

BUSEMANN-PETTY PROBLEM

1. WORDING

Problem ([3]). Given two convex bodies K and L in \mathbb{R}^n centered at the origin such that every central hyperplane section of K has an area less than or equal to the area of the section of L by the same hyperplane, does it follow that $\text{vol}(K) \leq \text{vol}(L)$?

Answer. The answer is positive for $n \leq 4$, negative for $n > 5$.

Final step given in [23].

The dual problem, known as *Shepard's problem*, is the corresponding problem replacing the areas of sections by areas of projections on hyperplane. It was solved independently by Petty and Schneider ([18] and [19]).

2. HISTORY: EXAMPLES AND COUNTER-EXAMPLES

- (1) Busemann [5] and [4]
- (2) Larman & Rogers [14]
- (3) Ball [1] and [2]

3. BETTER CONTEXT, CONDITIONS FOR A POSITIVE ANSWER

- (1) Definitions of *dual mixed volumes* and *intersection bodies* in [15] as well as an “almost characterization” of bodies giving a positive answer.
- (2) Generalizations of intersection bodies in [7] and [20]. Complete characterization for a positive answer in [20].

4. FINAL SOLUTION

- (1) \mathbb{R}^3 : [8]
- (2) \mathbb{R}^4 : [21], [12] and [23]

5. FOURIER ANALYTIC VIEWPOINT

- (1) Short version: [9]
- (2) Detailed version: [13]

6. DIFFERENT GENERALIZATIONS

- (1) Intersection body and intersection body of a starshape body: [20]. See also [24] and [6] for the case of polytopes.
- (2) Sections by planes with higher codimensions: [10], [22], [16] and [17].
- (3) Complex Busemann-Petty problem: [11]

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