us call $\mathsf{Bad}_\varepsilon^{+,-}$ the event that $\theta(\mathbf{x}, \mathbf{y}) \in [0, \varepsilon) \cup (\pi/2 - \varepsilon, \pi/2]$ for the $\varepsilon > 0$ given by Proposition 2.5. Write $\mathcal{E}_k := \mathsf{GoodHit}_k \cap (\mathsf{Bad}_\varepsilon^{+,-})^c \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}$. Resample Γ between **x** and **y** according to the uniform law among paths

---

[6] There might be ambiguity in the choice of such points and we solve this issue by asking a_j (resp. b_j) to lie on the right (resp. on the left) of ℓ_j^+ (resp. ℓ_j^-).

$\gamma \in \Lambda^{\mathbf{x} \to \mathbf{y}}$ such that $(\Gamma \setminus \Gamma_{\mathbf{x},\mathbf{y}}) \cup \gamma$ satisfies the area condition and call $\tilde{\Gamma}$ the resampling. Note that $\tilde{\Gamma}$ has law $\mathbb{P}_\lambda^{N^2}$. In the proof of Proposition 2.10, since we were resampling along the mean facet, it was clear that replacing the path below the facet by any path having a positive area (above the facet) kept the total area condition. However, in our case, the approximate "mean facet resampling" comes with a potential area loss. This possibility is ruled out for sufficiently small values of δ. For $\eta > 0$, introduce the (small deviation) event

$$\mathsf{SD}(\eta) := \big\{ \max(\mathsf{LR}(\mathbf{a}), \mathsf{LR}(\mathbf{b})) < \eta (\delta s(k+1))^{1/2} N^{1/3} \big\}. \tag{3.10}$$

Claim 3.6. *There exist $\delta_0 > 0$ sufficiently small and $c_0 = c_0(\delta_0, \eta) > 0$ such that the following holds: if $((\mathbf{x}, \mathbf{y}), \Gamma) \in \mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)$, then, if we replace $\Gamma_{\mathbf{x},\mathbf{y}}$ by a path $\gamma \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta)$, the excess area of the resulting path is at least $c_0(sk)^{3/2} N$. In particular, the new path lies in Λ^{N^2}.*

Proof of Claim 3.6. Notice that by definition $(\mathbf{x}, \mathbf{y}) = (\mathbf{a}, \mathbf{b})$ are located below the mean facet. Moreover, these points are at distance at most $\delta(k+1)tN^{2/3}$ from the endpoints of the mean facet. Changing Γ between $\mathbf{a}$ and $\mathbf{b}$ by an element of $\mathsf{GAC}(\mathbf{a}, \mathbf{b}, \eta)$ gives a path $\tilde{\Gamma}$ which gained an area at least $\eta \|\mathbf{a} - \mathbf{b}\|^{3/2} - \mathcal{A}(\mathbf{a}, \mathbf{b}, \Gamma)$ where $\mathcal{A}(\mathbf{a}, \mathbf{b}, \Gamma)$ is the area of the quadrilateral delimited by the extremities of the mean facet of Γ and the pair $(\mathbf{a}, \mathbf{b})$, see Figure 4. Given the hypothesis on Γ, we find that

$$\mathcal{A}(\mathbf{a}, \mathbf{b}, \Gamma) \le \eta \sqrt{\delta} (s(k+1))^{3/2} N. \tag{3.11}$$

Hence, for $\delta = \delta_0 > 0$ sufficiently small, there exists $c_0 = c_0(\delta_0, \eta) > 0$ such that (recall that $k \ge 3$)

$$\mathsf{ExcessArea}(\tilde{\Gamma}) \ge \eta s^{3/2}(k-2)^{3/2} N - \eta \sqrt{\delta_0}(s(k+1))^{3/2} N \ge c_0(sk)^{3/2} N. \tag{3.12}$$

$\square$

We now fix $\delta = \delta_0$ and let $\eta > 0$ to be fixed later. Using the same reasoning as in (2.40) we get that for $\|\mathbf{a} - \mathbf{b}\| \ge N_0$,

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\tilde{\Gamma} \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta) \mid ((\mathbf{x}, \mathbf{y}), \Gamma) \in \mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)]$$
$$\ge \mathbb{P}_{\mathbf{a},\mathbf{b}}[\mathsf{GAC}(\mathbf{a}, \mathbf{b}, \eta)] \ge c_1,$$

where $c_1, N_0 > 0$ are given by Lemma 3.4. Since $\|\mathbf{a} - \mathbf{b}\| \ge s(k-2)N \ge sN$, the above bound always hold as soon as $s \ge N_0/N$. Assume that $s \ge N_0$. By the claim above, one has

$$\big\{ \tilde{\Gamma} \in \mathsf{GAC}(\mathbf{x}, \mathbf{y}, \eta), \ \Gamma \in \mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta) \big\} \subset \big\{ \mathsf{ExcessArea}(\tilde{\Gamma}) \ge c_0(tk)^{3/2} N \big\}. \tag{3.13}$$

Putting all the pieces together, and using the fact that $\tilde{\Gamma}$ has law $\mathbb{P}_\lambda^{N^2}$,

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)] \le c_1^{-1} \mathbb{P}_\lambda^{N^2}[\mathsf{ExcessArea}(\Gamma) \ge c_0(tk)^{3/2} N]. \tag{3.14}$$

We now argue that $\mathsf{SD}(\eta)$ occurs with uniform non-zero probability under $\mathbb{P}_\lambda^{N^2}$.

Claim 3.7. *There exist $\eta > 0$ and $c_2 = c_2(\eta) > 0$ such that*

$$\mathbb{P}_\lambda^{N^2}[\mathsf{SD}(\eta) \mid \mathsf{BigMeanFL}_k(s)] \ge c_2. \tag{3.15}$$

Proof of Claim 3.7. The claim is a consequence of the coupling obtained in Proposition 2.6. Indeed, $\mathbf{a}$ and $\mathbf{b}$ belong to the excursion of Γ under the mean facet. Thus, conditioning on the extremities of the mean facet, which we call $\mathbf{a}_{\mathsf{fac}}$ and $\mathbf{b}_{\mathsf{fac}}$, the local roughness of $\mathbf{a}$ (resp. $\mathbf{b}$) is dominated by the height of a negative random walk excursion between $\mathbf{a}_{\mathsf{fac}}$ and $\mathbf{b}_{\mathsf{fac}}$ after $\mathbf{d}_{\mathbf{a}}$ steps, where $\mathbf{d}_{\mathbf{a}}$ is the number of steps from $\mathbf{a}_{\mathsf{fac}}$ to $\mathbf{a}$ (resp $\mathbf{d}_{\mathbf{b}}$ steps, where $\mathbf{d}_{\mathbf{b}}$ is the number of steps from $\mathbf{b}$ to $\mathbf{b}_{\mathsf{fac}}$) which is of order $\sqrt{\mathbf{d}_{\mathbf{a}}}$ (resp. $\sqrt{\mathbf{d}_{\mathbf{b}}}$). Since by choice of $\mathbf{a}$ and $\mathbf{b}$ we can deterministically bound $\mathbf{d}_{\mathbf{a}}$ and $\mathbf{d}_{\mathbf{b}}$ by $O(\delta sk N^{2/3})$, we get the claim choosing η large enough. $\square$

We are now able to conclude. Let $\eta > 0$ be given by the claim above. Notice that

$$\mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{Goodhit}_k \mid \mathsf{BigMeanFL}_k(s) \cap \mathsf{SD}(\eta)] \geq \left(\tfrac{\delta}{2}\right)^2. \tag{3.16}$$

Write

$$\mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s)] \leq \left(\tfrac{2}{\delta}\right)^2 \cdot c_2^{-1} \cdot \mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s) \cap \mathcal{E}_k \cap \mathsf{SD}(\eta)]$$

$$+ \left(\tfrac{2}{\delta}\right)^2 \cdot \mathbb{P} \otimes \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s) \cap \mathsf{GoodHit}_k \cap (\mathsf{Bad}_\varepsilon^{+,-}) \cap \{\Gamma \subset B_{K_1N} \setminus B_{K_2N}\}]$$

$$+ \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s) \cap \{\Gamma \subset B_{K_1N} \setminus B_{K_2N}\}^c]. \tag{3.17}$$

To bound the first term we use (3.14) together with Proposition 2.4. To bound the second term we use Proposition 2.5 and proceed as in the proof of Proposition 2.10. Finally, the last term is bounded using Lemma 2.2. Putting all the pieces together, we get the existence of $c, C > 0$ such that for $3 \leq k \leq k_N$ (recall that $s = t/3$),

$$\mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(s)] \leq Ce^{-c(tk)^{3/2}}. \tag{3.18}$$

As a consequence, setting $\tilde{c} = 3N_0$, there exists some constants $c', C' > 0$ such that, for all $\tilde{c} \leq t \leq N^\epsilon$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) \geq tN^{\frac{2}{3}}] \leq \sum_{k=3}^{k_N} \mathbb{P}_\lambda^{N^2}[\mathsf{BigMeanFL}_k(t/3)] + \mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) \geq (t/3)k_N N^{\frac{2}{3}}]$$

$$\leq C'e^{-c't^{3/2}}.$$

$\square$

3.1.2 Proof of Proposition 3.2

From Proposition 3.1, it is easy to deduce the bound of Proposition 3.2 using the coupling introduced in Subsection 2.3.

Proof of Proposition 3.2. Let $t > 0$ and $\delta > 0$ be a (small) constant to be fixed later . We split the event $\{\mathsf{MeanLR} > tN^{1/3}\}$ according to the value of MeanFL,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}] = \mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}, \mathsf{MeanFL}(\Gamma) \leq t^{2-\delta}N^{\frac{2}{3}}]$$

$$+ \mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}, \mathsf{MeanFL}(\Gamma) \geq t^{2-\delta}N^{\frac{2}{3}}].$$

By Proposition 3.1, we know that, for $\tilde{c} \leq t^{2-\delta} \leq N^{2/3}$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}, \mathsf{MeanFL}(\Gamma) \geq t^{2-\delta}N^{\frac{2}{3}}] \leq C \exp(-ct^{\frac{3}{2}(2-\delta)}). \tag{3.19}$$

Now for $j \geq 0$, let A_j be the following event:

$$A_j := \{\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}, \mathsf{MeanFL}(\Gamma) = j\}. \tag{3.20}$$

We can now make use of Lemma 2.7 to argue that for every $j \leq t^{2-\delta}N^{2/3}$,

$$\mathbb{P}_\lambda^{N^2}[A_j] = \sum_{\substack{\|a-b\|=j \\ \arg(a) > \frac{\pi}{4} \geq \arg(b)}} \mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) \geq tN^{\frac{1}{3}} \mid \mathsf{Fac}_{a,b}]\mathbb{P}_\lambda^{N^2}[\mathsf{Fac}_{a,b}]$$

$$\leq C \exp\left(-c\frac{t^2 N^{2/3}}{j}\right)\mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) = j]$$

$$\leq C \exp(-ct^\delta)\mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) = j].$$

The first equality comes from the fact that there exists exactly one pair a, b such that $\arg(a) > \frac{\pi}{4} \geq \arg(b)$ and $\mathsf{Fac}(a, b)$ occurs. Moreover, if $\mathsf{Fac}(a, b)$ occurs with such a choice of a and b, then $[a, b]$ will automatically be the mean facet, so Lemma 2.7 holds for the second line. Thus, we obtained,

$$
\begin{aligned}
\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) > tN^{\frac{1}{3}}] &\leq C\exp(-ct^{\frac{3}{2}(2-\delta)}) + \sum_{j=0}^{t^{2-\delta}N^{2/3}} C\exp(-ct^\delta)\mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) = j] \\
&\leq C\exp(-ct^{\frac{3}{2}(2-\delta)}) + C\exp(-ct^\delta).
\end{aligned}
$$

Equating the exponents yields an optimal value of $\delta = \frac{6}{5}$ and the result. $\qquad\square$

3.2 Lower bounds

Recall the setting of Theorem 1.3. Our target estimates are the following.

Proposition 3.8. *There exists a function $F_1 : \mathbb{R}^+ \to [0, 1]$ satisfying $\lim_{t \to 0^+} F_1(t) = 0$, such that for any $t > 0$ small enough,*

$$
\limsup_{N \to \infty} \mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) < tN^{\frac{1}{3}}] \leq F_1(t). \tag{3.21}
$$

Proposition 3.9. *There exists a function $F_2 : \mathbb{R}^+ \to [0, 1]$ satisfying $\lim_{t \to 0^+} F_2(t) = 0$, such that for any $t > 0$ small enough,*

$$
\limsup_{N \to \infty} \mathbb{P}_\lambda^{N^2}[\mathsf{MeanFL}(\Gamma) < tN^{\frac{2}{3}}] \leq F_2(t). \tag{3.22}
$$

Remark 3.10. As before, we obtain

$$
\liminf_{N \to \infty} \mathbb{E}_\lambda^{N^2}\left[\frac{\mathsf{MeanFL}(\Gamma)}{N^{2/3}}\right] > 0, \qquad \liminf_{N \to \infty} \mathbb{E}_\lambda^{N^2}\left[\frac{\mathsf{MeanLR}(\Gamma)}{N^{1/3}}\right] > 0. \tag{3.23}
$$

Let us briefly describe the strategy of the proof. We carefully analyse the marginal of $\mathbb{P}_\lambda^{N^2}$ in the cone $\mathbf{A}_N$ of apex 0 and angular opening $N^{-1/3}$. Due to the Brownian Gibbs property, conditionally on the outside of the cone, this marginal is the law of a random walk bridge conditioned on capturing a random amount of area. We first prove in Lemma 3.12 that this random area is of order N, with Gaussian tails. Hence, the marginal of $\mathbb{P}_\lambda^{N^2}$ in $\mathbf{A}_N$ is absolutely continuous with respect to the law of a random walk bridge without the area conditioning, in the large N limit. The strategy of proof is thus clear: we prove the corresponding statements for the unconditioned random walk bridge (see Lemmas 3.16 and 3.17), and transfer them to $\mathbb{P}_\lambda^{N^2}$ thanks to the latter observation.

Let us introduce a few quantities that we will be useful in the remainder of this section.

Definition 3.11. For $N \geq 1$, we define $\theta_N := N^{-1/3}$. We set $\mathbf{A}_N$ to be the cone of apex 0 and angular opening of θ_N centered around the line $\{y = x\}$. If $\gamma \in \Lambda^{N^2}$, let us define $\mathbf{x}_N(\gamma)$ (resp. $\mathbf{y}_N(\gamma)$) to be point of γ that lies in $\mathbf{A}_N$ and that is the closest to the line constituting the left (resp. right) boundary of $\mathbf{A}_N$. Finally, we define

$$
\mathcal{A}_N(\gamma) := \left|\mathsf{Enclose}(\gamma_{\mathbf{x}_N(\gamma),\mathbf{y}_N(\gamma)} \cap \mathbf{A}_{\mathbf{x}_N(\gamma),\mathbf{y}_N(\gamma)})\right| - \left|\mathbf{T}_{0,\mathbf{x}_N(\gamma),\mathbf{y}_N(\gamma)}\right|. \tag{3.24}
$$

We shall also make a slight abuse of notation in considering $\mathcal{A}_N(\widetilde{\gamma})$ for some oriented path $\widetilde{\gamma}$ linking $\mathbf{x}_N(\gamma)$ to $\mathbf{y}_N(\gamma)$ (for some $\gamma \in \Lambda^{N^2}$). In that case, the meaning of $\mathcal{A}_N(\widetilde{\gamma})$ is clear: it is the corresponding $\mathcal{A}_N(\gamma')$ for any $\gamma' \in \Lambda^{N^2}$ which extends $\widetilde{\gamma}$ outside the cone $\mathbf{A}_{\mathbf{x}_N(\gamma),\mathbf{y}_N(\gamma)}$.

The following lemma will play a central role in the proofs of Propositions 3.8 and 3.9.

Figure 5: One possible output of the map $T_{x,y}$. The shift of (x,y) is denoted $(\tilde{x}, \tilde{y})$. The initial path is the black path (after a shift to the left). The two modifications involved in the procedure described above are represented in red. The red path in the light green shaded region has been sampled uniformly amongst the paths that capture a non-negative area (computed with respect to the black dashed line). The area captured by the turquoise region is sufficient to compensate the possible area loss due to the resampling of the path between x and y.

Lemma 3.12. *There exist two constants $c, C > 0$ such that for any $\beta > 0$, and for N large enough,*

$$\mathbb{P}_\lambda^{N^2}\left[\mathcal{A}_N(\Gamma) \geq \beta N\right] \leq C e^{-c\beta^2}. \tag{3.25}$$

Proof. The proof of this statement heavily relies on the probabilistic multi-valued map principle stated in Lemma B.1. For $\alpha > 0$, we introduce

$$\mathsf{BadArea}_\alpha^N := \{\gamma \in \Lambda^{N^2}, \, \mathcal{A}_N(\gamma) \in [\alpha N, (\alpha+1)N]\}. \tag{3.26}$$

Define for $x, y \in \mathbb{N}^2$,

$$A_{x,y} := \mathsf{BadArea}_\alpha^N \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\} \cap \{\mathbf{x}_N(\Gamma) = x, \, \mathbf{y}_N(\Gamma) = y\}$$
$$\cap \{\Gamma \notin \mathsf{Bad}_{\varepsilon, \mathbf{x}_N(\Gamma), \mathbf{y}_N(\Gamma)}^{+,-}\},$$

where $\mathsf{Bad}_{\varepsilon, \mathbf{x}_N(\Gamma), \mathbf{y}_N(\Gamma)}^{+,-} := \mathsf{Bad}_{\varepsilon, \mathbf{A}_{\mathbf{x}_N(\Gamma), \mathbf{y}_N(\Gamma)}}^{+} \cup \mathsf{Bad}_{\varepsilon, \mathbf{A}_{\mathbf{x}_N(\Gamma), \mathbf{y}_N(\Gamma)}}^{-}$, and ε is given by Proposition 2.5.

Fix $x, y \in \mathbb{N}^2$ such that $\mathbb{P}_\lambda^{N^2}[A_{x,y}] \neq 0$.

Let us define a multi-valued map $T_{x,y} : A_{x,y} \to \mathcal{P}(\Lambda^{N^2})$ via the following two-step procedure (see Figure 5 for an illustration of the procedure). From some path $\gamma \in \mathsf{BadArea}_\alpha^N$, we can create a new oriented path of the first quadrant by erasing the portion of γ lying between x and y (previously called $\gamma_{x,y}$) and replacing it by any oriented path $\tilde{\gamma}_{x,y}$ linking x to y thus obtaining an intermediate path $\tilde{\gamma}$ which we require to satisfy $\mathcal{A}_N(\tilde{\gamma}) \geq 0$, and finally by adding $\lceil \frac{\alpha+1}{K_2} \rceil$ horizontal steps at the beginning of $\tilde{\gamma}$ (K_2 is the constant defined in 2.2). We define $T_{x,y}(\gamma)$ to be the subset of Λ of the paths that can be obtained from γ by this procedure. Notice that T is well defined since if $\gamma \in A_{x,y}$, the intermediate path $\tilde{\gamma}$ defined above satisfies $\mathcal{A}(\tilde{\gamma}) \geq N^2 - (\alpha+1)N$, and it is easy to check that the first $\lceil \frac{\alpha+1}{K_2} \rceil$ horizontal steps that have

been added to $\widetilde{\gamma}$ capture an area at least equal to $(\alpha + 1)N$ (this comes from the fact that $\gamma \in \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\}$). Hence, for any $\gamma \in A_{x,y}$, $T_{x,y}(\gamma) \subset \Lambda^{N^2}$.

We are going to apply Lemma B.1 to $T_{x,y}$. Before doing so, let us notice that we may co-restrict the map defined above to the set

$$B_{x,y} := \{\mathbf{x}_N(\Gamma^{\text{cut}}) = x, \mathbf{y}_N(\Gamma^{\text{cut}}) = y\} \cap \{\text{The first } \lceil \tfrac{\alpha+1}{K_2} \rceil \text{ steps are horizontal}\} \subset \Lambda^{N^2}, \quad (3.27)$$

where Γ^{cut} is the path Γ minus the first $\lceil \tfrac{\alpha+1}{K_2} \rceil$ steps, translated of $\lceil \tfrac{\alpha+1}{K_2} \rceil$ towards the left direction, see Figure 5. Now, applying Lemma B.1 yields

$$\mathbb{P}_\lambda^{N^2}[A_{x,y}] \leq \varphi(T_{x,y})\psi(T_{x,y})\mathbb{P}_\lambda^{N^2}[B_{x,y}], \quad (3.28)$$

where the quantities $\varphi(T_{x,y})$ and $\psi(T_{x,y})$ are defined in Lemma B.1. It is very easy to check that

$$\varphi(T_{x,y}) = \left(\frac{1}{\lambda}\right)^{\lceil \frac{\alpha+1}{K_2} \rceil}, \qquad \psi(T_{x,y}) = \frac{|\{\gamma \in \Lambda^{x \to y}, \mathcal{A}_N(\gamma) \in [\alpha N, (\alpha+1)N]\}|}{|\{\gamma \in \Lambda^{x \to y}, \mathcal{A}_N(\gamma) \geq 0\}|}. \quad (3.29)$$

Recall that $\mathbb{P}_{x,y}$ is the uniform measure on $\Lambda^{x \to y}$. We can write

$$\psi(T_{x,y}) = \mathbb{P}_{x,y}[\mathcal{A}_N(\gamma) \in [\alpha N, (\alpha+1)N] \mid \mathcal{A}_N(\gamma) \geq 0]. \quad (3.30)$$

Let us call $\theta(x,y)$ the positive angle formed by the segment $[x,y]$ and the horizontal line passing through x. Since we assumed that $\mathbb{P}_\lambda^{N^2}[A_{x,y}] \neq 0$, we have $\theta(x,y) \in [\varepsilon, \pi/2 - \varepsilon]$. Let $h_N := \|y - x\|$. As it turns out, h_N is of order $N^{2/3}$.

Claim 3.13. *There exist two constants $c = c(\varepsilon), C = C(\varepsilon) > 0$ such that*

$$h_N \in [cN^{2/3}, CN^{2/3}]. \quad (3.31)$$

Proof of Claim 3.13. The constraint $\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}$ enforces that for N large enough $h_N \geq \frac{1}{2}K_2 N\theta_N = \frac{1}{2}K_2 N^{2/3}$. Then, since we restricted ourselves to the case $\varepsilon \leq \theta(x,y) \leq \pi/2 - \varepsilon$, there exists $C_1 > 0$ such that

$$h_N \leq \frac{C_1}{\sin \varepsilon}N\theta_N = \frac{C_1}{\sin \varepsilon}N^{2/3}. \quad (3.32)$$

$\square$

By translation invariance, $\mathbb{P}_{x,y}$ only depends on $\theta = \theta(x,y)$ and h_N. Since these parameters are more relevant we write $\mathbb{P}_{h_N}^\theta := \mathbb{P}_{x,y}$. Moreover, with this observation and to the cost of rotating the picture by $\pi/4$, we can always assume that samples of $\mathbb{P}_{h_N}^\theta$ start at 0 and end at $e^{i\pi/4}(y - x)$, see Figure 6. We will denote by $X^{(\theta)}$ a sample of $\mathbb{P}_{h_N}^\theta$.

Now,

$$\begin{aligned}
\mathbb{P}_{h_N}^\theta[\mathcal{A}_N(X^{(\theta)}) \in [\alpha N, (\alpha+1)N]] &= \mathbb{P}_{h_N}^\theta[h_N^{-3/2}\mathcal{A}_N(X^{(\theta)}) \in [\alpha N h_N^{-3/2}, (\alpha+1)N h_N^{-3/2}]] \\
&\leq \mathbb{P}_{h_N}^\theta[h_N^{-3/2}\mathcal{A}_N(X^{(\theta)}) \in [C^{-3/2}\alpha, c^{-3/2}(\alpha+1)]] \\
&\leq \sup_{\vartheta \in [\varepsilon, \frac{\pi}{2} - \varepsilon]} \mathbb{P}_{h_N}^\vartheta[h_N^{-3/2}\mathcal{A}_N(X^{(\vartheta)}) \in [C^{-3/2}\alpha, c^{-3/2}(\alpha+1)]],
\end{aligned}$$

where the second inequality follows by Claim 3.13.

Classical invariance theorems for random walks (started at 0) conditioned to reach αk after k steps (see [Lig68] or [CC13, Remark 2.6]) yield the following convergence in distribution[7]:

---

[7]The convergence holds in the space $\mathcal{C}([0,1], \mathbb{R})$ of continuous functions on $[0,1]$ equipped with the topology of uniform convergence.
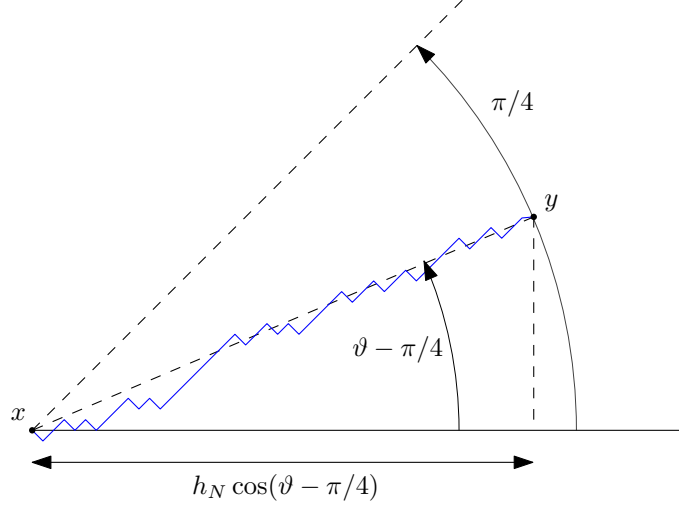
Figure 6: An illustration of a sample $X^{(\vartheta)}$ of $\mathbb{P}^{\vartheta}_{h_N}$ (in blue). This path corresponds to a symmetric random walk that does $\pm 2^{-1/2}$ jumps at times $k2^{-1/2}$ for $1 \leq k \leq \sqrt{2}h_N \cos(\vartheta - \pi/4)$.

calling $(X^{(\pi/4)}_{th_N})_{0\leq t\leq 1}$ the trajectory[8] of a sample of $\mathbb{P}^{\pi/4}_{h_N}$, one has that under $\mathbb{P}^{\pi/4}_{h_N}$

$$\lim_{N\to\infty} \left(\frac{X^{(\pi/4)}_{th_N}}{\sigma\sqrt{2^{1/2}h_N}} \right)_{t\in[0,1]} = (\mathsf{BB}_t)_{t\in[0,1]}, \tag{3.33}$$

where $(\mathsf{BB}_t)_{t\in[0,1]}$ is the standard Brownian bridge on $[0,1]$, and $\sigma = 2^{-1/2}$. This observation, together with the strategy described in [CC13, Remark 2.6], yields a similar invariance principle for the trajectory $(X^{(\vartheta)}_{th_N\cos(\vartheta-\pi/4)})_{0\leq t\leq 1}$ of a sample of $\mathbb{P}^{\vartheta}_{h_N}$. For $\vartheta \in (0, \pi/2)$ and $t \in [0,1]$, denote

$$f^{(\vartheta)}_N(t) := \frac{1}{\sigma(\vartheta)\sqrt{2^{1/2}h_N\cos(\vartheta-\pi/4)}}(X^{(\vartheta)}_{th_N\cos(\vartheta-\pi/4)} - th_N\sin(\vartheta-\pi/4)), \tag{3.34}$$

where $\sigma(\vartheta) := 2^{-1/2}\sqrt{1 - \tan^2(\vartheta - \pi/4)}$. Then, for $\vartheta \in (0, \pi/2)$, under $\mathbb{P}^{\vartheta}_{h_N}$,

$$\lim_{N\to\infty} \left(f^{(\vartheta)}_N(t) \right)_{t\in[0,1]} = (\mathsf{BB}_t)_{t\in[0,1]}, \tag{3.35}$$

where the convergence holds in distribution. In particular, one has that

$$h_N^{-3/2}\mathcal{A}_N(X^{(\vartheta)}) = h_N^{-3/2}\int_0^{h_N\cos(\vartheta-\pi/4)}(X^{(\vartheta)}_t - \tan(\vartheta-\pi/4)t)\mathrm{d}t$$

$$= 2^{1/4}\sigma(\vartheta)[\cos(\vartheta-\pi/4)]^{3/2}\int_0^1 f^{(\vartheta)}_N(t)\mathrm{d}t.$$

Hence, (3.35) implies the following convergence in distribution under $\mathbb{P}^{\vartheta}_{h_N}$:

$$\lim_{N\to\infty} h_N^{-3/2}\mathcal{A}_N(X^{(\vartheta)}) = c_0(\vartheta)\int_0^1 \mathsf{BB}_t\mathrm{d}t, \tag{3.36}$$

where $c_0(\vartheta) := 2^{1/4}\sigma(\vartheta)[\cos(\vartheta-\pi/4)]^{3/2}$.

---

[8]The sample is a priori only defined for the discrete times $k/\sqrt{2}$ for $0 \leq k \leq \sqrt{2}h_N$ but we extend it to $[0, h_N]$ by linear interpolation.

Remark 3.14. It follows from [CC13, Remark 2.6] that the latter convergence is uniform (for instance at the level of the convergence of cumulative distribution function) in $\vartheta \in [\varepsilon, \pi/2 - \varepsilon]$. Indeed, when the underlying random walk has increments with exponential moments, the authors identify the law of the random walk bridge conditioned to stay above a line of slope ϑ with a unconditioned random walk whose increments are given by a suitable exponential tilt of the increments of the former walk. Hence the Radon-Nikodym derivative of the ϑ-tilted walk with respect to the $\pi/4$-tilted walk is explicit and is a continuous function of ϑ. In particular, it is uniformly bounded on a compact interval such as $[\varepsilon, \pi/2 - \varepsilon]$. We shall use this observation several times in what follows.

Call P the law of the Brownian bridge $(\mathsf{BB}_t)_{t\in[0,1]}$. The above observations yield that for N large enough,

$$\mathbb{P}^\theta_{h_N}[\mathcal{A}_N(X^\theta) \in [\alpha N, (\alpha + 1)N]] \leq 2\mathsf{P}\Big[\int_0^1 \mathsf{BB}_t \mathrm{d}t \geq c_0(\varepsilon)^{-1}C^{-3/2}\alpha\Big]. \tag{3.37}$$

A similar reasoning yields the lower bound, for N large enough,

$$\mathbb{P}^\theta_{h_N}[\mathcal{A}_N(X^\theta) \geq 0] \geq \frac{1}{2}\mathsf{P}\left[\int_0^1 \mathsf{BB}_t \mathrm{d}t \geq 0\right] =: \eta. \tag{3.38}$$

Gathering (3.37) and (3.38), we get

$$\psi(T_{x,y}) \leq 2\eta^{-1}\mathsf{P}\Big[\int_0^1 \mathsf{BB}_t \mathrm{d}t \geq c_0(\varepsilon)^{-1}C^{-3/2}\alpha\Big]. \tag{3.39}$$

Using the following estimate on the tail area of a Brownian bridge (see [LJ07, Theorem 1.2]), we obtain $c_1 = c_1(\varepsilon), C_1 = C_1(\varepsilon) > 0$ such that

$$\mathsf{P}\Big[\int_0^1 \mathsf{BB}_t \mathrm{d}t \geq c_0(\varepsilon)^{-1}C^{-3/2}\alpha\Big] \leq C_1\mathrm{e}^{-c_1\alpha^2}. \tag{3.40}$$

Using (3.28), (3.38), and (3.40), we get that for N large enough,

$$\begin{aligned}\mathbb{P}^{N^2}_\lambda[A_{x,y}] &\leq 2C_1\eta^{-1}\mathrm{e}^{-c_1\alpha^2}\lambda^{-\lceil\frac{\alpha+1}{K_2}\rceil}\mathbb{P}^{N^2}_\lambda[B_{x,y}] \\ &\leq C_2\mathrm{e}^{-c_2\alpha^2}\mathbb{P}^{N^2}_\lambda[B_{x,y}].\end{aligned}$$

To obtain the lemma, it remains to sum over $\alpha \geq \beta$ and over all possible values for x and y,

$$\mathbb{P}^{N^2}_\lambda[\mathcal{A}_N(\Gamma) \geq \beta N, \Gamma \subset B_{K_1N} \setminus B_{K_2N}, \Gamma \notin \mathsf{Bad}^{+,-}_{\varepsilon,\mathbf{x}_N(\Gamma),\mathbf{y}_N(\Gamma)}] \leq C_2 \sum_{\alpha=\lfloor\beta\rfloor}^{+\infty} \exp(-c_2\alpha^2). \tag{3.41}$$

Then, for some $C_3 > 0$,

$$\mathbb{P}^{N^2}_\lambda[\mathcal{A}_N(\Gamma) \geq \beta N, \Gamma \subset B_{K_1N} \setminus B_{K_2N}, \Gamma \notin \mathsf{Bad}^{+,-}_{\varepsilon,\mathbf{x}_N(\Gamma),\mathbf{y}_N(\Gamma)}] \leq C_3\mathrm{e}^{-c_2\beta^2}. \tag{3.42}$$

Using Lemma 2.2 and Proposition 2.5, there exists $C_4 > 0$ such that for N large enough (namely $N \geq \beta^3$),

$$\mathbb{P}^{N^2}_\lambda[\mathcal{A}_N(\Gamma) \geq \beta N] \leq C_4\mathrm{e}^{-c_2\beta^2}. \tag{3.43}$$

$\square$

Definition 3.15. If $\gamma_{u,v} \in \Lambda^{u\to v}$ and $z \in \gamma_{u,v}$, we define $\mathsf{LR}(z)(\gamma_{u,v})$, *the local roughness of the vertex* $z \in \gamma_{u,v}$, to be the local roughness of z in the path $\gamma_{u,v}$ extended to the left with only horizontal steps to up to the y-axis and to the right with only vertical steps up to the x-axis. We define the mean facet length of $\gamma_{u,v}$ similarly.

The two next lemmas study the typical behaviours of the mean facet length and of the local roughness of a uniform path between $\mathbf{x}_N(\Gamma)$ and $\mathbf{y}_N(\Gamma)$ when Γ is sampled according to $\mathbb{P}_\lambda^{N^2}$. For simplicity, in the rest of the section, when Γ is a sample of $\mathbb{P}_\lambda^{N^2}$, we shall abbreviate $\mathbf{x}_N(\Gamma)$ (resp. $\mathbf{y}_N(\Gamma)$) by $\mathbf{x}_N$ (resp. $\mathbf{y}_N$). The proofs of these lemmas rely on Gaussian computations and are not specific to the considered model. They are deferred to the end of the section.

We will write $\mathcal{G}_N := \{\Gamma \in B_{K_1 N} \setminus B_{K_2 N}\} \cap (\mathsf{Bad}_{\varepsilon, \mathbf{x}_N, \mathbf{y}_N}^{+,-})^c$.

Lemma 3.16. *There exists a function Ψ independent of every parameter of the problem with* $\Psi(t) \xrightarrow[t \to 0^+]{} 0$ *such that, for $t > 0$, if N is large enough,*

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N} \mathbb{P}_{\mathbf{x}_N, \mathbf{y}_N}[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) < t N^{1/3}]\big] \leq \Psi(t). \tag{3.44}$$

The corresponding statement for the mean facet length is the following

Lemma 3.17. *There exists a function Φ independent of every parameter of the problem with* $\Phi(t) \xrightarrow[t \to 0^+]{} 0$ *such that, for $t > 0$, if N is large enough,*

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N} \mathbb{P}_{\mathbf{x}_N, \mathbf{y}_N}[\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) < t N^{2/3}]\big] \leq \Phi(t). \tag{3.45}$$

These two lemmas in hand, we can now prove the lower bound of Proposition 3.8. Proposition 3.9 will be proved by the same method.

Proof of Proposition 3.8. Fix some $t > 0$, and some $\beta > 0$ that is going to be chosen later as a function of t. We implement the following resampling procedure: let us start with a sample Γ of $\mathbb{P}_\lambda^{N^2}$. As usual we condition on the configuration outside of the cone $\mathbf{A}_{\mathbf{x}_N, \mathbf{y}_N}$ that we name $\Gamma_{\mathsf{ext}} := \Gamma \setminus \Gamma_{\mathbf{x}_N, \mathbf{y}_N}$. By the Brownian Gibbs property, the distribution of $\Gamma_{\mathbf{x}_N, \mathbf{y}_N}$ is uniform among the oriented paths linking $\mathbf{x}_N$ and $\mathbf{y}_N$ that enclose at least a certain amount of area which is measurable with respect to the exterior configuration. Let us call A_{ext} the area already enclosed by Γ_{ext} (namely, $A_{\mathsf{ext}} = \mathrm{Enclose}(\Gamma \cap \mathbf{A}_{\mathbf{x}_N, \mathbf{y}_N}^c \cap \mathbb{N}^2)$), so that the conditional distribution of $\Gamma_{\mathbf{x}_N, \mathbf{y}_N}$ is nothing but $\mathbb{P}_{\mathbf{x}_N, \mathbf{y}_N}[\, \cdot \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}]$.

Before starting, let us point out a crucial fact. With our definition of $\mathsf{LR}(z)(\Gamma_{u,v})$, it is a deterministic fact that for any $u, v \in \Gamma, z \in \Gamma_{u,v}$,

$$\mathsf{LR}(z)(\Gamma_{u,v}) \leq \mathsf{LR}(z)(\Gamma) \quad \text{and thus} \quad \mathsf{MeanLR}(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) \leq \mathsf{MeanLR}(\Gamma). \tag{3.46}$$

Thus, we have

$$\mathbb{P}_\lambda^{N^2}\big[\mathsf{MeanLR}(\Gamma) < t N^{1/3}\big]$$
$$\leq \mathbb{E}_\lambda^{N^2}\big[\mathbb{P}_{\mathbf{x}_N, \mathbf{y}_N}[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) < t N^{1/3} \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}]\big].$$

Introduce
$$\mathcal{F}_N := \{N^2 - A_{\mathsf{ext}} \leq \beta N\} \cap \{\Gamma \subset B_{K_1 N} \setminus B_{K_2 N}\} \cap (\mathsf{Bad}_{\varepsilon, \mathbf{x}_N, \mathbf{y}_N}^{+,-})^c, \tag{3.47}$$

where $\mathsf{Bad}_{\varepsilon, \mathbf{x}_N, \mathbf{y}_N}^{+,-}$ was introduced in the proof of Lemma 3.12. Note that by Lemma 2.2, Proposition 2.5 and Lemma 3.12, one has that for $N \geq \beta^3$,

$$\mathbb{P}_\lambda^{N^2}[\mathcal{F}_N^c] \leq C e^{-c \beta^2}. \tag{3.48}$$

Hence,

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{P}_{\mathbf{x}_N, \mathbf{y}_N}[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) < t N^{1/3} \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}]\big]$$
$$\leq \mathbb{E}_\lambda^{N^2}\big[\mathbb{P}_{\mathbf{x}_N, \mathbf{y}_N}[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) < t N^{1/3} \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}] \mathbb{1}_{\mathcal{F}_N}\big]$$
$$+ C e^{-c \beta^2}.$$

Now, note that

$$\mathbb{1}_{\mathcal{F}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\left[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3} \mid \mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq N^2 - A_{\mathsf{ext}}\right]$$
$$\leq \mathbb{1}_{\mathcal{F}_N}\frac{\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\left[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3}\right]}{\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\left[\mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq \beta N\right]}.$$

Using the methods developed above[9] we could also show that for some $c', C' > 0$, if N is large enough,

$$\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\left[\mathcal{A}_N(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \geq \beta N\right] \geq C'\mathsf{e}^{-c'\beta^2}. \tag{3.49}$$

By Lemma 3.16, we obtain that, integrating over Γ_{ext}, and putting everything together,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) < tN^{1/3}] \leq C\mathsf{e}^{-c\beta^2} + \frac{\Psi(t)}{C'\mathsf{e}^{-c'\beta^2}}. \tag{3.50}$$

It remains to carefully chose β in terms of t. We optimize the above equation by setting

$$\beta(t) = \sqrt{(c + c')^{-1}\log(\tfrac{c}{c'}\tfrac{CC'}{\Psi(t)})}. \tag{3.51}$$

Observe that $\beta(t) \xrightarrow[t\to 0]{} +\infty$. Choosing N large enough, we obtain that for some $C'' > 0$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MeanLR}(\Gamma) < tN^{1/3}] \leq C''(\Psi(t))^{\frac{c}{c+c'}} \tag{3.52}$$

which is the announced result. □

Proposition 3.9 is a consequence of the exact same argument.

Proof of Proposition 3.9. We proceed exactly as in the proof of Proposition 3.8. Indeed, observe that as previously,

$$\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) \leq \mathsf{MeanFL}(\Gamma). \tag{3.53}$$

Hence, the proof can be reproduced *mutatis mutandi* using this time Lemma 3.17. □

We now turn to the proofs of the two Gaussian lemmas, namely Lemma 3.16 and Lemma 3.17. The process of the concave majorant of a Brownian bridge has been extensively studied in [Gro83, Sui01, BP11]. Let us briefly summarise their results.

- Groeneboom showed in [Gro83] that conditionally on the convex majorant of a Brownian motion, the difference process was a succession of independent Brownian excursions between the extremal points of the concave majorant. The result was later extended to the standard Brownian bridge in [BP11] via a Doob tranformation.

- Groeneboom also gave an explicit representation of the distribution of the process of the slopes of the concave majorant. Later, Suidan [Sui01] was able to derive the joint law of the ordered lengths of the segments of $[0,1]$ in the partition of $[0,1]$ induced by the extremal points of the concave majorant of a Brownian bridge.

We refer to the introduction of [BP11] for a clear and detailed presentation of these results. In particular, we import the following statement from the three cited articles.

Theorem 3.18. *Let* $(\mathsf{BB}_t)_{0\leq t\leq 1}$ *be the standard Brownian bridge and let* $(\mathcal{C}(t))_{0\leq t\leq 1}$ *be its smallest concave majorant. Let* $s \in (0,1)$. *Let us define the two following random variables:*

- *$L(s)$ is the length of the almost surely unique facet of $\mathcal{C}$ intersecting the line* $\{x = s\}$,

---

[9]This is again an application of Donsker's Theorem and the use of the asymptotic results of [LJ07], we do not write the full proof but rather refer to the proof of Lemmas 3.16 and 3.17 for details.

- $R(s)$ *is the difference process defined by* $R(s) := \mathcal{C}(s) - B(s)$.

Fix $\epsilon > 0$. Then, one has

$$\sup_{s \in [\epsilon, 1-\epsilon]} \mathbb{P}\left[L(s) < r\right] \xrightarrow[r \to 0^+]{} 0, \tag{3.54}$$

and

$$\sup_{s \in [\epsilon, 1-\epsilon]} \mathbb{P}\left[R(s) < r\right] \xrightarrow[r \to 0^+]{} 0. \tag{3.55}$$

We briefly sketch the proof of Theorem 3.18 and refer to [Gro83,Sui01,BP11] for more details.

Sketch of proof of Theorem 3.18. Fix some $s \in [\epsilon, 1 - \epsilon]$. For $L(s)$, we remark that the results of [Gro83] imply that $\mathcal{C}$ almost surely has a finite number of slope changes (or extremal points) in $[\epsilon, 1 - \epsilon]$, and [BP11] gives an explicit description of their lengths in terms of a size-biased uniform stick-breaking process. In particular, it implies that the distribution of $L(s)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+$, which implies that $\mathbb{P}\left[L(s) < r\right] \xrightarrow[r \to 0]{} 0$. The statement follows by the observation that $s \in (0, 1) \mapsto \mathbb{P}\left[L(s) < r\right]$ is continuous.

For $R(s)$, let us denote by $a \leq s \leq b$ the horizontal coordinates of the endpoints of the facet intersecting the line $\{x = s\}$. The exact same argument as above implies that $\mathbb{P}\left[\min(s - a, b - s) < \varepsilon\right]$ goes to 0 when ε goes to 0. On the complementary of the latter event, by [Gro83], $R(s)$ is the height of a Brownian excursion at a positive distance of its endpoints, and is indeed continuous with respect to the Lebesgue measure on $\mathbb{R}^+$. We conclude as previously, observing that $s \in (0, 1) \mapsto \mathbb{P}\left[R(s) < r\right]$ is continuous. $\square$

Now observe that in Lemmas 3.16 and 3.17, $\|\mathbf{y}_N - \mathbf{x}_N\|$ is of order $N^{2/3}$ so that the scaling is Gaussian. Hence, Donsker's Theorem suggests that we may use the above-mentioned results to obtain lower tails for the random variables $N^{-1/3}\mathsf{MeanLR}$ and $N^{-2/3}\mathsf{MeanFL}$.

Proof of Lemma 3.16. As in the proof of Lemma 3.12, we write $h_N = \|\mathbf{y}_N - \mathbf{x}_N\|$. Moreover, let us call $\mathbf{t}_N^0$ the intersection between the line segment $[\mathbf{x}_N, \mathbf{y}_N]$ and the line $\{y = x\}$. Remember that we work under the event $\{\Gamma \in B_{K_1 N} \setminus B_{K_2 N}\} \cap (\mathsf{Bad}_{\varepsilon, \mathbf{x}_N, \mathbf{y}_N}^{+,-})^c$, which allows us to reuse Claim 3.13, namely

$$cN^{2/3} \leq h_N \leq CN^{2/3}. \tag{3.56}$$

Moreover, there exists a constant $c_\varepsilon > 0$ such that

$$\min(\|\mathbf{t}_N^0 - \mathbf{x}_N\|_2, \|\mathbf{y}_N - \mathbf{t}_N^0\|) \geq c_\varepsilon h_N. \tag{3.57}$$

Recall the parametrisation of the probability measures $\mathbb{P}_{h_N}^\theta = \mathbb{P}_{\mathbf{x}_N, \mathbf{y}_N}$ introduced in the proof of Lemma 3.12. Similarly as before,

$$
\begin{aligned}
\mathbb{E}_\lambda^{N^2} \big[\mathbb{1}_{\mathcal{G}_N} \mathbb{P}_{\mathbf{x}_N, \mathbf{y}_N} \big[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N, \mathbf{y}_N}) &< t N^{1/3} \big] \big] \\
&= \mathbb{E}_\lambda^{N^2} \big[\mathbb{1}_{\mathcal{G}_N} \mathbb{P}_{h_N}^\theta \big[h_N^{-1/2} \mathsf{LR}(\mathbf{t}_N^0) < t h_N^{-1/2} N^{1/3} \big] \big] \\
&\leq \mathbb{E}_\lambda^{N^2} \big[\mathbb{1}_{\mathcal{G}_N} \mathbb{P}_{h_N}^\theta \big[h_N^{-1/2} \mathsf{LR}(\mathbf{t}_N^0) < c^{-1/2} t \big] \big] \\
&\leq \mathbb{E}_\lambda^{N^2} \big[\mathbb{1}_{\mathcal{G}_N} \sup_{\vartheta \in [\varepsilon, \pi/2 - \varepsilon]} \mathbb{P}_{h_N}^\vartheta \big[h_N^{-1/2} \mathsf{LR}(\mathbf{t}_N^0) < c^{-1/2} t \big] \big],
\end{aligned}
$$

where c is the constant appearing in (3.56). Now, observe that $\mathbf{t}_0^N$ is independent of the random path sampled according to $\mathbb{P}_{h_N}^\vartheta$, and we can write

$$\mathbb{P}_{h_N}^\vartheta \big[h_N^{-1/2} \mathsf{LR}(\mathbf{t}_N^0) < c^{-1/2} t \big] \leq \sup_{s \in [c_\varepsilon, 1 - c_\varepsilon]} \mathbb{P}_{h_N}^\vartheta \big[h_N^{-1/2} \mathsf{LR}(h_N s) < c^{-1/2} t \big], \tag{3.58}$$

Thus, we obtained that

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3}\big]\big]$$

$$\leq \mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\sup_{\vartheta\in[\varepsilon,\pi/2-\varepsilon]}\sup_{s\in[c_\varepsilon,1-c_\varepsilon]}\mathbb{P}_{h_N}^\vartheta\big[h_N^{-1/2}\mathsf{LR}(sh_N) < c^{-1/2}t\big]\big], \quad (3.59)$$

where c_ε is the constant appearing in (3.57). Reasoning as above, one has the following convergence in distribution (the random variable R was introduced in Theorem 3.18), under $\mathbb{P}_{h_N}^\vartheta$ and assuming $\mathcal{G}_N$ occurs, for $s\in[c_\varepsilon, 1-c_\varepsilon]$,

$$\lim_{N\to\infty} h_N^{-1/2}\mathsf{LR}(sh_N) = \gamma(\vartheta)R(s), \quad (3.60)$$

where $\gamma(\vartheta) = \sigma(\vartheta)\sqrt{2^{1/2}\cos(\vartheta - \pi/4)}$ and $\sigma(\vartheta)$ was defined below (3.34). Once again, we claim that this convergence is uniform in $\vartheta\in[\varepsilon,\pi/2-\varepsilon]$ (see Remark 3.14). Hence, the limsup of the right-hand side of (3.59) is bounded by

$$\sup_{s\in[c_\varepsilon,1-c_\varepsilon]}\mathsf{P}[R(s) < tc^{-1/2}\gamma(\varepsilon)^{-1}]. \quad (3.61)$$

We argue as in the proof of Lemma 3.12 to get for N large enough

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanLR}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{1/3}\big]\big] \leq C_1\sup_{s\in[c_\varepsilon,1-c_\varepsilon]}\mathsf{P}\big[R(s) < tc^{-1/2}\gamma(\varepsilon)^{-1}\big]$$
$$=: \Psi(t).$$

The fact that Ψ goes to 0 is a consequence of Theorem 3.18. □

We turn to the proof of Lemma 3.17.

Proof of Lemma 3.17. The proof follows the exact same strategy as above. Indeed, keeping the notations of the proof of Lemma 3.16, we may write

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{2/3}\big]\big]$$

$$\leq \mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\sup_{\vartheta\in[\varepsilon,\pi/2-\varepsilon]}\mathbb{P}_{h_N}^\vartheta\big[h_N^{-1}\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tc^{-1}\big]\big]. \quad (3.62)$$

We are going to use the same method as in the proof of Lemma 3.16. We refer to [Sui01, Theorems 4 and 5] to observe that the facet length scales as the time h_N in a random walk bridge. Some extra care is needed due to the fact that the function $\phi\in\mathcal{C}([0,1],\mathbb{R})\mapsto\mathsf{MeanFL}(\phi)\in\mathbb{R}^+$ is not continuous in ϕ; however it is easy to check that its points of discontinuity are included in the set of continuous functions having at least three aligned local maxima. Using the fact that the set of such functions has measure 0 under P and the above-mentioned uniformity in ϑ, we obtain from arguments similar to that of proof of Lemma 3.16 that there exists a constant $\delta(\varepsilon) > 0$ such that, for N large enough,

$$\mathbb{E}_\lambda^{N^2}\big[\mathbb{1}_{\mathcal{G}_N}\mathbb{P}_{\mathbf{x}_N,\mathbf{y}_N}\big[\mathsf{MeanFL}(\Gamma_{\mathbf{x}_N,\mathbf{y}_N}) < tN^{2/3}\big]\big] \leq C_1'\sup_{s\in[c_\varepsilon,1-c_\varepsilon]}\mathsf{P}[L(s) < t\delta(\varepsilon)c^{-1}]$$
$$=: \Phi(t).$$

We conclude using Theorem 3.18. □

4 Analysis of the maximal facet length and of the maximal local roughness

This section is devoted to the proof of Theorem 1.5. The corresponding statements in the context of the random-cluster have been obtained by Hammond in [Ham12a, Ham12b] (see Section 5 for a detailed discussion). We will essentially adapt the robust arguments developed by Hammond in these papers although some new difficulties, coming from the oriented feature of the model, will emerge.

The proof of the upper bound of (1.7) is in spirit the same as the one of Proposition 2.10 observing that the proof can be reproduced (with care) replacing the N^ϵ by $C(\log N)^{1/3}$ with a large C.

The proof of the lower bound is more technical and follows the lines of [Ham12b]. We use a different strategy of resampling consisting in successively resampling Γ in deterministic angular sectors. We then show that each one of these resampling has a sufficiently large probability of producing a favourable output (meaning with a large local roughness) — and thus that the final output has a large local roughness with very high probability. However, we will encounter several technical issues, the main one being the fact that a favourable ouptut could be destroyed by further resamplings. This technical point will be solved by carefully analysing the Markovian dynamic induced by the successive resamplings in Section 4.2. The bound on the maximal facet length will be a simple byproduct of the bound on the maximal local roughness.

4.1 Upper bounds

We start by proving the upper bound on MaxFL. The correct control is given by the following proposition.

Proposition 4.1. *There exist constants $\tilde{c}, c, C > 0$ such that, for any $\tilde{c} \le t \le N^{2/3}(\log N)^{-2/3}$,*

$$\mathbb{P}_\lambda^{N^2}\big[\mathsf{MaxFL}(\Gamma) \ge t N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\big] \le C e^{-ct^{3/2}\log N}. \tag{4.1}$$

The proof of this proposition follows the same lines as the proof of Proposition 2.10 with the exception that we modify the event $\mathsf{GAC}\,(x, y, \eta)$ of Subsection 3.1 to require a poly-logarithmic deviation of the area from its typical value.

Definition 4.2 (Logarithmic good area capture). Let $\gamma \in \Lambda$ be a path, $\eta > 0$ and $x, y \in \mathbb{N}^2$. We say that γ realizes the event $\mathsf{LogGAC}(x, y, \eta)$ (meaning "logarithmic good area capture") if

(*i*) $\gamma \cap \mathbf{A}_{x,y} \in \Lambda^{x \to y}$, or in words, γ connects x and y by a oriented path,

(*ii*) $|\mathrm{Enclose}(\gamma \cap \mathbf{A}_{x,y})| - |\mathbf{T}_{0,x,y}| \ge \eta \|x - y\|^{\frac{3}{2}} (\log \|x - y\|)^{\frac{1}{2}}$.

As previously, we need to estimate the probability of $\mathsf{LogGAC}(x, y, \eta)$ under $\mathbb{P}_{x,y}$, the uniform measure on $\Lambda^{x \to y}$. The following result, whose proof is postponed to the Appendix A, gives us a lower bound on $\mathbb{P}_{x,y}[\mathsf{LogGAC}(x, y, \eta)]$. Recall that $\theta(x, y) \in [0, \pi/2]$ the angle formed by the horizontal axis and the segment joining x and y.

Lemma 4.3. *Let $\varepsilon, \eta > 0$. There exist $C = C(\varepsilon) > 0$ and $N_0 = N_0(\varepsilon, \eta) > 0$ such that for any $x, y \in \mathbb{N}^2$ satisfying $\Lambda^{x \to y} \ne \emptyset$, $\varepsilon \le \theta(x, y) \le \pi/2 - \varepsilon$ and $\|x - y\| \ge N_0$,*

$$\mathbb{P}_{x,y}\left[\mathsf{LogGAC}(x, y, \eta)\right] \ge \|x - y\|^{-C\eta^2}. \tag{4.2}$$

Proposition 2.5 will ensure that the condition on $\theta(x, y)$ holds with high probability.

Proof of Proposition 4.1. We repeat the same strategy as in the proof of Proposition 2.10 considering this time the event

$$\mathsf{BigFacet}(t) := \big\{\mathsf{MaxFL}(\Gamma) \ge t N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\big\} \tag{4.3}$$

instead of $\mathsf{BigMeanFL}(t)$, LogGAC instead of GAC, and replacing $\mathsf{GoodHit}$ by the event that the random pair $(\mathbf{x}, \mathbf{y})$ hits the extremities of the largest facet. The control on the probability of LogGAC is given by Lemma 4.3 above. $\qquad\square$

With this result, it is now easy to estimate the correct scale for MaxLR. The statement is given in the following proposition.

Proposition 4.4. *There exist constants $\tilde{c}, c, C > 0$ such that, for any $\tilde{c} \leq t \leq N^{5/6}(\log N)^{-5/6}$, then,*

$$\mathbb{P}_\lambda^{N^2}\big[\mathsf{MaxLR}(\Gamma) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}}\big] \leq Ce^{-ct^{6/5}\log N}. \tag{4.4}$$

Proof. Denote by $\mathsf{MaxFac}(\Gamma)$ the facet where $\mathsf{MaxLR}(\Gamma)$ is attained, and by $\mathsf{MLRF}(\Gamma)$ its length. It is clear that

$$\mathsf{MLRF}(\Gamma) \leq \mathsf{MaxFL}(\Gamma). \tag{4.5}$$

With this remark and Proposition 4.1, we can conclude using the exact same method as in the proof of Proposition 3.2, the only difference being that we now use the coupling along the facet which has the largest local roughness. More precisely, let $\delta > 0$ to be fixed later. Let $t > 0$, then

$$\big\{\mathsf{MaxLR}(\Gamma) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}}\big\}$$

$$\subset \big\{\mathsf{MLRF}(\Gamma) \geq t^{2-\delta}N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\big\} \cup A \cup \big\{\Gamma \subset (B_{K_1N})^c\big\} \cup \bigcup_{j=N^{\frac{1}{3}}}^{t^{2-\delta}N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}} A_j, \tag{4.6}$$

where

$$A_j = \big\{\mathsf{MaxLR}(\Gamma) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}},\ \mathsf{MLRF}(\Gamma) = j,\ \Gamma \subset B_{K_1N}\big\}, \tag{4.7}$$

$$A = \big\{\mathsf{MaxLR}(\Gamma) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}},\ \mathsf{MLRF}(\Gamma) < N^{\frac{1}{3}}\big\}, \tag{4.8}$$

and K_1 is defined as in Lemma 2.2. It is easy to check that $A = \emptyset$ for $t \geq 1$, which is now our assumption. Thanks to Lemma 2.2 and Proposition 4.1, we already know how to control the probability of the first three terms in the right-hand side of (4.6). The proof will follow from obtaining an upper bound on $\mathbb{P}_\lambda^{N^2}[A_j]$. This bound is (again) given by the Lemma 2.7 (see Remark 2.8).

Let us denote by $B(j, x)$ the event that x belong to Γ and the facet above x has length j. Then, for $N^{1/3} \leq j \leq t^{2-\delta}N^{2/3}(\log N)^{1/3}$,

$$\begin{aligned}
\mathbb{P}_\lambda^{N^2}[A_j] &\leq \sum_{x \in B_{K_1N}} \mathbb{P}_\lambda^{N^2}\big[\mathsf{LR}(x) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}},\ B(j, x)\big] \\
&\leq \sum_{x \in B_{K_1N}} \sum_{\substack{a, b \in B_{K_1N} \\ \|b-a\|=j \\ \arg(a) > \arg(x) \geq \arg(b)}} \mathbb{P}_\lambda^{N^2}\big[\mathsf{LR}(x) \geq tN^{\frac{1}{3}}(\log N)^{\frac{2}{3}} \mid \mathsf{Fac}(a, b)\big]\mathbb{P}_\lambda^{N^2}[\mathsf{Fac}(a, b)] \\
&\leq O(N^4 j) \exp\Big(-c\frac{t^2 N^{2/3}(\log N)^{4/3}}{2j}\Big) \\
&\leq e^{-ct^\delta \log N}
\end{aligned}$$

where the third inequality is a consequence of Lemma 2.7 (and more particularly Remark 2.8). The proof follows from setting $\delta = \frac{6}{5}$ and taking $t_0 \leq t \leq N^{5/6}(\log N)^{-5/6}$, where t_0 is large enough. $\qquad\square$

4.2 Lower bounds

We start with the proof of the lower bound for MaxLR and then see how we can deduce the lower bound for MaxFL.

4.2.1 The strategy

We rely again on the strategy of resampling. We will divide the first quadrant of $\mathbb{Z}^2$ in angular sectors of angle

$$\theta_N = \theta_N(\chi) := \chi N^{-\frac{1}{3}} (\log N)^{\frac{1}{3}}, \tag{4.9}$$

where $\chi > 0$ is a small constant that we will specify later. Then, we will randomly process a resampling in some of these sectors. We will argue that each one of these sectors has a probability of producing a "favourable" output (i.e. with a large local roughness) which is decaying slowly (see Lemma 4.19) in N, i.e. roughly like $N^{-\delta}$ for some (small) $\delta > 0$. By iterating these resamplings on (sufficiently many) different sectors we ensure to recover a new path which presents at least one sector with a large local roughness. There is a crucial point we will have to deal with: if a given sector produces a favorable output, this could be undone by a further resample. That is why we will resample on sufficiently distant sectors (see Proposition 4.20).

Let us first fix some notations. Define

$$m_N := \left\lfloor \frac{\pi}{2\theta_N} \right\rfloor \tag{4.10}$$

to be the number of sectors subject to resampling. Note that

$$m_N \underset{N \to +\infty}{\sim} \frac{\pi}{2\chi} N^{\frac{1}{3}} (\log N)^{-\frac{1}{3}}. \tag{4.11}$$

Denote by $\mathbf{A}_j(N)$ the sector of the first quadrant defined by the angles $(j-1)\theta_N$ and $j\theta_N$ for $j \in \{1, \ldots, m_N\}$. Note that the first quadrant may contain a narrower sector which we will not make use of. Finally, we introduce the following notation: given two points a and b in the first quadrant, we denote by $\angle(a, b)$ the (non negative[10]) angle between the associated vectors $\overrightarrow{a}$ and $\overrightarrow{b}$[11]. For a closed shape S, Enclose(S) is the region of the first quadrant enclosed by S (and if needed the coordinate axes).

4.2.2 Defining the resampling procedure

We want to resample in the sectors $\mathbf{A}_j(N)$. However, choosing the beginning and ending points of the pieces subject to resampling in each sector must be done precociously.

Definition 4.5. Let $x, y \in \mathbb{N}^2$ be such that $\Lambda^{x \to y} \neq \emptyset$. Denote by $\mathbf{A}_{x,y}$ the cone of apex 0 bounded by x and y. For all $\gamma \in \Lambda^{N^2}$, $\Psi_{x,y}(\gamma)$ is the random element of Λ that is equal to γ outside $\mathbf{A}_{x,y}$, and that is random on $\mathbf{A}_{x,y}$ having the marginal law of $\mathbb{P}_\lambda^{N^2}$ on the considered sector, given the area condition and the path outside the sector.

Remark 4.6. In words, the formation of $\Psi_{x,y}(\gamma)$ can be summed up in two steps:

- Step A: we condition on the marginal of γ on $\mathbf{A}_{x,y}$ by the information of $\gamma \cap \mathbf{A}_{x,y}^c$.

- Step B: we condition on the area: $|\text{Enclose}((\gamma \cap \mathbf{A}_{x,y}^c) \cup \gamma_{x,y})| \geq N^2$.

The strategy is now the following: for each sector $\mathbf{A}_j(N)$, we pick some points x_j, y_j that belong to this sector, and to the random path we are considering, and we apply Ψ_{x_j, y_j}.

Definition 4.7 (Definition of RES$_j$). Let $j \in \{1, \ldots, m_N\}$. We define a procedure RES$_j$: $\Lambda^{N^2} \to \Lambda$ that only acts on $\mathbf{A}_j(N)$. Let $\mathbf{B}_j(N)$ be the cone of apex 0 defined by the angles $(j-1)\theta_N + \theta_N/4$ and $j\theta_N - \theta_N/4$. Notice that $\mathbf{B}_j(N) \subset \mathbf{A}_j(N)$. Let $\gamma \in \Lambda^{N^2}$. Let $x_j = x_j(\gamma)$ (resp. $y_j = y_j(\gamma)$) be the left-most (resp. right-most) point of $\gamma \cap \mathbf{B}_j(N)$. The procedure RES$_j$ consists in resampling γ between x_j and y_j. More precisely, we set RES$_j(\gamma) := \Psi_{x_j, y_j}(\gamma)$.

---

[10]We will always choose $\angle(a, b) \in [0, \pi/2]$.

[11]Sometimes we will also write $\angle(\overrightarrow{a}, \overrightarrow{b})$, depending on the context.

We will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space in which RES_j acts on an input Γ having the distribution $\mathbb{P}_\lambda^{N^2}$.

The following proposition is a straightforward consequence of the definition of $\Psi_{x,y}$,

Proposition 4.8 ($\mathbb{P}_\lambda^{N^2}$ is invariant under the resampling procedure)**.** *Let $j \in \{1, \ldots, m_N\}$. Then, if $\Gamma \sim \mathbb{P}_\lambda^{N^2}$, $\mathsf{RES}_j(\Gamma) \sim \mathbb{P}_\lambda^{N^2}$. In words, the law $\mathbb{P}_\lambda^{N^2}$ is invariant under the map RES_j.*

We have now properly defined a resampling procedure on each sector $\mathbf{A}_j(N)$. As we saw, this procedure can be realized in two steps. Our next objective is to find a sufficient condition for step B to be realized: how can we ensure that the resampled path captures enough area.

4.2.3   A sufficient condition to capture enough area

Let a and b be two points in $\mathbb{N}^2$ (with $\arg(a) > \arg(b)$). Assume we are given $\gamma \in \Lambda^{N^2}$ that passes through a and b. Modify γ between these two points replacing the original piece of path $\gamma_{a,b}$ by a new element $\tilde{\gamma}_{a,b} \in \Lambda^{a \to b}$. We are looking for a condition on $\tilde{\gamma}_{a,b}$ that is sufficient for the modified path $\tilde{\gamma}$ to belong to Λ^{N^2}. It is clear that a sufficient condition for $\tilde{\gamma}$ to enclose a sufficiently large area is

$$|\mathrm{Enclose}\,(\tilde{\gamma}_{a,b} \cup [0,a] \cup [0,b])| \geq |\mathrm{Enclose}(\gamma) \cap \mathbf{A}_{a,b}|\,. \tag{4.12}$$

As in [Ham12b], we obtain a quantitative sufficient condition (see Corollary 4.13) for (4.12) to be satisfied. We directly import without a proof the result from there as they trivially extend to our setting.

We will need a notation: for $x, y \in \mathbb{N}^2$ distinct, we denote by $\ell_{x,y}$ the unique line that passes through x and y. The following result is illustrated in Figure 7.

Lemma 4.9. *Let $\gamma \in \Lambda^{N^2}$. Let $j \in \{1, \ldots, m_N\}$. Denote by z_j the point of $\mathcal{C}(\gamma)$ of argument $j\theta_N$. Let ℓ_j be the tangent line of $\mathcal{C}(\gamma)$ at z_j. Let $x_j, y_j \in \gamma \cap \mathbf{A}_{z_{j-1},z_j}$ and $\mathbf{E}_{x_j,y_j} := \mathbf{E}_j$ be the pentagon delimited by the lines $\ell_{j-1}, \ell_j, \ell_{x_j,y_j}, \ell_{0,x_j}$ and ℓ_{0,y_j}. Then,*

$$(\mathrm{Enclose}(\gamma) \cap \mathbf{A}_{x_j,y_j}) \subset \mathbf{T}_{0,x_j,y_j} \cup \mathbf{E}_j, \tag{4.13}$$

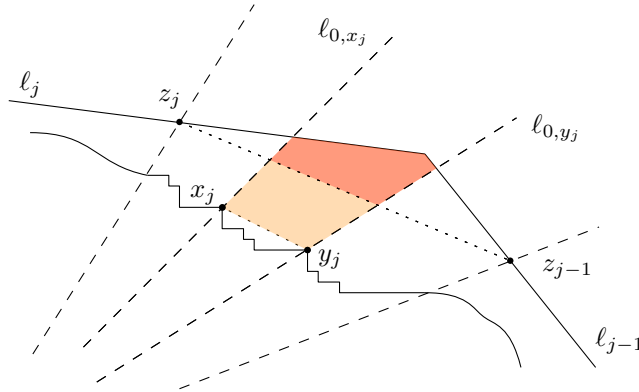*where $\mathbf{T}_{0,x_j,y_j}$ is the triangle of apexes $0, x_j, y_j$.*

Figure 7: Illustration of Lemma 4.9. The light orange shaded region corresponds to $\mathbf{E}_j^0$, and the dark orange shaded region corresponds to $\mathbf{E}_j^1$.

An immediate consequence of the lemma is the fact that (4.12) is satisfied if

$$|\mathrm{Enclose}\,(\tilde{\gamma}_{a,b} \cup [0,a] \cup [0,b])| \geq |\mathbf{T}_{0,a,b}| + |\mathbf{E}_{a,b}|\,. \tag{4.14}$$

We now write $\mathbf{E}_j = \mathbf{E}_j^0 \cup \mathbf{E}_j^1$ (where $\mathbf{E}_j^0$ denotes the part of $\mathbf{E}_j$ that lies below ℓ_{z_{j-1},z_j}), see Figure 7. It seems natural to seek an upper bound for $|\mathbf{E}_j|$. We start by determining an upper bound on $|\mathbf{E}_j^1|$. This is what we perform in the next lemma.

Lemma 4.10 ([Ham12b, Lemma 3.9]). *Let $\gamma \in \Lambda^{N^2}$. Assume $\gamma \subset B_{K_1 N}$. We keep the notations of the preceding lemma. Let $\overrightarrow{w_j}$ be the unit vector tangent to $\mathcal{C}(\gamma)$ at z_j oriented in the counterclockwise sense. We define the set of sectors with* moderate boundary turning *to be*

$$\text{MBT} = \left\{ j \in \{1, \dots, m_N\}, \ \|z_j - z_{j-1}\| \leq \frac{10 \pi K_1 N}{m_N} \text{ and } \angle(\overrightarrow{w_j}, \overrightarrow{w_{j-1}}) \leq \frac{10\pi}{m_N} \right\}. \tag{4.15}$$

Then,

$$|\text{MBT}| \geq \frac{9 m_N}{10}, \tag{4.16}$$

and if $j \in \text{MBT}$,

$$|\mathbf{E}_j^1| \leq \frac{1}{2}(20)^3 K_1^2 \chi^3 N \log N. \tag{4.17}$$

We now need to find an upper bound on $|\mathbf{E}_j^0|$. It is clear that such bound can only be obtained under the assumption that the points x_j and y_j defined above are not to far from $\mathcal{C}(\gamma)$. This remark motivates the following definition. Recall that if $\gamma \in \Lambda$ and $v \in \gamma$, $\mathsf{LR}(v, \gamma) = \mathsf{d}(v, \mathcal{C}(\gamma))$ denotes the local roughness of v in γ.

Definition 4.11 (Favourable sectors). Let $\gamma \in \Lambda^{N^2}$. Let φ be a function that maps $(0, \infty)$ to itself and that satisfies, as $t \to 0$,

$$\varphi(t) = o(\sqrt{t}). \tag{4.18}$$

For $\chi > 0$, we say that the sector $\mathbf{A}_j(N)$ is *favourable under γ* if there exists $v \in \gamma \cap \mathbf{A}_j(N)$ such that

$$\mathsf{LR}(v, \gamma) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}}. \tag{4.19}$$

We define $\text{UNFAV}(\gamma, \chi) = \text{UNFAV}$ to be the set of $j \in \{1, \dots, m_N\}$ such that $\mathbf{A}_j(N)$ is not favourable under γ.

Lemma 4.12 ([Ham12b, Lemma 3.10]). *Let $\gamma \in \Lambda^{N^2}$. Assume that $\gamma \cap B_{K_2 N} = \emptyset$. Let $j \in \text{MBT} \cap \text{UNFAV}$. Then,*

$$|\mathbf{E}_j^0| \leq 20 K_1 \chi \varphi(\chi) N \log N. \tag{4.20}$$

From the preceding results, we can deduce the sufficient condition to capture enough area we were looking for,

Corollary 4.13 (Sufficient condition to capture a large area). *Let $\gamma \in \Lambda^{N^2}$ be such that $\gamma \subset B_{K_1 N} \setminus B_{K_2 N}$. Let $j \in \{1, \dots, m_N\}$. Assume $j \in \text{MBT} \cap \text{UNFAV}$. Let $x_j, y_j \in \gamma \cap \mathbf{A}_j(N)$ be chosen accordingly to the procedure described in Definition 4.7. Then, if we resample γ between x_j and y_j, replacing the path γ_{x_j, y_j} by a new path $\tilde{\gamma}_{x_j, y_j}$, the new path $\tilde{\gamma}$ will be an element of Λ^{N^2} if the following condition is satisfied,*

$$\left| \text{Enclose}(\tilde{\gamma}_{x_j, y_j} \cup [0, x_j] \cup [0, y_j]) \right| - \left| \mathbf{T}_{0, x_j, y_j} \right| \geq \left(\tfrac{1}{2}(20)^3 K_1^2 \chi^3 + 20 K_1 \varphi(\chi) \chi \right) N \log N. \tag{4.21}$$

4.2.4   The right shape

Now that we have a sufficient condition for a resampled path to be accepted, we need to exhibit an event that will ensure this sufficient condition. We make good use of the event $\mathsf{LogGAC}(x, y, \eta)$ introduced in Definition 4.2. As we are about to see, the realisation of this event will ensure that the condition of Corollary 4.13 will be satisfied. Nevertheless, we also need our resamplings to have a large local roughness and that is what motivates Definition 4.14.

Definition 4.14 (Significant inward deviation). Let $\eta > 0$ and $x, y \in \mathbb{N}^2$ with $\arg(x) > \arg(y)$. Let $\gamma \in \Lambda$ be a path which passes by x and y. We say that γ realizes the event $\mathsf{LogSID}(x, y, \eta)$ (meaning "logarithmic significant inward deviation") if there exists $z \in \gamma_{x, y}$ such that

$$\mathsf{d}\left(z, \partial \text{conv}([0, x] \cup [0, y] \cup \gamma_{x, y})\right) \geq \eta \|x - y\|^{\frac{1}{2}} (\log \|x - y\|)^{\frac{1}{2}}, \tag{4.22}$$

where ∂conv is the border of the convex hull.

We are going to resample in the sectors $\mathbf{A}_j(N)$ introduced above. For a path $\gamma \in \Lambda^{N^2}$, call $x_j = x_j(\gamma), y_j = y_j(\gamma)$ the extremities of the portion of γ subject to resampling in $\mathbf{A}_j(N)$ (as described in Definition 4.7). We first check that provided $\chi > 0$ is small enough, for a path γ satisfying "nice" properties, if we replace the portion of γ between x_j and y_j by a path satisfying $\mathsf{LogGAC}(x_j, y_j, \eta)$ and $\mathsf{LogSID}(x_j, y_j, \eta)$, we obtain an element of Λ^{N^2} with a large local roughness.

Definition 4.15 (Successful action of RES_j). Let $\eta > 0$ and $1 \leq j \leq m_N$. Let $\gamma \in \Lambda^{N^2}$. We say that our resampling operation in the j-th sector RES_j **acts η-successfully** on γ if the new path γ' obtained after the operation realises the event $\mathsf{LogGAC}(x_j, y_j, \eta) \cap \mathsf{LogSID}(x_j, y_j, \eta)$.

We start by proving that the occurrence of $\mathsf{LogGAC}(x_j, y_j, \eta)$ forces the area condition to be satisfied.

Lemma 4.16. *Let $\eta > 0$. There exists $\chi_1 = \chi_1(\eta) > 0$ such that for $\chi \in (0, \chi_1)$ the following assertion holds. Let $\gamma \in \Lambda^{N^2}$ such that $\gamma \subset B_{K_1 N} \setminus B_{K_2 N}$. Let $j \in \{1, \dots, m_N\}$. Assume that $j \in \mathrm{MBT} \cap \mathrm{UNFAV}$. Assume that we modify γ between x_j and y_j and that the modified path γ' satisfies $\gamma' \in \mathsf{LogGAC}(x_j, y_j, \eta)$. Then, $\gamma' \in \Lambda^{N^2}$.*

Proof. It is easy to see from Definition 4.7 that

$$\angle(x_j, y_j) \geq \frac{\theta_N}{4}. \tag{4.23}$$

Using this and the fact that $x_j, y_j \notin B_{K_2 N}$, we get

$$\|x_j - y_j\| \geq \frac{K_2 \chi}{2\pi} N^{\frac{2}{3}} (\log N)^{\frac{1}{3}}. \tag{4.24}$$

The occurence of $\mathsf{LogGAC}(x_j, y_j, \eta)$ implies that

$$\left| \mathrm{Enclose}(\gamma'_{x_j, y_j} \cup [0, x_j] \cup [0, y_j]) \right| - |\mathbf{T}_{0, x_j, y_j}| \geq \frac{\eta}{2} \left(\frac{2}{3}\right)^{\frac{1}{2}} (K_2/2\pi)^{\frac{3}{2}} \chi^{\frac{3}{2}} N \log N. \tag{4.25}$$

Since $\varphi(t) = o(\sqrt{t})$ as $t \to 0$, we see that condition in Corollary 4.13 is ensured by fixing $\chi > 0$ small enough. $\qquad\square$

A direct consequence of the proof is the following identity: if $\gamma \in \Lambda^{N^2}$ and $j \in \{1, \dots, m_N\}$ satisfy the properties above, and $\chi > 0$ is small enough,

$$\mathbb{P}\left[\mathsf{RES}_j \text{ acts } \eta\text{-successfully on } \gamma\right] =$$
$$\frac{|\{\gamma'_{x_j, y_j} \in \Lambda^{x_j \to y_j}, \ (\gamma \setminus \gamma_{x_j, y_j}) \cup \gamma'_{x_j, y_j} \in \mathsf{LogGAC}(x_j, y_j, \eta) \cap \mathsf{LogSID}(x_j, y_j, \eta)\}|}{|\{\gamma'_{x_j, y_j} \in \Lambda^{x \to y}, \ (\gamma \setminus \gamma_{x_j, y_j}) \cup \gamma'_{x_j, y_j} \in \Lambda^{N^2}\}|}. \tag{4.26}$$

Lemma 4.17. *Let $\eta > 0$. There exists $\chi_2 = \chi_2(\eta) > 0$ such that for $\chi \in (0, \chi_2)$ the following assertion holds. Let $\gamma \in \Lambda^{N^2}$ such that $\gamma \subset B_{K_1 N} \setminus B_{K_2 N}$. Let $j \in \{1, \dots, m_N\}$. Assume that $j \in \mathrm{MBT} \cap \mathrm{UNFAV}$ and that RES_j acts η-successfully on γ. Then, $\mathbf{A}_j(N)$ is favourable under $\gamma' = \mathsf{RES}_j(\gamma)$.*

Proof. Since $\gamma' \in \mathsf{LogSID}(x_j, y_j, \eta)$, there exists $z \in \gamma'_{x_j, y_j}$ such that

$$\mathrm{d}\left(z, \partial\mathrm{conv}\left([0, x_j] \cup [0, y_j] \cup \gamma'_{x_j, y_j}\right)\right) \geq \eta \|x_j - y_j\|^{\frac{1}{2}} (\log \|x_j - y_j\|)^{\frac{1}{2}}. \tag{4.27}$$

One may write $\gamma' = \left(\gamma \cap \mathbf{A}_{x_j,y_j}^c \right) \cup \gamma'_{x_j,y_j}$, so that we have

$$
\begin{aligned}
\mathrm{d}\left(z, \mathcal{C}\left(\gamma'\right)\right) &= \mathrm{d}\left(z, \mathcal{C}\left(\left(\gamma \cap \mathbf{A}_{x_j,y_j}^c\right) \cup \gamma'_{x_j,y_j}\right)\right) \\
&\geq \mathrm{d}\left(z, \partial\mathrm{conv}\left([0, x_j] \cup [0, y_j] \cup \gamma'_{x_j,y_j}\right)\right) \\
&\geq \eta\|x_j - y_j\|^{\frac{1}{2}}(\log \|x_j - y_j\|)^{\frac{1}{2}} \\
&\geq \frac{1}{2}\sqrt{K_2}\left(\frac{2}{3}\right)^{\frac{1}{2}} \eta \chi^{\frac{1}{2}} N^{\frac{1}{3}}(\log N)^{\frac{2}{3}}.
\end{aligned}
$$

Since $\varphi(t) = o(\sqrt{t})$ as $t \to 0$, we get the result choosing $\chi > 0$ small enough (in terms of η). $\qquad\square$

We now fix $\eta > 0$ and $\chi = \chi(\eta) > 0$ sufficiently small so that Lemmas 4.16 and 4.17 are true. Combining Lemma 4.17 and Equation (4.26) we get, if $\gamma \in \Lambda^{N^2}$ satisfies the conditions above and if $j \in \mathrm{MBT} \cap \mathrm{UNFAV}$,

$$
\mathbb{P}[\mathbf{A}_j(N) \text{ is favourable under } \mathsf{RES}_j(\gamma)] \geq \mathbb{P}_{x_j,y_j}[\mathsf{LogGAC}(x_j, y_j, \eta) \cap \mathsf{LogSID}(x_j, y_j, \eta)], \quad (4.28)
$$

where we recall that $\mathbb{P}_{x_j,y_j}$ is the uniform law on $\Lambda^{x_j \to y_j}$. The last step is then to estimate the probability on the right-hand side of the above inequality. This step is actually done using the following result, whose proof is postponed to the appendix.

Lemma 4.18. *Let $\eta, \varepsilon > 0$. There exist $C = C(\varepsilon), N_0 = N_0(\varepsilon, \eta) > 0$ such that for any $N \geq N_0$, any $x, y \in \mathbb{N}^2$ such that $\arg(x) > \arg(y)$, $\theta(x, y) \in [\varepsilon, \pi/2 - \varepsilon]$ and $\|y - x\| \geq N_0$,*

$$
\mathbb{P}_{x,y}\left[\mathsf{LogGAC}(x, y, \eta) \cap \mathsf{LogSID}(x, y, \eta)\right] \geq N^{-C\eta^2}. \quad (4.29)
$$

We can now state the main result of this section.

Lemma 4.19 (Successful resampling). *Let $\eta > 0$. Let $\chi = \chi(\eta) > 0$ be sufficiently small so that Lemmas 4.16 and 4.17 hold true. Let $\gamma \in \Lambda^{N^2}$ such that $\gamma \subset B_{K_1 N} \setminus B_{K_2 N}$. Let $j \in \{1, \dots, m_N\}$. Assume that $j \in \mathrm{MBT} \cap \mathrm{UNFAV}$ and that $\gamma \notin \mathsf{Bad}_{\varepsilon,\mathbf{A}}^- \cup \mathsf{Bad}_{\varepsilon,\mathbf{A}}^+$ for $\mathbf{A} = \mathbf{B}_j(N)$ where $\varepsilon > 0$ is given by Proposition 2.5. Then, there exist $C = C(\varepsilon) > 0$, and $N_0 = N_0(\varepsilon, \eta) \in \mathbb{N}$ such that for $N \geq N_0$,*

$$
\mathbb{P}[\mathbf{A}_j(N) \text{ is favourable under } \mathsf{RES}_j(\gamma)] \geq \mathbb{P}\left[\mathsf{RES}_j \text{ acts } \eta\text{-successfully on } \gamma\right] \geq N^{-C\eta^2}. \quad (4.30)
$$

4.2.5   How not to undo a large local roughness

As described earlier, we want to perform our random surgery successively on different sectors $\mathbf{A}_j(N)$ in order to maximise the probability to obtain a favourable output. However, if we do it without caution, might it be that we will lose in the process favourable sectors. In order to avoid such problem we must only resample sufficiently distant sectors. This problem was handled by Hammond in [Ham12b]. Once again, the proof immediately extends to our setup.

Proposition 4.20 ([Ham12b, Lemmas 3.14 and 3.16]). *Let $s_1(N) \in \{1, \dots, m_N\}$. Let $s_2(N) \in (0, \infty)$ such that*

$$
s_2(N) < \frac{\chi}{2}s_1(N). \quad (4.31)
$$

Set $s_3(N) = s_1(N) + 1$. Let $(k, j) \in \{1, \dots, m_N\}^2$ be such that $|k - j| \geq s_3(N)$. Let $\gamma \in \Lambda^{N^2}$ be such that $\gamma \cap B_{K_2 N} = \emptyset$. Define $\gamma' = \mathsf{RES}_j(\gamma)$. Assume that

$$
\max\left\{\mathsf{MaxFL}(\gamma), \mathsf{MaxFL}(\gamma')\right\} \leq s_2(N)N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}. \quad (4.32)
$$

Then, the sector $\mathbf{A}_k(N)$ is favourable under γ if and only if it is favourable under γ'. Moreover,

$$
\mathcal{C}(\gamma) \cap \mathbf{A}_k(N) = \mathcal{C}(\gamma') \cap \mathbf{A}_k(N). \quad (4.33)
$$

4.2.6 The proof of the lower bound for MaxLR

A naive approach to the proof of the lower bound would be to apply the resampling successively on all the sectors $\mathbf{A}_j(N)$. As we saw in the preceding section, this raises a major problem. Applying the random surgery consecutively on two successive sectors might have for consequence the disappearance of a favourable sector, which is something we clearly seek to avoid. This problem can be solved working on distant sectors as seen in the last section. This is why we decide to resample roughly one in $s_3(N)$ sectors. We decide to randomly choose the sectors we resample: the reason is obvious, we do not want to resample a sector where there is already a favourable sector. We now explain in detail our strategy.

We keep the notations of the above sections. We want to define a complete resampling RES that will involve all the procedures RES_j we constructed before and the quantities $\{s_i(N)\}_{i=1,2,3}$ introduced in Proposition 4.20, with their form being specified later.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in which are defined: a random path Γ of law $\mathbb{P}_\lambda^{N^2}$, and the operations of resampling RES_j for $1 \leq j \leq m_N$, which act independently. We also generate a sequence of i.i.d random variables $\{X_k\}_{1 \leq k \leq m_N}$ that have the law of a Bernoulli of parameter $\frac{1}{s_3(N)}$. We now properly define our entire resampling procedure RES. This operation will be defined under $(\Omega, \mathcal{F}, \mathbb{P})$. We build it by induction. Set $\Gamma_0 = \Gamma$ (the input). For $1 \leq j \leq m_N$, if $X_j = 1$ then set $\Gamma_j = \mathsf{RES}_j(\Gamma_{j-1})$, else set $\Gamma_j = \Gamma_{j-1}$ (no action is taken and the sector $\mathbf{A}_j(N)$ remains unchanged). We also set $\mathrm{UNFAV}_j = \mathrm{UNFAV}(\Gamma_j, \chi)$ and $\mathrm{MBT}_j = \mathrm{MBT}(\Gamma_j)$. As we previously saw, the law $\mathbb{P}_\lambda^{N^2}$ is invariant under RES. We will analyse $\mathbb{P}_\lambda^{N^2}$ by identifying it with the law of the output Γ_{m_N} of RES under the measure $\mathbb{P}$. However, the only way to analyse the output of the procedure is if it the input lies in a space $\mathcal{G}$ of "good" paths. Introduce for $0 \leq i \leq m_N$,

$$\mathcal{G}_{1,i} := \{\mathsf{MaxFL}(\Gamma_i) \leq s_2(N) N^{\frac{2}{3}} (\log N)^{\frac{1}{3}}\}, \tag{4.34}$$

$$\mathcal{G}_{2,i} := \{\Gamma_i \subset B_{K_1 N} \setminus B_{K_2 N}\}, \tag{4.35}$$

$$\mathcal{G}_{3,i} := \{\Gamma_i \notin \mathsf{Bad}_{\varepsilon, \mathbf{B}_i(N)}^- \cup \mathsf{Bad}_{\varepsilon, \mathbf{B}_i(N)}^+\}, \tag{4.36}$$

where $\varepsilon > 0$ is given by Proposition 2.5. We also define for $0 \leq i \leq m_N$,

$$\mathcal{G}_i := \bigcap_{1 \leq j \leq 3} \mathcal{G}_{j,i}, \text{ and } \qquad \mathcal{G}_{(i)} := \bigcap_{0 \leq j \leq i} \mathcal{G}_j. \tag{4.37}$$

Finally, the space of good outcomes that is interesting is nothing but

$$\mathcal{G} := \mathcal{G}_{(m_N)}. \tag{4.38}$$

For $\varepsilon_1 \in (0, \frac{2}{3})$ to be fixed later, we define

$$s_3(N) := N^{\varepsilon_1}, \qquad \text{and} \qquad s_2(N) := \frac{\chi}{4} N^{\varepsilon_1}, \tag{4.39}$$

so that the conditions of Proposition 4.20 are satisfied. Let us first prove that the set of good outcomes $\mathcal{G}$ happens with sufficiently large probability.

Lemma 4.21. *There exist two constants $c, C > 0$ such that*

$$\mathbb{P}\left[\mathcal{G}^c\right] \leq C \exp(-c\chi^{\frac{3}{2}} N^{\frac{3\varepsilon_1}{2}} \log N) + C \exp(-cN) + C \exp(-cN\theta_N). \tag{4.40}$$

Proof. By definition, we have that

$$\mathbb{P}[\mathcal{G}^c] \leq m_N \mathbb{P}_\lambda^{N^2} \left[\mathsf{MaxFL}(\Gamma) > s_2(N) N^{\frac{2}{3}} (\log N)^{\frac{1}{3}}\right]$$
$$+ m_N \mathbb{P}_\lambda^{N^2}[\Gamma \subset (B_{K_1 N} \setminus B_{K_2 N})^c] + m_N \mathbb{P}_\lambda^{N^2}[\mathsf{Bad}_{\varepsilon, \mathbf{B}_1(N)}^- \cup \mathsf{Bad}_{\varepsilon, \mathbf{B}_1(N)}^+].$$

We then conclude by applying Proposition 4.1, Lemma 2.2 and Proposition 2.5. $\qquad \square$

Our second goal is to control the evolution of the sets MBT and UNFAV during the procedure (under the assumption of $\mathcal{G}$). This control is obtained in the following lemma.

Lemma 4.22 ([Ham12b, Lemma 3.17]). *For $1 \leq j \leq m_N - s_3(N)$, if $\mathcal{G}$ occurs then*

$$\text{UNFAV}_j \cap \{j + s_3(N), \dots, m_N\} = \text{UNFAV}_0 \cap \{j + s_3(N), \dots, m_N\}, \qquad (4.41)$$

and

$$\text{MBT}_j \cap \{j + s_3(N), \dots, m_N\} = \text{MBT}_0 \cap \{j + s_3(N), \dots, m_N\}. \qquad (4.42)$$

Proof. Let $(j, k) \in \{1, \dots, m_N\}^2$ satisfy $j + s_3(N) \leq k \leq m_N$. If $\mathcal{G}$ occurs, then we may apply Proposition 4.20 to each of the first j stages of the procedure RES. In particular, this tells us that $\mathbf{A}_k(N)$ is favourable under Γ_j if and only if it is favourable under Γ_0: this is exactly (4.41).

Then, the condition $k \in \text{MBT}_j$ is determined by the data $\mathcal{C}(\Gamma_j) \cap \mathbf{A}_k(N)$. However, one may also successively apply Proposition 4.20 to the first j stages of the procedure RES to obtain that

$$\mathcal{C}(\Gamma_j) \cap \mathbf{A}_k(N) = \mathcal{C}(\Gamma_0) \cap \mathbf{A}_k(N), \qquad (4.43)$$

which yields (4.42). $\qquad\qquad\square$

Now, recall from Lemma 4.10 that if $\Gamma_0 \subset B_{K_1 N}$ then

$$|\text{MBT}_0| \geq \frac{9m_N}{10} \geq \frac{m_N}{2}. \qquad (4.44)$$

By the occurence of $\mathcal{G}_{2,0}$, we may then define a set $\overline{\text{MBT}_0} \subset \text{MBT}_0$ that satisfies the following properties:

- $|\overline{\text{MBT}_0}| \geq \frac{m_N}{4s_3(N)}$,

- each pair of consecutive elements of $\overline{\text{MBT}_0}$ differ by at least $2s_3(N) + 1$.

We also write $\overline{\text{MBT}_0} \cap \text{UNFAV}_0 = \{p_1, \dots, p_{r_1}\}$ and $\overline{\text{MBT}_0} \cap \text{UNFAV}_0^c = \{q_1, \dots, q_{r_2}\}$ with $r_1 + r_2 = |\overline{\text{MBT}_0}|$. For $1 \leq r \leq r_1$, let P_r denote the event that in the action of RES, at stage p_r, RES_{p_r} is chosen to act and acts successfully while no action is taken at stages j for $j \in \{p_r - s_3(N), \dots, p_r - 1\} \cup \{p_r + 1, \dots, p_r + s_3(N)\}$. Similarly, for $1 \leq r \leq r_2$, let Q_r denote the event that in the action of RES, at stage q_r no action is taken, and same for the stages j with $j \in \{q_r - s_3(N), \dots, q_r\} \cup \{q_r + 1, \dots, q_r + s_3(N)\}$. We make good use of these events to make favourable sectors appear.

Lemma 4.23. *We keep the notations introduced above. Then,*

(i) *for each $r \in \{1, \dots, r_1\}$, $\mathcal{G} \cap P_r$ implies that the sector $\mathbf{A}_{p_r}(N)$ is favourable under the output Γ_{m_N},*

(ii) *for each $r \in \{1, \dots, r_2\}$, $\mathcal{G} \cap Q_r$ implies that the sector $\mathbf{A}_{q_r}(N)$ is favourable under the output Γ_{m_N}.*

Proof.   (i) By Lemma 4.22, we have $p_r \in \text{UNFAV}_{p_r - s_3(N)} \cap \text{MBT}_{p_r - s_3(N)}$, since $p_r \in \text{UNFAV}_0 \cap \text{MBT}_0$. Given P_r, we then have $p_r \in \text{UNFAV}_{p_r - 1} \cap \text{MBT}_{p_r - 1}$ because $\Gamma_{p_r - s_3(N)} = \Gamma_{p_r - 1}$. Applying Lemma 4.17 to the η-successful action of RES_{p_r} on $\Gamma_{p_r - 1}$, we find that $\mathbf{A}_{p_r}(N)$ is favourable under Γ_{p_r}. This remains the case for $\Gamma_{p_r + s_3(N)}$ because $\Gamma_{p_r} = \Gamma_{p_r + s_3(N)}$. Now, thanks to Proposition 4.20, $\mathbf{A}_{p_r}(N)$ stays a favourable sector during the remaining stages of the procedure RES (recall that we also work under $\mathcal{G}$ in which $\mathcal{G}_{1,i}$ occurs for $1 \leq i \leq m_N$).

(*ii*) This is essentially the same argument as the one depicted above. We have that $q_r \notin$ $\mathrm{UNFAV}_{q_r - s_3(N)}$ so that the inaction of RES_j for $j \in \{q_r - s_3(N), \ldots, q_r + s_3(N)\}$ entails $q_r \notin \mathrm{UNFAV}_{q_r + s_3(N)}$. We can then conclude as above using Proposition 4.20.

$\square$

An immediate consequence of Lemma 4.23 is that

$$\left(\bigcup_{i=1}^{r_1} P_i \cup \bigcup_{i=1}^{r_2} Q_i \right) \cap \mathcal{G} \subset \left\{ \mathsf{MaxLR}\left(\Gamma_{m_N}\right) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}} \right\}. \tag{4.45}$$

Also, note that by the discussion above, if $\mathcal{G}$ happens then $r_1 + r_2 \geq \frac{m_N}{4 s_3(N)}$. Then,

$$\left(\bigcup_{i=1}^{r_1} P_i \cup \bigcup_{i=1}^{r_2} Q_i \right)^c \cap \mathcal{G} \subset$$

$$\bigcap_{i=1}^{\frac{m_N}{8 s_3(N)}} \left(\{r_1 \geq i\} \cap P_i^c \cap \mathcal{G}_{(p_i - s_3(N))} \right) \cup \bigcap_{i=1}^{\frac{m_N}{8 s_3(N)}} \left(\{r_2 \geq i\} \cap Q_i^c \cap \mathcal{G}_{(q_i - s_3(N))} \right).$$

Using all the material above, we claim that, for any $K \in \mathbb{N}$, given $\{r_1 \geq K\} \cap \mathcal{G}_{(p_K - s_3(N))}$ and the values of $\mathbb{1}_{P_1}, \ldots, \mathbb{1}_{P_{K-1}}$, the conditional probability that P_K occurs is at least

$$N^{-C\eta^2} \frac{1}{s_3(N)} \left(1 - \frac{1}{s_3(N)} \right)^{2 s_3(N)}, \tag{4.46}$$

where $\eta > 0$ and $C = C(\varepsilon)$ is given by Lemma 4.19. Indeed, the event on which we condition is measurable with respect to $\sigma(\Gamma_0, \ldots, \Gamma_{p_K - s_3(N)})$, and if it occurs, $\Gamma_{p_K - s_3(N)}$ satisfies the hypothesis of Lemma 4.19. The claim then follows by this lemma. As a result,

$$\mathbb{P}\left[\bigcap_{i=1}^{\frac{m_N}{8 s_3(N)}} \left(\{r_1 \geq i\} \cap P_i^c \cap \mathcal{G}_{(p_i - s_3(N))} \right) \right] \leq \left(1 - N^{-C\eta^2} \frac{1}{s_3(N)} \left(1 - \frac{1}{s_3(N)} \right)^{2 s_3(N)} \right)^{\frac{m_N}{8 s_3(N)}}. \tag{4.47}$$

Similarly,

$$\mathbb{P}\left[\bigcap_{i=1}^{\frac{m_N}{8 s_3(N)}} \left(\{r_2 \geq i\} \cap Q_i^c \cap \mathcal{G}_{(q_i - s_3(N))} \right) \right] \leq \left(1 - \left(1 - \frac{1}{s_3(N)} \right)^{2 s_3(N) + 1} \right)^{\frac{m_N}{8 s_3(N)}}. \tag{4.48}$$

Now, Proposition 4.8 implies that

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MaxLR}(\Gamma) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}} \right] = \mathbb{P}\left[\mathsf{MaxLR}\left(\Gamma_{m_N}\right) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}} \right]. \tag{4.49}$$

Using (4.45) we then obtain,

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MaxLR}(\Gamma) \geq \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}} \right] \geq 1 - \mathbb{P}[\mathcal{G}^c] - \mathbb{P}\left[\left(\bigcup_{i=1}^{r_1} P_i \cup \bigcup_{i=1}^{r_2} Q_i \right)^c \cap \mathcal{G} \right]. \tag{4.50}$$

Using Lemma 4.21, Equations (4.47), (4.48), (4.11) and Definition 4.39, we obtain the vanishing of $\mathbb{P}_\lambda^{N^2}\left[\mathsf{MaxLR}(\Gamma) < \varphi(\chi) N^{\frac{1}{3}} (\log N)^{\frac{2}{3}} \right]$ to 0 as N goes to infinity. More precisely, we have a quantitative result. Recall that one may take $\eta > 0$ sufficiently small so that it always satisfies $C\eta^2 < \varepsilon_1$. For some constant $c(\chi) > 0$,

$$\mathbb{P}[\mathcal{G}^c] \leq \exp(-c(\chi) N^{\frac{3\varepsilon_1}{2}} \log N), \tag{4.51}$$

$$\left(1 - N^{-C\eta^2}\frac{1}{s_3(N)}\left(1 - \frac{1}{s_3(N)}\right)^{2s_3(N)}\right)^{\frac{m_N}{8s_3(N)}} \leq \exp(-c(\chi)N^{\frac{1}{3}-3\varepsilon_1}(\log N)^{\frac{1}{3}}), \qquad (4.52)$$

and

$$\left(1 - \left(1 - \frac{1}{s_3(N)}\right)^{2s_3(N)+1}\right)^{\frac{m_N}{8s_3(N)}} \leq \exp(-c(\chi)N^{\frac{1}{3}-\varepsilon_1}(\log N)^{\frac{1}{3}}). \qquad (4.53)$$

Taking $\varepsilon_1 = \frac{2}{27}$, we obtain the following result,

Proposition 4.24. *For any $\chi > 0$ sufficiently small, there exist $c(\chi), N_0 > 0$ such that for $N \geq N_0$,*

$$\mathbb{P}_\lambda^{N^2}\left[\mathsf{MaxLR}(\Gamma) < \varphi(\chi)N^{\frac{1}{3}}(\log N)^{\frac{2}{3}}\right] \leq \exp(-c(\chi)N^{\frac{1}{9}}\log N). \qquad (4.54)$$

Remark 4.25. Recall that above we first consider values of ε_1 sufficiently small (typically $\varepsilon_1 < 1/9$), and then fix a value of $\eta > 0$ such that $C\eta^2 \ll \varepsilon_1$ (where C is the constant given by Lemma 4.19. Finally, we choose $\chi = \chi(\eta)$ sufficiently small such that Lemmas 4.16 and 4.17 hold. We did not try to obtain an optimal bound in Proposition 4.24.

4.2.7 The proof of the lower bound for MaxFL

Once (1.8) is obtained, getting a lower bound on MaxFL is immediate with the coupling described in Section 2.

Proposition 4.26. *Let $\chi > 0$ be sufficiently small. There exist a function ϕ such that $\phi(t) \xrightarrow[t\to0^+]{} +\infty$ such that for N large enough,*

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}] \leq N^{-\phi(\chi)}. \qquad (4.55)$$

Proof. We split the event $\{\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}\}$ into the union of two events, according to the value of the maximal local roughness,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}] \leq \mathbb{P}_\lambda^{N^2}[\mathsf{MaxLR} < \varphi(\chi)(\log N)^{\frac{2}{3}}N^{\frac{1}{3}}]$$
$$+ \mathbb{P}_\lambda^{N^2}[\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}, \ \mathsf{MaxLR} \geq \varphi(\chi)(\log N)^{\frac{2}{3}}N^{\frac{1}{3}}]. \quad (4.56)$$

The first term of this sum has been shown to be smaller than $\exp(-c(\chi)N^{1/9}\log N)$ for χ sufficiently small and N sufficiently large. To bound the second term, we use the coupling described in Proposition 2.6. Indeed, we invoke the proof of Proposition 4.4 to get that for some $c > 0$,

$$\mathbb{P}_\lambda^{N^2}[\mathsf{MaxFL}(\Gamma) < \varphi(\chi)^3 N^{\frac{2}{3}}(\log N)^{\frac{1}{3}}, \ \mathsf{MaxLR} \geq \varphi(\chi)(\log N)^{\frac{2}{3}}N^{\frac{1}{3}}] \leq$$
$$O(N^6)\exp(-\tfrac{c}{\varphi(\chi)}\log N) + \exp(-cN), \quad (4.57)$$

where the second term comes from the bound on $\mathbb{P}_\lambda^{N^2}[\Gamma \cap (B_{K_1N})^c \neq \emptyset]$. This gives the result provided $\chi > 0$ is sufficiently small in which case one can set $\phi(\chi) := -(6 - c\varphi(\chi)^{-1}) > 0$. $\qquad\square$

5 Discussion: extending the results to the Wulff setting

In that section, we briefly explain how this model — despite being simple — can be seen as a toy model for the phase separation interface in the Wulff setting. In particular, we argue that it shares all the crucial properties of the latter object. We choose to illustrate this fact through the example of the outermost circuit in a constrained subcritical random-cluster model, a setting that was introduced by Hammond in [Ham11, Ham12a, Ham12b].

5.1 Definition of the random-cluster model

We quickly recall the definition of the random-cluster model (also known as FK percolation) and give a few of its basic properties (see [DC20] for more details).

The random-cluster model on $\mathbb{Z}^2$ is a model of random sub-graphs of $\mathbb{Z}^2$. Its law is described by two parameters, $p \in [0,1]$ and $q > 0$. We start by defining it on a finite graph. Let $G = (V, E)$ be a finite sub-graph of $\mathbb{Z}^2$. The *boundary* of G is defined by

$$\partial G := \left\{ x \in V, \exists y \notin V, \{x, y\} \in E(\mathbb{Z}^2) \right\}. \tag{5.1}$$

A *percolation configuration* on G is an element ω of $\{0,1\}^E$. Each edge of G takes the value 0 or 1. We say that an edge $e \in G$ is *open* if $\omega(e) = 1$ and is *closed* otherwise. Two vertices $x, y \in G$ are said to be connected in ω if there exists a collection of vertices of G $x = x_0, x_1, \ldots, x_n = y$ such that for all $0 \leq i \leq n-1$, $\{x_i, x_{i+1}\} \in E$ and $\omega(\{x_i, x_{i+1}\}) = 1$. We call this event $\{x \leftrightarrow y\}$. A *cluster* of ω is a maximal connected component of the set of vertices (it can be an isolated vertex). Given a percolation configuration ω, we let $o(\omega)$ denote its number of open edges, and let $k(\omega)$ denote its number of vertex clusters. A *boundary condition* on G is a partition $\eta = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k$ of ∂G. From a configuration $\omega \in \{0,1\}^E$, we create a configuration ω^η by identifying together the vertices that belong to the same set $\mathcal{P}_i$ of η. We shall call the *free boundary* condition (resp. wired boundary condition) the partition made of singletons (resp. of the whole set ∂G). We will denote it $\eta = 0$ (resp. $\eta = 1$).

Definition 5.1. Let $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^2$, and η be a boundary condition on G. Let $p \in [0,1]$ and $q > 0$. The random-cluster measure on G with boundary condition η is the following measure on $\{0,1\}^E$,

$$\phi_{p,q,G}^\eta(\omega) = \frac{1}{Z_{p,q,G}^\eta} \left(\frac{p}{1-p} \right)^{o(\omega)} q^{k(\omega^\eta)}, \tag{5.2}$$

where $Z_{p,q,G}^\eta > 0$ is the normalisation constant ensuring that $\phi_{p,q,G}^\eta$ is a probability measure.

For $\eta = 0$ and $\eta = 1$, this measure can actually be extended to the whole $\mathbb{Z}^2$ by taking the weak limit of the measures ϕ_{p,q,G_n}^η over any exhaustion $(G_n)_{n \in \mathbb{N}}$ of $\mathbb{Z}^2$. Moreover, the limit measure does not depend on the choice of the exhaustion. Below, we will simply write $\phi_{p,q}^\eta$ instead of $\phi_{p,q,\mathbb{Z}^2}^\eta$. A fundamental feature of this model is that it undergoes a *phase transition*: for any $q \geq 1$, there exists a critical parameter $p_c = p_c(q) \in (0,1)$ such that:

- $\forall p < p_c(q), \phi_{p,q}^1(0 \leftrightarrow \infty) = 0$.

- $\forall p > p_c(q), \phi_{p,q}^0(0 \leftrightarrow \infty) > 0$.

We will focus on the first case — called *the subcritical regime*. In this regime, it is classical that the choice of boundary conditions does not affect the infinite volume measure. Hence, we drop η from the notation and simply write $\phi_{p,q}$ for the unique infinite volume measure when $p < p_c(q)$. Finally, in the subcritical regime, the following limit — called *the correlation length* — exists and is strictly positive: for any $x \in \mathbb{S}^1$,

$$\xi_{p,q}(x) := -\lim_{n \to \infty} \frac{n}{\log \phi_{p,q}[0 \leftrightarrow \lfloor nx \rfloor]} > 0. \tag{5.3}$$

5.2 Extension of the results to the random-cluster model

For the rest of this section, let us fix $q \geq 1$ and $0 < p < p_c(q)$. Due to the exponential decay of the size of the cluster of 0, it is easy to see that almost surely, the outermost open circuit surrounding 0 is well-defined. Following [Ham12a], we call it Γ_0. Γ_0 sampled according to $\phi_{p,q}$ has to be seen as the analog of Γ sampled according to $\mathbb{P}_\lambda$ in our model. It remains to introduce

the analog of the event $\{\mathcal{A}(\Gamma) \geq N^2\}$. Indeed introduce the area trapped by the circuit $\mathcal{A}(\Gamma_0)$, and define the conditioned measure $\phi_{p,q}^{N^2} := \phi_{p,q} \left[\, \cdot \mid \mathcal{A}(\Gamma_0) \geq N^2 \right]$. This measure is the analog of $\mathbb{P}_\lambda^{N^2}$. We now compare the properties of Γ sampled according to $\mathbb{P}_\lambda^{N^2}$ with the properties of Γ_0 sampled according to $\phi_{p,q}^{N^2}$.

Global curvature: the Wulff shape. The first essential feature exhibited by $\mathbb{P}_\lambda^{N^2}$ is the convergence of its sample paths at the macroscopic scale towards a deterministic curve given by the solution of a variational problem. This is a very classical result in statistical mechanics known as the *Wulff phenomenon* (see [DKS92, Cer06] for detailed monographs about this theory). Define the following compact set:

$$\mathcal{W} = \nu \bigcap_{u \in \mathbb{S}^1} \{t \in \mathbb{R}^2, \langle t, u \rangle \leq \xi_{p,q}^{-1}(u)\}, \tag{5.4}$$

with the constant $\nu > 0$ being chosen so that the set $\mathcal{W}$ is of volume 1 (and $\langle \cdot, \cdot \rangle$ denoting the usual scalar product on $\mathbb{R}^2$). The boundary $\partial \mathcal{W}$ of the Wulff shape plays the role of the limit shape f_λ in our model. Indeed, one has the following result, which is the exact analog of Theorem 1.6, and appears in [Ham12a, Proposition 1]: for any $\varepsilon > 0$,

$$\phi_{p,q}^{N^2}[d_\mathcal{H}\left(\partial \mathcal{W}, N^{-1}\Gamma_0\right) > \varepsilon] \xrightarrow[n \to \infty]{} 0, \tag{5.5}$$

where $d_\mathcal{H}$ is the Hausdorff distance on compact sets of $\mathbb{R}^2$.

Moreover, we used at several occasions the strict concavity (and differentiability) of f_λ (this is more or less the content of Lemma 2.5): it is well known (see [CIV08, Theorem B]), that $\mathcal{W}$ is indeed a strictly convex set and that its boundary $\partial \mathcal{W}$ is analytic.

Local Brownian nature of the interface. A second dramatic feature of the model is the Brownian nature of the interface under a scaling of the type $N^{-1/3}\Gamma(tN^{2/3})$. Indeed, Donsker's invariance principle is heavily used in Subsection 3.2 to argue that the area captured by Γ in a cone of angular opening of order $N^{-1/3}$ is of order at most βN, with some Gaussian tails on β. This is also the case in the random-cluster model, and is an instance of the celebrated *Ornstein–Zernike theory.* Indeed, it is known since the breakthrough work of [CIV08] that a subcritical cluster of FK percolation conditioned to link two distant points is asymptotically Brownian whenever the distance between the points goes to infinity. Hence, an analog of Donsker's invariance principle holds for a percolation interface at the scale $(N^{1/3}, N^{2/3})$, and the arguments of Subsection 3.2 can be reproduced *mutatis mutandi.* For precise statements, see [CIV08, Theorem A, Theorem C] for the local Gaussian asymptotic for a subcritical cluster and the invariance principle towards a Brownian bridge for a rescaled subcritical cluster.

Brownian Gibbs property in the random-cluster model. Perhaps the most essential tool used through the work is what was referred to — according to [CH14] — as the *Brownian Gibbs property* of the model, stating that if one forgets some portion of the walk Γ, then conditionally on the remaining portion of Γ, the distribution of the erased part is simply the distribution of a random walk conditioned to link both parts of the remaining non-erased walk Γ, conditioned on the event that the total enclosed area is at least N^2.

For the random-cluster model, the existence of this Brownian Gibbs property is less clear, for two distinct reasons. The first reason is that Γ_0 lacks — at least locally — the oriented structure of our random walk model. Thus, due to possible backtracks of Γ_0 it is not *a priori* clear that one is able to perform the resampling operation between any two points, which could be a possible obstruction to the strategy described in this work. The second possible obstacle to the Brownian Gibbs property is the lack of independence: indeed the Spatial Markov property of the random-cluster model (see [DC20]) is the exact analog of the Brownian Gibbs property. However, opposed to our random walk model, one has to take into consideration the *boundary conditions* enforced by the conditioning on the circuit outside of some fixed region. This could be

a problem, as our strategy strongly relies on independence of the resampled piece — conditionally on the area constraint.

These two obstructions have successfully been overcome by Hammond in [Ham11]. We refer to this work to observe that one can indeed implement the resampling strategy in the context of the random-cluster model. However, in the remainder of this section, we quickly explain the strategy to overcome the two difficulties pointed out previously.

(i) **Renewal structure and regularity of the outermost circuit.** It was observed in [Ham11] that the first difficulty can be overcome by implementing the resampling strategy only between *renewal points* of the outermost circuit. For the precise definition of these points, see [Ham11, Definition 1.9]: these are precisely the sites where the circuit can be resampled without any backtrack problem. Then, the main result of [Ham11], stated in Theorem 1.1, is that the probability of not seeing such a point in a cone of angular opening uN^{-1} decays faster than $\exp(-u^2)$. Since all our resamplings take place in sectors of angular opening of order $N^{-1/3}$, one sees that imposing that the resampling occurs between renewals does not change the scaling of the quantities that we are computing.

(ii) **Exponential mixing versus independence.** The second obstruction is ruled out by the exponential mixing property of subcritical random-cluster models (see [DC20]). Again, since our reasoning takes place at scales of order $N^{1/3}$ and the latter property ensures that random-cluster configurations decorrelate at a polynomial rate between spatial scales of logarithmic order, the lack of independence is not a problem in our setting (see the discussion in [Ham12a, Section 2.2]).

We hope to have convinced the careful reader that our techniques, even though written in a simple context, are robust and suitable to the analysis of any subcritical statistical mechanics models exhibiting the features discussed above (which in turn should be the case at least in a wide range of models). Finally let us conclude this section by mentioning that thanks to the well-known Edwards–Sokal coupling, this analysis might possibly allow the identification of the fluctuations scale of the facets of a droplet in a supercritical Potts model with q colours, which is maybe a more physically appealing conclusion.

References

[Ale01]   Kenneth S. Alexander, *Cube-root boundary fluctuations for droplets in random cluster models*, Comm. Math. Phys. **224** (2001), no. 3, 733–781. MR 1871907

[BIV00]   T. Bodineau, D. Ioffe, and Y. Velenik, *Rigorous probabilistic analysis of equilibrium crystal shapes*, J. Math. Phys. **41** (2000), no. 3, 1033–1098, Probabilistic techniques in equilibrium and nonequilibrium statistical physics. MR 1757951

[BP11]   Fadoua Balabdaoui and Jim Pitman, *The distribution of the maximal difference between a Brownian bridge and its concave majorant*, Bernoulli **17** (2011), no. 1, 466–483. MR 2798000

[CC13]   Francesco Caravenna and Loïc Chaumont, *An invariance principle for random walk bridges conditioned to stay positive*, Electron. J. Probab. **18** (2013), no. 60, 32. MR 3068391

[Cer06]   R. Cerf, *The Wulff crystal in Ising and percolation models*, Lecture Notes in Mathematics, vol. 1878, Springer-Verlag, Berlin, 2006, Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With a foreword by Jean Picard. MR 2241754

[CH14]   Ivan Corwin and Alan Hammond, *Brownian Gibbs property for Airy line ensembles*, Invent. Math. **195** (2014), no. 2, 441–508. MR 3152753

[Chu75]   Kai Lai Chung, *Maxima in Brownian excursions*, Bull. Amer. Math. Soc. **81** (1975), 742–745. MR 373035

[CIV08]   Massimo Campanino, Dmitry Ioffe, and Yvan Velenik, *Fluctuation theory of connectivities for subcritical random cluster models*, Ann. Probab. **36** (2008), no. 4, 1287–1321. MR 2435850

[CKL23]   Patrizio Caddeo, Yujin H. Kim, and Eyal Lubetzky, *On level line fluctuations of SOS surfaces above a wall*, arXiv preprint, arXiv:2309.09106 (2023).

[CLM+16] Pietro Caputo, Eyal Lubetzky, Fabio Martinelli, Allan Sly, and Fabio Lucio Toninelli, *Scaling limit and cube-root fluctuations in SOS surfaces above a wall*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 5, 931–995. MR 3500829

[CP00] Raphaël Cerf and Ágoston Pisztora, *On the Wulff crystal in the Ising model*, Ann. Probab. **28** (2000), no. 3, 947–1017. MR 1797302

[dA92] A. de Acosta, *Moderate deviations and associated Laplace approximations for sums of independent random vectors*, Trans. Amer. Math. Soc. **329** (1992), no. 1, 357–375. MR 1046015

[DC20] Hugo Duminil-Copin, *Lectures on the Ising and Potts models on the hypercubic lattice*, Random graphs, phase transitions, and the Gaussian free field, Springer Proc. Math. Stat., vol. 304, Springer, Cham, [2020] ©2020, pp. 35–161. MR 4043224

[DH96] R. Dobrushin and O. Hryniv, *Fluctuations of shapes of large areas under paths of random walks*, Probab. Theory Related Fields **105** (1996), no. 4, 423–458. MR 1402652

[DH97] _____, *Fluctuations of the phase boundary in the 2D Ising ferromagnet*, Comm. Math. Phys. **189** (1997), no. 2, 395–445. MR 1480026

[DKS92] Roland Dobrushin, Roman Kotecký, and Senya Shlosman, *Wulff construction: a global shape from local interactions*, Translations of mathematical monographs, vol. 104, American Mathematical Society, Providence, R. I, 1992.

[DZ10] Amir Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, Stochastic Modelling and Applied Probability, vol. 38, Springer-Verlag, Berlin, 2010, Corrected reprint of the second (1998) edition. MR 2571413

[GG21] Shirshendu Ganguly and Reza Gheissari, *Local and global geometry of the 2D Ising interface in critical prewetting*, Ann. Probab. **49** (2021), no. 4, 2076–2140. MR 4260475

[Gri10] Geoffrey Grimmett, *Probability on graphs: Random processes on graphs and lattices*, Institute of Mathematical Statistics Textbooks, no. Book 1, Cambridge University Press, Cambridge, 2010.

[Gro83] Piet Groeneboom, *The concave majorant of Brownian motion*, Ann. Probab. **11** (1983), no. 4, 1016–1027. MR 714964

[Ham11] Alan Hammond, *Phase separation in random cluster models III: circuit regularity*, J. Stat. Phys. **142** (2011), no. 2, 229–276. MR 2764125

[Ham12a] _____, *Phase separation in random cluster models I: Uniform upper bounds on local deviation*, Comm. Math. Phys. **310** (2012), no. 2, 455–509. MR 2890306

[Ham12b] _____, *Phase separation in random cluster models II: The droplet at equilibrium, and local deviation lower bounds*, Ann. Probab. **40** (2012), no. 3, 921–978. MR 2962083

[Ham19] _____, *Minerva lectures at Columbia university, February–March 2019: Universality in models of local random growth via Gibbs resamplings*, 2019.

[HP08] Alan Hammond and Yuval Peres, *Fluctuation of a planar Brownian loop capturing a large area*, Trans. Amer. Math. Soc. **360** (2008), no. 12, 6197–6230. MR 2434284

[HV04] O. Hryniv and Y. Velenik, *Universality of critical behaviour in a class of recurrent random walks*, Probab. Theory Related Fields **130** (2004), no. 2, 222–258. MR 2093763

[IOSV22] Dmitry Ioffe, Sébastien Ott, Senya Shlosman, and Yvan Velenik, *Critical prewetting in the 2D Ising model*, Ann. Probab. **50** (2022), no. 3, 1127–1172. MR 4413213

[ISV15] Dmitry Ioffe, Senya Shlosman, and Yvan Velenik, *An invariance principle to Ferrari-Spohn diffusions*, Comm. Math. Phys. **336** (2015), no. 2, 905–932. MR 3322390

[Ken18] Richard Kenyon, *Limit shapes and their analytic parameterizations*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 3137–3151. MR 3966526

[Lig68] Thomas M Liggett, *An invariance principle for conditioned sums of independent random variables*, Journal of Mathematics and Mechanics **18** (1968), no. 6, 559–570.

[LJ07] Guy Louchard and Svante Janson, *Tail estimates for the Brownian excursion area and other Brownian areas*, Electron. J. Probab. **12** (2007), no. 58, 1600–1632. MR 2365879

[Oko16] Andrei Okounkov, *Limit shapes, real and imagined*, Bull. Amer. Math. Soc. (N.S.) **53** (2016), no. 2, 187–216. MR 3474306

[Sui01] T. M. Suidan, *Convex minorants of random walks and Brownian motion*, Teor. Veroyatnost. i Primenen. **46** (2001), no. 3, 498–512. MR 1978665
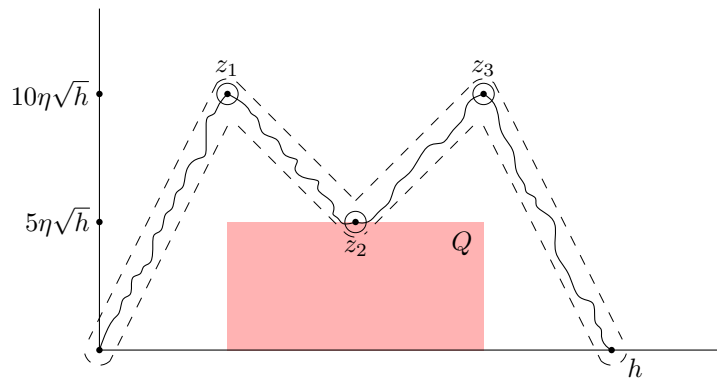
Figure 8: Illustration of the event Shape introduced in the proof of Lemma 3.4. The θ-path is allowed to fluctuate in the region surrounded by the dashed line.

[UA03]   Hasan B. Uzun and Kenneth S. Alexander, *Lower bounds for boundary roughness for droplets in Bernoulli percolation*, Probab. Theory Related Fields **127** (2003), no. 1, 62–88. MR 2006231

[Vel04]   Y. Velenik, *Entropic repulsion of an interface in an external field*, Probab. Theory Related Fields **129** (2004), no. 1, 83–112. MR 2052864

[Wul01]   Georg Wulff, *Xxv. zur frage der geschwindigkeit des wachsthums und der auflösung der krystallflächen*, Zeitschrift für Kristallographie - Crystalline Materials **34** (1901), 449 – 530.

Appendix A   Computations on the simple random walk

This subsection is devoted to the proofs of Lemmas 3.4, 4.3 and 4.18 . The proofs are adapted from [Ham12a].

Proof of Lemma 3.4. We assume that there exists some $\varepsilon > 0$ such that $\varepsilon \leq \theta(x, y) \leq \pi/2 - \varepsilon$. Let $h = \|x - y\|$. Let $\mathbf{R}$ be the rectangle whose up-left corner is x, down-right corner is y and which has horizontal and vertical sides. Since $\theta(x, y) \in [\varepsilon, \pi/2 - \varepsilon]$, $\mathbf{R}$ is not degenerate. We start by changing the coordinates. We set $z_0 = (0, 0)$, $z_1 = (h/4, 10\eta\sqrt{h})$, $z_2 = (h/2, 5\eta\sqrt{h})$, $z_3 = (3h/4, 10\eta\sqrt{h})$ and $z_4 = (h, 0)$. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plane which maps $[x, y]$ to an horizontal segment and T the translation which maps x to 0. Let $\tilde{\mathbf{R}} = (T \circ \mathbb{R}_\theta)(\mathbf{R})$. To ensure that $z_1, z_3 \in \tilde{\mathbf{R}}$, we set $N_0 = N_0(\varepsilon, \eta)$ large enough so that for $h \geq N_0$,

$$\arctan(40\eta h^{-\frac{1}{2}}) \leq \varepsilon. \tag{A.1}$$

We now assume that $h \geq N_0$. We will need the following definitions.

A θ-*path* is an element of $(T \circ \mathbb{R}_\theta)(\Lambda^{x \to y})$. Given $u, v \in \mathbb{R}^2$ and $\gamma \in \Lambda^{x \to y}$, we say that u and v are θ-*connected* in γ if u and v belong to $(T \circ \mathbb{R}_\theta)(\gamma)$. For $i \in \{0, \ldots, 3\}$, let H_i be the event that z_i and z_{i+1} are θ-connected by a θ-path which fluctuates less than $10\|z_{i+1} - z_i\|^{\frac{1}{2}}$ around the segment $[z_i, z_{i+1}]$. Finally, define the event $\mathsf{Shape} = H_0 \cap H_1 \cap H_2 \cap H_3$, see Figure 8.

Notice that

$$\mathsf{Shape} \subset \mathsf{GAC}\,(x, y, \eta). \tag{A.2}$$

Indeed, let Q be the rectangle $[h/4, 3h/4] \times [0, 5\eta\sqrt{h}]$ (see Figure 8). Then, if γ realises Shape, $(T \circ \mathbb{R}_\theta)^{-1}(Q)$ and γ do not cross, except maybe on a small region of area $O(h)$ around $(T \circ \mathbb{R}_\theta)^{-1}(z_2)$. Hence, provided that N is large enough,

$$
\begin{aligned}
|\mathrm{Enclose}(\gamma \cap \mathbf{A}_{x,y})| &\geq |\mathbf{T}_{0,x,y}| + \frac{5}{2}\eta h^{\frac{3}{2}} - O(h) \\
&\geq |\mathbf{T}_{0,x,y}| + \eta h^{\frac{3}{2}}.
\end{aligned}
$$

Now, it remains to get a good estimate on $\mathbb{P}_{x,y}[\mathsf{Shape}]$. A simple computation coming from Gaussian fluctuations gives that there exists a constant $c > 0$ such that for $i \in \{0, \ldots, 3\}$,

$$\mathbb{P}_{x,y}[H_i] \geq c\mathbb{P}_{x,y}[z_i \leftrightarrow z_{i+1}]^{12}. \tag{A.3}$$

We then have,

$$\begin{aligned}
\mathbb{P}_{x,y}[\mathsf{Shape}] &= \mathbb{P}_{x,y}\left[\bigcap_{i=0}^{3} H_i\right] \\
&\geq c^4 \frac{\prod_{i=0}^{3} |\{\theta\text{-paths from } z_i \text{ to } z_{i+1}\}|}{|\{\theta\text{-paths from } z_0 \text{ to } z_3\}|}.
\end{aligned}$$

Let $a = \cos\theta$ and $b = \sin\theta$. Define

$$\begin{cases} \alpha_0 = \frac{h}{4}a + 10\eta h^{\frac{1}{2}}b \\ \beta_0 = \frac{h}{4}b - 10\eta h^{\frac{1}{2}}a \end{cases} , \qquad \begin{cases} \alpha_1 = \frac{h}{4}a - 5\eta h^{\frac{1}{2}}b \\ \beta_1 = \frac{h}{4}b + 5\eta h^{\frac{1}{2}}a \end{cases} , \tag{A.4}$$

$$\begin{cases} \alpha_2 = \frac{h}{4}a + 5\eta h^{\frac{1}{2}}b \\ \beta_2 = \frac{h}{4}b - 5\eta h^{\frac{1}{2}}a \end{cases} , \quad \begin{cases} \alpha_3 = \frac{h}{4}a - 10\eta h^{\frac{1}{2}}b \\ \beta_3 = \frac{h}{4}b + 10\eta h^{\frac{1}{2}}a \end{cases} , \quad \begin{cases} \alpha_4 = ha \\ \beta_4 = hb \end{cases} . \tag{A.5}$$

It is nothing but a little combinatorial fact that the cardinal of the set of θ-paths linking z_i and z_{i+1} is $\binom{\alpha_i + \beta_i}{\alpha_i}$. So that[13],

$$\frac{\prod_{i=0}^{3} |\{\theta\text{-paths from } z_i \text{ to } z_{i+1}\}|}{|\{\theta\text{-paths from } z_0 \text{ to } z_3\}|} = \frac{\prod_{i=0}^{3} \binom{\alpha_i + \beta_i}{\alpha_i}}{\binom{\alpha_4 + \beta_4}{\alpha_4}}. \tag{A.6}$$

A quite tedious computation using Stirling's estimate yields the existence of some constant $C = C(\varepsilon, \eta) > 0$ such that

$$\mathbb{P}_{x,y}[\mathsf{Shape}] \geq Ch^{-\frac{3}{2}}. \tag{A.7}$$

We are a factor $h^{-3/2}$ away from the desired result. This factor can be removed by considering variants of the event Shape where the vertical coordinates of z_1, z_2, z_3 may differ from the original ones by at most $2\eta h^{1/2}$. Since these variants of Shape are still included in $\mathsf{GAC}\,(x, y, \eta)$ and there being $(2\eta)^3 h^{3/2}$ such events, we get the desired conclusion. We obtained ,

$$\mathbb{P}_{x,y}\left[\mathsf{GAC}\,(x, y, \eta)\right] \geq C(\eta, \varepsilon). \tag{A.8}$$

$\square$

Remark A.1. It is clear that the constant $C(\eta, \varepsilon)$ degenerates when ε goes to 0, which explains the need for the *a priori* estimate given by Proposition 2.5.

Proof of Lemmas 4.3 *and* 4.18. Note that the statement of Lemma 4.18 implies the one of Lemma 4.3. We then focus on the first one. The proof of Lemma 3.4 can be reproduced with a minor change of parameters. Indeed, define this time $z_0 = (0, 0)$, $z_1 = (\frac{h}{4}, 10\eta(h\log h)^{\frac{1}{2}})$, $z_2 = (\frac{h}{2}, 5\eta(h\log h)^{\frac{1}{2}})$, $z_3 = (\frac{3h}{4}, 10\eta(h\log h)^{\frac{1}{2}})$ and $z_4 = (h, 0)$.

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plane which maps $[x, y]$ on an horizontal segment and T the translation which maps x to 0. Let $\tilde{\mathbf{R}} = T \circ R_\theta(\mathbf{R})$. To ensure that $z_1, z_3 \in \tilde{\mathbf{R}}$, we set $N_0 = N_0(\varepsilon, \eta)$ large enough so that for $h \geq N_0$,

$$\arctan(40\eta(h\log h)^{-\frac{1}{2}}) \leq \varepsilon. \tag{A.9}$$

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[12]$\{z_i \leftrightarrow z_{i+1}\}$ is the event that z_i and z_{i+1} are θ-connected.

[13]We omit integer rounding and the fact that there might not be a θ-path from z_i to z_{i+1} (but rather a θ-path from $z_i + B_1$ to $z_{i+1} + B_1$ where B_1 is the unit ball).

Assume that $h \geq N_0$. As previously, let, for $i \in \{0, \ldots, 3\}$, H_i be the event that z_i and z_{i+1} are θ-connected by a θ-path which fluctuates less than $10\|z_{i+1} - z_i\|^{\frac{1}{2}}$ around the segment $[z_i, z_{i+1}]$, and define the event $\mathsf{LogShape} = H_0 \cap H_1 \cap H_2 \cap H_3$. As previously, we notice that

$$\mathsf{LogShape} \subset \mathsf{LogGAC}(x, y, \eta). \tag{A.10}$$

Moreover,

$$\mathsf{LogShape} \subset \mathsf{LogSID}(x, y, \eta). \tag{A.11}$$

Indeed, it is straightforward to check that $(T \circ \mathbb{R}_\theta)^{-1}(z_2)$ accomplishes the born on the local roughness:

$$\begin{aligned} \mathrm{d}((T \circ \mathbb{R}_\theta)^{-1}(z_2), \mathcal{C}(\gamma)) &\geq \mathrm{d}(z_2, [z_1, z_3]) \\ &= 5\eta(h \log h)^{\frac{1}{2}} + O(1). \end{aligned}$$

Finally, the estimate of $\mathbb{P}_{x,y}[\mathsf{LogShape}]$ follows the one of $\mathbb{P}_{x,y}[\mathsf{Shape}]$ conducted in the preceding lemma. Defining $a = \cos \theta, b = \sin \theta$ and

$$\begin{cases} \alpha_0 = \frac{h}{4}a + 10\eta(h \log h)^{\frac{1}{2}}b \\ \beta_0 = \frac{h}{4}b - 10\eta(h \log h)^{\frac{1}{2}}a \end{cases}, \qquad \begin{cases} \alpha_1 = \frac{h}{4}a - 5\eta(h \log h)^{\frac{1}{2}}b \\ \beta_1 = \frac{h}{4}b + 5\eta(h \log h)^{\frac{1}{2}}a \end{cases}, \tag{A.12}$$

$$\begin{cases} \alpha_2 = \frac{h}{4}a + 5\eta(h \log h)^{\frac{1}{2}}b \\ \beta_2 = \frac{h}{4}b - 5\eta(h \log h)^{\frac{1}{2}}a \end{cases}, \quad \begin{cases} \alpha_3 = \frac{h}{4}a - 10\eta(h \log h)^{\frac{1}{2}}b \\ \beta_3 = \frac{h}{4}b + 10\eta(h \log h)^{\frac{1}{2}}a \end{cases}, \quad \begin{cases} \alpha_4 = ha \\ \beta_4 = hb \end{cases}. \tag{A.13}$$

As previously, we can use Stirling's estimate to infer that there exists $C = C(\varepsilon) > 0$ such that

$$\mathbb{P}_{x,y}[\mathsf{LogShape}] \geq h^{-\frac{3}{2}} h^{-C\eta^2}. \tag{A.14}$$

We recover the result by considering $h^{3/2}$ variants of the event $\mathsf{LogShape}$, as above. $\qquad \square$

Appendix B Multivalued map principle

If A and B are two finite sets, we say that any function $T : A \to \mathcal{P}(B)$ is a *multivalued map* and for any $b \in B$, we write $T^{-1}(b) = \{a \in A, b \in T(a)\}$.

Lemma B.1 (Probabilistic multivalued map principle). *Let* $(\Omega_1, \mathbb{P}_1), (\Omega_2, \mathbb{P}_2)$ *be two discrete probability spaces, let* A *(resp. B) be a measurable subset of* Ω_1 *(resp. Ω_2) and let* $T : A \to \mathcal{P}(B)$ *be a multivalued map. We assume that the following quantities are finite:*

$$\varphi(T) := \max_{a \in A} \max_{b \in T(a)} \frac{\mathbb{P}_1(a)}{\mathbb{P}_2(b)} < \infty, \tag{B.1}$$

$$\psi(T) := \frac{\max_{b \in B} |T^{-1}(b)|}{\min_{a \in A} |T(a)|} < \infty. \tag{B.2}$$

Then,

$$\mathbb{P}_1[A] \leq \varphi(T)\psi(T)\mathbb{P}_2[B]. \tag{B.3}$$

Proof. The lemma follows by the following simple computation:

$$\begin{aligned} \mathbb{P}_1[A] &= \sum_{a \in A} \sum_{b \in T(a)} \frac{\mathbb{P}_1[a]}{\mathbb{P}_2[b]} \mathbb{P}_2[b] |T(a)|^{-1} \\ &\leq \varphi(T) \sum_{b \in B} \mathbb{P}_2[b] \sum_{a \in T^{-1}(b)} |T(a)|^{-1} \\ &\leq \varphi(T)\psi(T)\mathbb{P}_2[B]. \end{aligned}$$

$\square$