BUSEMANN-PETTY PROBLEM

1. Wording

Problem ([3]). Given two convex bodies $K$ and $L$ in $\mathbb{R}^n$ centered at the origin such that every central hyperplane section of $K$ has an area less than or equal to the area of the section of $L$ by the same hyperplane, does it follow that $vol(K) \leq vol(L)$?

Answer. The answer is positive for $n \leq 4$, negative for $n > 5$.

Final step given in [23].

The dual problem, known as Shephard’s problem, is the corresponding problem replacing the areas of sections by areas of projections on hyperplane. It was solved independently by Petty and Schneider ([18] and [19]).

2. History: examples and counter-examples

(1) Busemann [5] and [4]
(2) Larman & Rogers [14]
(3) Ball [1] and [2]

3. Better context, conditions for a positive answer

(1) Definitions of dual mixed volumes and intersection bodies in [15] as well as an “almost characterization” of bodies giving a positive answer.
(2) Generalizations of intersection bodies in [7] and [20]. Complete characterization for a positive answer in [20].

4. Final solution

(1) $\mathbb{R}^3$: [8]
(2) $\mathbb{R}^4$: [21], [12] and [23]

5. Fourier analytic viewpoint

(1) Short version: [9]
(2) Detailed version: [13]

6. Different generalizations

(1) Intersection body and intersection body of a starshape body: [20]. See also [24] and [6] for the case of polytopes.
(2) Sections by planes with higher codimensions: [10], [22], [16] and [17].

References


