

CONVEXITY OF L_p -INTERSECTION BODIES

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ABSTRACT. We extend the classical Brunn theorem to *symmetric moments* of convex bodies and use it to prove the convexity of the L_p -intersection bodies of centered convex bodies.

1. INTRODUCTION

Since Lutwak introduced them in his leading paper [18], intersection bodies and their generalizations received a fast growing attention while naturally appearing in various contexts. As major outcome, this concept led to the solution of the Busemann-Petty problem: in a 4 (or less) dimensional space, if the intersection bodies $\mathbb{I}(K)$ and $\mathbb{I}(L)$ of two centered convex bodies K and L satisfy $\mathbb{I}(K) \subset \mathbb{I}(L)$, then necessarily $\text{Vol}(K) \leq \text{Vol}(L)$ (see e.g. [5], [22] and [8]).

From a valuation theory perspective, Ludwig characterized in [17] the intersection body operator as the only non trivial $Gl(n)$ -contravariant star body valued valuation for the radial sum. Haberl and Ludwig then extended this result for the L_p -intersection body operator and the p -radial sum (see [12]). One should note that this last result characterizes the L_p -intersection body operator amongst valuations with range in the set of *centered* bodies.

The L_p -intersection bodies appeared in different contexts and, as usual in such a case, have been known under different names. In particular, it turns out that up to normalization the L_p -intersection body of a star body K coincides with $\Gamma_{-p}^*(K)$, its polar $-p$ -centroid body. Polar p -centroid bodies were notably used to derive new affine isoperimetric inequalities (see [20]). In particular, for *centered* bodies, these inequalities embed the well known Blaschke-Santaló inequality in a whole family of inequalities comparing the volume of a body with the volumes of its polar p -centroid bodies. These inequalities were then strengthened in [19] providing the whole family of L^p -Busemann-Petty centroid inequalities (see also [4]). Polar p -centroid bodies are also the geometric core of the solution to the L_p -Busemann-Petty problem (see [21]).

Non-symmetric L_p -intersection body operators were introduced in [11]. They coincide with the classical operators on the set of centered star bodies, but have been shown to be the right concept to deal with non-symmetric bodies. In particular the result of Haberl and Ludwig on valuations extends naturally to these operators if one removes the symmetry assumption (see [11]). Also, non-symmetric p -centroid and projection bodies were used in [13] to sharpen previously known L_p -Petty projection and L_p -Busemann-Petty centroid inequalities.

The notion of L_p -intersection bodies used in this paper will be a slight modification of the concept of *non-symmetric* L_p -intersection bodies I_p^+ from [11] (see

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Section 2 for a precise definition). Allowing the non-symmetry will ease the description of L_p -intersection bodies in terms of distributions, while changing the sign of the parameter p implies that L_p -intersection bodies of symmetric bodies then coincide up to normalization with polar p -centroid bodies.

Note that the concept of L_p -intersection bodies should not be mixed up with the one of k -intersection bodies (see e.g. [16]).

Up to normalization, the L_p -intersection bodies converge to the classical intersection body as p goes to -1 (see [7, Proposition 3.1] or [11, Theorem 1]). However, while it is well known, and easy to prove, that the L_p -intersection bodies of *star* bodies are convex if $p \geq 1$, the proof that intersection bodies of *centered convex* bodies are convex is more involved and is due to Busemann (see [3]). This classical result has a major consequence on the minimization properties of the Hausdorff area in normed and Finsler spaces: it is indeed equivalent to the fact that affine disks of codimension 1 have less Hausdorff area than hypersurfaces with same boundaries (see [2], or [1] for a modern treatment).

The problem of the convexity of the L_p -intersection bodies for $-1 < p < 1$, $p \neq 0$ was still open. Our aim is to give a unified proof for the whole range of p 's.

Theorem. *For $-1 \leq p \neq 0$, the L_p -intersection body of a centered convex body is a centered convex body.*

Busemann's proof for the classical intersection body uses the Brunn-Minkowski inequality. To extend his result to L_p -intersection bodies, we use a generalization of this classical inequality to *moments* \mathcal{M}_p of convex bodies that are defined as follows: for η a linear form which is non-negative on a convex body K , $p \geq 0$, and Ω a constant volume form,

$$\mathcal{M}_p(K) := \int_{x \in K} (\eta \cdot x)^p \Omega \quad .$$

Theorem (Hadwiger). *Let K_0, K_1 be two convex bodies in an n -dimensional space V and $\eta \in V^*$ a linear form which is non-negative on both bodies. Then, for every $p \geq 0$,*

$$\mathcal{M}_p(K_0 + K_1)^{\frac{1}{n+p}} \geq \mathcal{M}_p(K_0)^{\frac{1}{n+p}} + \mathcal{M}_p(K_1)^{\frac{1}{n+p}} \quad .$$

This theorem can be found in [14, p.266]. The Brunn-Minkowski inequality corresponds to the case $p = 0$. The classical inequality has for geometric consequence that central hyperplane sections of a centered convex body have maximal volume amongst sections with parallel hyperplanes. This is known as Brunn's theorem (see e.g. [16, Theorem 2.3]). Loosely speaking, this result extends stating that symmetric moments of central sections are maximal (see Section 3 for a precise definition); but one must be aware that the linear forms used to define the moments cannot be chosen arbitrarily.

Theorem (Brunn's theorem for moments). *Let K be a centered convex body in an n -dimensional space V , $p \geq 0$, and H_t a family of parallel hyperplanes defined by the equations $\xi \cdot x = t$. Then any linear form η_0 on the hyperplane H_0 may be extended to a linear form η on V such that the $(n-1)$ -dimensional symmetric p^{th} -moment*

$$\mathcal{M}_p^s(K \cap H_t)$$

is maximal at $t = 0$.

This result will play a key role in the proof of the convexity of L_p -intersection bodies.

As it has remarkably been shown in the analytical solution to the Busemann-Petty problem in [8], the distributions are a powerful tool to deal with intersection bodies. They will be used here to give a unified proof of the convexity of L_p -intersection bodies for the whole range $-1 < p \neq 0$. The case $p = -1$ corresponds to the classical Busemann's theorem. However we have deliberately decided to separate the proof of this last theorem from the others and to present it first. The reason is that it only requires the δ distribution which is better known than the distribution s_+^p giving the fractional derivative and needed for the other cases. It also allows us to avoid introducing normalizations that would make the notations more cumbersome. The use of distributions for the proof of our main result also emphasizes the need of the additional assumptions of symmetry and convexity on K for the cases $-1 \leq p < 1$, $p \neq 0$.

Our use of the distributions is somewhat different than in the classical literature on convex geometry, hence we give a small introduction to it beginning of Section 4.

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2. NOTATIONS AND PRELIMINARIES

For a matter of taste, we have chosen to write every formula, definition, ... in an invariant way. Let V be a real n -dimensional vector space and Ω a constant volume form on it. The orientation of V is chosen so that integrals of positive functions are positive. The Greek letter ω will represent a constant, non-zero $(n-1)$ -form on any given hyperplane of V ; if needed this notation will be made more precise adding a subscript. Hyperplanes are oriented with the same convention as V .

As usual, a *star body* $K \subset V$ is a compact subset of V , star-shaped with respect to the origin and with the property that for all $x \in K$ the open segment (o, x) lies in the interior of K . A star body is called *centered* if the origin is a center of symmetry. Except at a very few places in Section 3, *convex bodies* will be assumed to contain the origin in their interior, hence will belong to the set of star bodies.

The *Minkowski functional* of a star or convex body $K \subset V$ is defined by

$$\|x\|_K := \min\{\lambda \geq 0 \mid x \in \lambda K\} \quad .$$

This is a positive, continuous and homogeneous function of degree 1. Clearly, $\|\cdot\|_K$ is a norm if and only if K is a centered convex body. A Minkowski functional will be called *smooth* if it belongs to $\mathcal{C}^\infty(V \setminus \{o\})$. In such a case, the boundary ∂K is a smooth hypersurface of V which is everywhere transversal to the radial direction: $V = T_x \partial K \oplus \mathbb{R} \cdot x$.

We will need the following easy lemma.

Lemma 2.1. *If $\|\cdot\|_K$ is smooth and symmetric, then it is a norm if and only if for all $x \in \partial K$ and all $v \in T_x \partial K$ the following holds true*

$$\frac{d^2}{dt^2} \|x + tv\|_K|_{t=0} \geq 0 \quad .$$

Proof. By hypothesis, it suffices to prove that $\|\cdot\|_K$ is convex, or equivalently that the Hessian $D_x^2\|\cdot\|_K$ is positive semi-definite for every $x \in V$. Since $\|\cdot\|_K$ is homogeneous of degree 1, one has for its first and second derivatives:

$$\forall \lambda > 0, \quad D_{\lambda x} \|\cdot\|_K = D_x \|\cdot\|_K \quad \text{and} \quad D_{\lambda x}^2 \|\cdot\|_K = \lambda^{-1} D_x^2 \|\cdot\|_K \quad .$$

As consequence of the second equality, one may consider $x \in \partial K$ to check if the Hessian is positive semi-definite. Deriving the first equality gives that the radial direction is in the kernel of the Hessian:

$$\forall x \in \partial K, \quad \forall v \in V, \quad x^t \cdot D_x^2 \|\cdot\|_K \cdot v = 0 \quad .$$

Therefore, the Hessian is positive semi-definite if and only if its restriction to the tangent $T_x \partial K$ is positive semi-definite. \square

The L_p -intersection bodies appeared in different contexts and as is usual in that case have been known and studied under different names. Up to normalization and a slight change for the parameter p , the notion of L_p -intersection bodies used here coincides with the notion of non-symmetric L_p -intersection bodies $I_p^+ K$ of [11]. Also, since we do not use any inner product, L_p -intersection bodies are naturally subsets of the dual space V^* .

We use the following notation: for a non-zero linear form $\xi \in V^*$, the set $\{x \in K \mid \xi \cdot x \geq 0\}$ is denoted by K_ξ^+ .

Definition 2.2. Let $K \subset V$ be a star body. For $-1 < p \neq 0$, the L_p -intersection body of K is the star body $\mathbb{I}_p K \subset V^*$ whose Minkowski functional is defined by

$$\|\xi\|_{\mathbb{I}_p K} = \left(\int_{x \in K_\xi^+} (\xi \cdot x)^p \Omega \right)^{\frac{1}{p}} \quad .$$

One easily checks that $\mathbb{I}_p K = \Gamma(1-p) \cdot I_{-p}^+$. Moreover, if K is centered, then $\mathbb{I}_p K$ is also the polar p -centroid body up to normalization:

$$\mathbb{I}_p K = \left(\frac{\text{Vol}(K)}{2} \right)^{\frac{1}{p}} \Gamma_p^* K \quad ,$$

with the normalization of [7], see also [19].

For every non-zero linear form ξ , there exists an $(n-1)$ -form ω_ξ well defined on the hyperplanes $\xi \cdot x = cst$ and such that $\Omega = \xi \wedge \omega_\xi$. Obviously, the map $\xi \mapsto \omega_\xi$ is homogeneous of degree -1 .

Definition 2.3. Let $K \subset V$ be a star body. The intersection body of K is the star body $\mathbb{I}K \subset V^*$ whose Minkowski functional is defined by

$$\|\xi\|_{\mathbb{I}K} = \left(\int_{K \cap \ker \xi} \omega_\xi \right)^{-1} \quad .$$

It was first proved in [7, Proposition 3.1] that up to normalization $\mathbb{I}_p K$ goes to $\mathbb{I}K$ as p goes to -1 (see also [11, Theorem 1]). We will easily recover this result after having translated both previous definitions in terms of distributions.

3. BRUNN-MINKOWSKI INEQUALITIES FOR MOMENTS

In this section we present Hadwiger's extension of the classical Brunn-Minkowski inequality and give a proof of it for the sake of completeness. We then derive a generalization of Brunn's theorem to moments of convex bodies.

Let K be a convex body in an n -dimensional vector space V equipped with a constant volume form Ω . Let η be a linear form which is non-negative on K and $p \geq 0$. The p^{th} -moment of the convex body K with respect to η is the following quantity:

$$\mathcal{M}_p(K) := \int_{x \in K} (\eta \cdot x)^p \Omega \quad .$$

Theorem 3.1 (Brunn-Minkowski inequality for moments). *Let K_0, K_1 be two convex bodies in V and $\eta \in V^*$ a linear form which is non-negative on both bodies. Then, for every $p \geq 0$*

$$\mathcal{M}_p(K_0 + K_1)^{\frac{1}{n+p}} \geq \mathcal{M}_p(K_0)^{\frac{1}{n+p}} + \mathcal{M}_p(K_1)^{\frac{1}{n+p}} \quad .$$

The Brunn-Minkowski inequality corresponds to $p = 0$. The proof of this theorem mimics the proof of the classical inequality given in [6]. We will use for it two known inequalities which we briefly recall (see [6, pp. 362-368]).

Proposition 3.2 (Prékopa-Leindler inequality). *Let $0 < \lambda < 1$ and let f, g and h be non-negative integrable functions on \mathbb{R}^n satisfying*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda \quad \forall x, y \in \mathbb{R}^n \quad .$$

Then

$$\int_{\mathbb{R}^n} h(x) dx \geq \left(\int_{\mathbb{R}^n} f(x) dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x) dx \right)^\lambda \quad .$$

For $0 < \lambda < 1$ and $p \neq 0$, the p -mean of two non-negative real numbers a and b is defined by

$$m_p(a, b, \lambda) := ((1-\lambda)a^p + \lambda b^p)^{\frac{1}{p}}$$

if $ab \neq 0$ and $m_p(a, b, \lambda) = 0$ if not. One also defines

$$m_{-\infty}(a, b, \lambda) := \min\{a, b\}, \quad m_0(a, b, \lambda) := a^{1-\lambda} b^\lambda \quad \text{and} \quad m_\infty(a, b, \lambda) := \max\{a, b\} \quad .$$

Proposition 3.3 (p -means inequality). *For $-\infty \leq p < q \leq \infty$,*

$$m_p(a, b, \lambda) \leq m_q(a, b, \lambda) \quad .$$

Proof of Theorem 3.1. We assume first that both $\mathcal{M}_p(K_i) \neq 0$, or equivalently $\dim(K_i) = n$. Denote by H_t the hyperplanes defined by the equations $\eta \cdot x = t$. Let ω be a constant $(n-1)$ -form on V such that $\Omega = \eta \wedge \omega$. Note that the restriction of ω to H_t is uniquely defined. We also define $K_\lambda = (1-\lambda)K_0 + \lambda K_1$ and $A(t, \lambda) = \int_{H_t \cap K_\lambda} \omega$ for any $\lambda \in (0, 1)$. The classical Brunn-Minkowski then ensures that $\forall t_0, t_1 \geq 0$

$$A((1-\lambda)t_0 + \lambda t_1, \lambda)^{\frac{1}{n-1}} \geq (1-\lambda)A(t_0, 0)^{\frac{1}{n-1}} + \lambda A(t_1, 1)^{\frac{1}{n-1}} \quad .$$

Therefore, by the p -mean inequality, one has

$$A((1-\lambda)t_0 + \lambda t_1, \lambda) \geq A(t_0, 0)^{1-\lambda} \cdot A(t_1, 1)^\lambda \quad .$$

Another simple use of the p -mean inequality then gives that $\forall t_0, t_1 \geq 0$ and $\forall p \geq 0$

$$((1-\lambda)t_0 + \lambda t_1)^p A((1-\lambda)t_0 + \lambda t_1, \lambda) \geq (t_0^p A(t_0, 0))^{1-\lambda} \cdot (t_1^p A(t_1, 1))^\lambda \quad .$$

Let

$$B(t, \lambda) = \begin{cases} t^p A(t, \lambda) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The previous inequality ensures that for any $\lambda \in (0, 1)$ the functions $h = B(\cdot, \lambda)$, $f = B(\cdot, 0)$ and $g = B(\cdot, 1)$ satisfy the hypothesis of the Prékopa-Leindler inequality. Therefore

$$\mathcal{M}_p(K_\lambda) \geq \mathcal{M}_p(K_0)^{1-\lambda} \mathcal{M}_p(K_1)^\lambda \geq \min\{\mathcal{M}_p(K_0), \mathcal{M}_p(K_1)\} \quad .$$

Note finally that the p^{th} -moment is positively homogeneous of degree $n + p$, therefore the result follows applying the last inequality to

$$\bar{K}_0 = \frac{K_0}{\mathcal{M}_p(K_0)^{\frac{1}{n+p}}}, \quad \bar{K}_1 = \frac{K_1}{\mathcal{M}_p(K_1)^{\frac{1}{n+p}}} \quad \text{and} \quad \lambda = \frac{\mathcal{M}_p(K_1)^{\frac{1}{n+p}}}{\mathcal{M}_p(K_0)^{\frac{1}{n+p}} + \mathcal{M}_p(K_1)^{\frac{1}{n+p}}} \quad .$$

If $\dim K_0, \dim K_1 < n$, then $\mathcal{M}_p(K_0) = \mathcal{M}_p(K_1) = 0$ and the inequality holds.

If, say, $\dim(K_0) < n$ and $\dim(K_1) = n$, then $\mathcal{M}_p(K_0) = 0$ and for any fixed $x \in K_0$,

$$K_0 + K_1 \supseteq x + K_1 \quad .$$

Therefore, since for all $y \in K_1$ one has $\eta \cdot (x + y) \geq \eta \cdot y$,

$$\mathcal{M}_p(K_0 + K_1)^{\frac{1}{n+p}} \geq \mathcal{M}_p(x + K_1)^{\frac{1}{n+p}} \geq \mathcal{M}_p(K_1)^{\frac{1}{n+p}} \quad .$$

□

Since the p^{th} -moment of any n -dimensional convex body is positively homogeneous of degree $n + p$, it directly follows that for any $\lambda \in [0, 1]$,

$$\mathcal{M}_p((1 - \lambda)K_0 + \lambda K_1)^{\frac{1}{n+p}} \geq (1 - \lambda)\mathcal{M}_p(K_0)^{\frac{1}{n+p}} + \lambda\mathcal{M}_p(K_1)^{\frac{1}{n+p}} \quad .$$

Hence the following corollary is a simple consequence of the obvious geometric fact that if K_0 and K_1 are parallel hyperplanes sections at 'height' 0 and 1 of an n -dimensional convex body K , then $(1 - \lambda)K_0 + \lambda K_1$ is contained in the section of K at height λ .

Corollary 3.4. *Let K be a convex body in V , $\eta \in V^*$ a linear form which is non-negative on K and H_t a family of parallel hyperplanes defined by the equations $\xi \cdot x = t$ for some fixed linear form ξ . Then, calling $\mathcal{M}_p(t)$ the p^{th} -moment of the section $K_t := K \cap H_t$, the function $\mathcal{M}_p(t)^{\frac{1}{n-1+p}}$ is concave on the set of t 's for which $K_t \neq \emptyset$.*

To study the convexity of intersection bodies in the next section as well as to extend Brunn's theorem to moments of centered convex bodies, we will have to consider bodies that contain the origin in their interiors. Obviously the hypothesis that the linear form η is non-negative on these bodies does not hold. To by-pass this problem, we use the *positive* and the *symmetric* p^{th} -moments:

$$\mathcal{M}_p^+(K) := \int_{x \in K_\eta^+} (\eta \cdot x)^p \Omega \quad \text{and} \quad \mathcal{M}_p^s(K) := \int_{x \in K} |\eta \cdot x|^p \Omega \quad .$$

Then Brunn's theorem extends as follows:

Theorem 3.5 (Brunn's theorem for moments). *Let K be a centered convex body in V , $p \geq 0$ and H_t a family of parallel hyperplanes defined by the equations $\xi \cdot x = t$.*

Then any linear form η_0 on the hyperplane H_0 may be extended to a linear form η on V such that the symmetric p^{th} -moment

$$\mathcal{M}_p^s(K \cap H_t)$$

is maximal at $t = 0$.

To prove the theorem, we will need the following lemma:

Lemma 3.6. *Let K be a convex body in V with smooth boundary and containing the origin in its interior, and let $H_t \equiv \xi \cdot x = t$. Then any linear form η_0 on H_0 may be extended to a linear form η on V such that*

$$\frac{d}{dt} \mathcal{M}_p^+(K \cap H_t)|_{t=0} = 0 \quad .$$

The proof of this lemma uses distributions, so we leave it for the end of the next section.

Proof of Theorem 3.5. We assume the boundary of K to be smooth, the general result will follow by approximation. Using Corollary 3.4 where η is the extension of η_0 given by the previous lemma, we obtain that $\mathcal{M}_p^+(t)^{\frac{1}{p-1+p}}$ is concave and maximal at $t = 0$. Hence $\mathcal{M}_p^+(t)$ is also maximal at $t = 0$ (while in general not concave).

The symmetry of K then implies that for the same extension η , the function

$$\mathcal{M}_p^-(t) := \int_{x \in K \cap H_t, \eta \cdot x \leq 0} (-\eta \cdot x)^p \omega$$

is also maximal at $t = 0$. Hence the result follows from the equality

$$\mathcal{M}_p^s(t) = \mathcal{M}_p^+(t) + \mathcal{M}_p^-(t) \quad .$$

□

4. INTERSECTION BODIES

We start this section recalling some basic notions and facts about distributions, and use them later to prove our convexity results. There is no real difference between distribution theory on \mathbb{R}^n or on an oriented vector space V equipped with a constant volume form Ω (see also [15, pp.142-146] or [10, Chapter VI] for a more general setting). Briefly, the spaces of test functions will be $\mathcal{C}_0^\infty(\mathbb{R})$ and $\mathcal{C}_0^\infty(V)$: the spaces of smooth compactly supported functions. These are topologized in the usual way: a sequence of test functions (φ_j) converges to 0 if their supports are all contained in a fixed compact set and if the φ_j 's as well as all their partial derivatives uniformly converge to 0. The integration of functions on \mathbb{R} will be with respect to the standard Lebesgue measure denoted here by ds , while the integration of functions on the oriented vector space V will be with respect to the constant volume form Ω . The spaces of distributions, i.e. continuous linear functionals on the space of test functions, will be denoted by $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}'(V)$.

4.1. Distributions. The main two distributions on \mathbb{R} we will use are δ and s_+^p , the last one being defined for $\Re p > -1$ as

$$(s_+^p, \varphi) := \int_0^\infty s^p \varphi(s) ds, \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}) \quad .$$

It is a well known fact that this distribution may be extended by analytical continuation to all $p \in \mathbb{C} \setminus (-1 \cdot \mathbb{N}^*)$. We refer to [9, pp. 46-52] for this construction and useful formulas, in particular for $-k - 1 < \Re p < -k$, $k \in \mathbb{N}^*$,

$$(1) \quad (s_+^p, \varphi) = \int_0^\infty s^p \left(\varphi(s) - \varphi(0) - s\varphi'(0) - \dots - \frac{s^{k-1}}{(k-1)!} \varphi^{(k-1)}(0) \right) ds \quad .$$

Also,

$$(2) \quad (s_+^p)' := \frac{d}{ds} s_+^p = p s_+^{p-1}$$

and

$$\lim_{p \rightarrow 1} (s_+^p)'' = \delta, \quad \lim_{p \rightarrow 0} (s_+^p)'' = -\delta' \quad .$$

The pull-back of distributions by submersions is well defined (see e.g. [10, Chapter 6, §1]). We briefly recall the construction, specializing it for our needs. Every non-zero linear form $\xi \in V^*$ is by definition a surjective linear map from V to \mathbb{R} . We will use it to pull-back the distributions on \mathbb{R} . However, to make formulas involving such pull-backs more readable, it seems adequate to us to introduce the notation $\pi_\xi: V \rightarrow \mathbb{R}$ for the linear surjection $x \mapsto \xi \cdot x$.

For every non-zero linear form ξ , there exists an $(n-1)$ -form ω_ξ well defined on the hyperplanes $\xi \cdot x = cst$ and such that $\Omega = \xi \wedge \omega_\xi$. As one easily sees, the *Radon transform* of a test function $\varphi \in \mathcal{C}_0^\infty(V)$

$$\mathcal{R}_\xi \varphi(s) := \int_{\pi_\xi^{-1}(s)} \varphi \omega_\xi$$

is a test function on \mathbb{R} . We use it to define the pull-back of distributions:

Definition 4.1. Given a non-zero linear form $\xi \in V^*$, the pull-back $\pi_\xi^* f$ of a distribution $f \in \mathcal{D}'(\mathbb{R})$ is defined by

$$(\pi_\xi^* f, \varphi) := (f, \mathcal{R}_\xi \varphi) \quad .$$

In Proposition 4.2 below which is probably folklore, we compute the directional derivatives of a pull-back with respect to the parameter ξ . To make this operation more precise, let's assume first that f_t , $t \in \mathbb{R}$, is a one-parameter family of distributions. If it exists, we will call (*first*) *derivative of the family* f_t the only distribution g which satisfies

$$\frac{d}{dt} (f_t, \varphi) |_{t=0} = (g, \varphi), \quad \forall \varphi \in \mathcal{C}_0^\infty \quad .$$

Note that one may define higher order derivatives in a similar way.

◀ *Example* Let δ_t be the delta distribution on \mathbb{R} supported at t . Then,

$$(3) \quad \left(\frac{d}{dt} \delta_t \right) |_{t=0} = -\delta' \quad ,$$

since

$$(\delta_t, \varphi) = (\delta, \varphi(\cdot + t)) \quad .$$

►

Proposition 4.2. *Let ξ and η be linear forms with $\xi \neq 0$ and f a distribution on \mathbb{R} . Then for any positive integer q ,*

$$\frac{d^q}{dt^q} (\pi_{\xi+t\eta}^* f) |_{t=0} = \eta^q \cdot \pi_{\xi}^* f^{(q)} \quad ,$$

where $f^{(q)}$ stands for the usual q^{th} -derivative of the fixed distribution f .

Proof. Let $x_0 \in V$ be such that $\xi \cdot x_0 = 1$ and $\eta \cdot x_0 = 0$. Denote by ω the $(n-1)$ -form $\Omega(x_0 \wedge \cdot)$. Then

$$\Omega = (\xi + t\eta) \wedge \omega \quad .$$

Indeed, one obviously has $\Omega = \xi \wedge \omega$ and $\eta \wedge \omega = 0$ since $\eta \cdot x_0 = \omega(x_0 \wedge \cdot) = 0$. Therefore,

$$\mathcal{R}_{\xi+t\eta} \varphi(s) = \int_{\pi_{\xi+t\eta}^{-1}(s)} \varphi \omega \quad .$$

Consider the affine map

$$\begin{aligned} A_t : V &\longrightarrow V \\ x &\longmapsto x + t(\eta \cdot x)x_0 \end{aligned}$$

It maps the hyperplane $\pi_{\xi+t\eta}^{-1}(s)$ on the hyperplane $\pi_{\xi}^{-1}(s)$. Moreover, A_t maps the $(n-1)$ -form ω onto itself. Then,

$$\int_{\pi_{\xi+t\eta}^{-1}(s)} \varphi(x) \omega = \int_{\pi_{\xi}^{-1}(s)} \varphi(x - t(\eta \cdot x)x_0) \omega \quad .$$

Deriving q times this expression with respect to t , one gets

$$\frac{d^q}{dt^q} \mathcal{R}_{\xi+t\eta} \varphi(s) |_{t=0} = (-1)^q \int_{\pi_{\xi}^{-1}(s)} (\eta \cdot x)^q \frac{d^q}{ds^q} \varphi(x + tx_0) |_{t=0} \omega \quad .$$

Since $\eta \cdot x_0 = 0$ and $\xi \cdot x_0 = 1$, this last expression is also equal to

$$(-1)^q \frac{d^q}{ds^q} \mathcal{R}_{\xi}(\eta^q \varphi)(s) \quad .$$

Finally,

$$\frac{d^q}{dt^q} (\pi_{\xi+t\eta}^* f, \varphi) |_{t=0} = \left(f, (-1)^q \frac{d^q}{ds^q} \mathcal{R}_{\xi}(\eta^q \varphi) \right)$$

which is also

$$(\eta^q \pi_{\xi}^* f^{(q)}, \varphi) \quad .$$

□

We now define the product of two pull-backs. Two linearly independent linear forms ξ and η on V define a linear surjection:

$$\begin{aligned} \pi_{\xi \times \eta} : V &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (\xi \cdot x, \eta \cdot x) \end{aligned}$$

Hence one may define the Radon transform $\mathcal{R}_{\xi \times \eta}$ as an integral on codimension-2 planes parallel to $\ker \xi \cap \ker \eta$ in a similar way as we did for hyperplanes, and also define the pull-back $\pi_{\xi \times \eta}^* h \in \mathcal{D}'(V)$ of any distribution h on \mathbb{R}^2 .

In [9, pp. 98-100], the direct product of two distributions f and g on \mathbb{R} is defined as follows:

$$\forall \varphi \in C_0^\infty(\mathbb{R}^2), (f \times g, \varphi) := (f(x), (g(y), \varphi(x, y))) \quad .$$

This leads to the following definition:

Definition 4.3. Let ξ and η be two linearly independent linear forms on V and f, g two distributions on \mathbb{R} . Then the product of the pull-backs of the two distributions is defined by

$$\pi_\xi^* f \cdot \pi_\eta^* g := \pi_{\xi \times \eta}^* f \times g \quad .$$

Note that since the direct product is commutative, the same holds for the product of pull-backs.

Finally, recall that a distribution on an n -dimensional space is called *homogeneous of degree p* if

$$\forall \alpha > 0, \quad \left(f, \varphi \left(\frac{\cdot}{\alpha} \right) \right) = \alpha^{p+n} (f, \varphi) \quad .$$

On \mathbb{R} , δ is homogeneous of degree -1 and s_+^p of degree p (see [9, pp. 79-82]). The following proposition will be used to express the Minkowski functionals of the L_p -intersection bodies in terms of distributions.

Proposition 4.4. *If f is a distribution homogeneous of degree p , so is its pull-back $\pi_\xi^* f$. Moreover, the map $\xi \mapsto \pi_\xi^* f$ is homogeneous of degree p .*

Proof. Note first that applying the change of variables $y = \alpha^{-1} \cdot x$ in the integral defining the Radon transform, one obtains

$$\mathcal{R}_\xi \left(\varphi \left(\frac{\cdot}{\alpha} \right) \right) (s) = \alpha^{n-1} \cdot (\mathcal{R}_\xi \varphi) \left(\frac{s}{\alpha} \right) \quad .$$

Hence

$$\left(\pi_\xi^* f, \varphi \left(\frac{\cdot}{\alpha} \right) \right) = \alpha^{n-1} \cdot \left(f, (\mathcal{R}_\xi \varphi) \left(\frac{\cdot}{\alpha} \right) \right) \quad .$$

By the homogeneity of f , this is also equal to

$$\alpha^{n-1} \cdot \alpha^{p+1} \cdot (f, \mathcal{R}_\xi \varphi) = \alpha^{n+p} \cdot (\pi_\xi^* f, \varphi)$$

what proves the first affirmation.

Since $\Omega = \alpha \xi \wedge \omega_{\alpha \xi} = \xi \wedge \omega_\xi$, one has $\omega_{\alpha \xi} = \alpha^{-1} \omega_\xi$ (actually the equality holds whenever both sides are restricted to any hyperplane parallel to $\ker \xi$). Also, $\pi_{\alpha \xi}^{-1}(s) = \pi_\xi^{-1}(\alpha^{-1}s)$. Therefore,

$$(\mathcal{R}_{\alpha \xi} \varphi)(s) = \alpha^{-1} \cdot (\mathcal{R}_\xi \varphi)(\alpha^{-1}s) \quad .$$

Then

$$(\pi_{\alpha \xi}^* f, \varphi) = \alpha^{-1} \cdot \left(f, \mathcal{R}_\xi \varphi \left(\frac{\cdot}{\alpha} \right) \right)$$

which by homogeneity of f is also equal to

$$\alpha^{-1} \cdot \alpha^{p+1} \cdot (f, \mathcal{R}_\xi \varphi) = \alpha^p \cdot (\pi_\xi^* f, \varphi) \quad .$$

□

We give an example to illustrate this proposition:

◀ *Example* On \mathbb{R} the distribution δ is homogeneous of degree -1 . Also, with its derivative it satisfies the following equality: $s \cdot \delta' = -\delta$. Considering the pull-back on both sides, the following equation holds:

$$(4) \quad \xi \cdot \pi_\xi^* \delta' = -\pi_\xi^* \delta \quad .$$

Therefore, with Proposition 4.2, one has

$$\frac{d}{dt} \left(\pi_{(1+t)\xi}^* \delta \right) \Big|_{t=0} = -\pi_\xi^* \delta$$

what also shows that the map $\xi \mapsto \pi_\xi^* \delta$ is homogeneous of degree -1 . ▶

4.2. Convexity of intersection bodies. To measure the area of central sections of a convex body as well as its moments, will apply the pull-backs of δ and s_+^p to the characteristic function of the body $\mathbb{1}_K$. This is not a test function, but a limit of test functions. Hence this has to be understood as a limit process.

Also to apply the derivatives of these distributions to the characteristic function, we need to assume that ∂K is smooth. So the general statement on convexity will follow by approximating general convex bodies by convex bodies with smooth boundaries. Note also that it is important that the origin lies inside the interior of the bodies. Indeed, $(\pi_\xi^* \delta', \mathbb{1}_K)$ would not be defined if $\ker \xi$ was a supporting hyperplane.

The following example gives explicit formulas for computing moments and their derivatives using distributions.

◀ *Example* With the notations of the first section, if K is a convex body with smooth boundary, the following formulas hold:

$$\mathcal{M}_p^+(K) = (\pi_\eta^* s_+^p, \mathbb{1}_K), \quad \frac{d}{dt} (\mathcal{M}_p^+(t))|_{t=0} = (-\pi_\xi^* \delta' \cdot \pi_\eta^* s_+^p, \mathbb{1}_K) \quad .$$

► We may now give the alternative definitions of intersection bodies and L_p -intersection bodies.

Definition 4.5. Let K be a star body in V . The *intersection body* $\mathbb{I}K$ of K is the star body in V^* whose Minkowski functional is defined by

$$(5) \quad \|\xi\|_{\mathbb{I}K} := (\pi_\xi^* \delta, \mathbb{1}_K)^{-1} \quad .$$

Note that this definition makes sense since both sides of (5) are homogeneous of degree 1 (see Proposition 4.4).

Definition 4.6. Let K be a star body in V . For $-1 < p \neq 0$, the L_p -*intersection body* $\mathbb{I}_p K$ of K is the star body in V^* whose Minkowski functional is

$$\|\xi\|_{\mathbb{I}_p K} := (\pi_\xi^* s_+^p, \mathbb{1}_K)^{\frac{1}{p}} \quad .$$

As for the classical intersection body, this definition of the L_p -intersection bodies makes sense since $\pi_\xi^* s_+^p$ is homogeneous of degree p in ξ (see Proposition 4.4).

It was first proved in [7, Proposition 3.1] that up to normalization $\mathbb{I}_p K$ goes to $\mathbb{I}K$ as p goes to -1 (see also [11, Theorem 1]). The definition given here in terms of distributions makes it easy to recover this result since $(p+1)^{-1} s_+^p$ and $(\Gamma(p+1))^{-1} \cdot s_+^p$ go to δ as p goes to -1 (see [9, p. 56]).

We first focus on the classical intersection body. Assume the star body K to be centered with smooth Minkowski functional, then the same holds for its intersection body and the tangent space to the intersection body has the following geometric description:

Theorem 4.7. *If K is centered with smooth Minkowski functional, the tangent space $T_\xi \partial \mathbb{I}K$ consists of all linear forms η that vanish on the tangent line to the curve of centers of mass of sections of K by hyperplanes parallel to $\ker \xi$.*

Proof. Let $c(t)$ be the center of mass of the section of K by the hyperplane $\xi \cdot x = t$ and call δ_t the δ distribution with support at $t \in \mathbb{R}$. Then, by definition of the center of mass, for every non-zero linear form $\eta \neq \lambda \cdot \xi$ one has

$$\eta \cdot c(t) = \frac{(\pi_\xi^* \delta_t, \mathbb{1}_K \cdot \eta)}{(\pi_\xi^* \delta_t, \mathbb{1}_K)} \quad .$$

Deriving this equation with respect to t at $t = 0$, one obtains

$$\eta \cdot \dot{c}(0) = \frac{(-\pi_\xi^* \delta', \mathbf{1}_K \cdot \eta)}{(\pi_\xi^* \delta, \mathbf{1}_K)}$$

since K is symmetric. Hence, by Proposition 4.2

$$\frac{d}{dt} (\|\xi + t\eta\|_{\mathbb{I}K}^{-1})|_{t=0} = -(\pi_\xi^* \delta, \mathbf{1}_K) \eta \cdot \dot{c}(0) \quad .$$

□

Theorem 4.8 (Busemann). *The intersection body of a centered convex body is convex.*

Proof. We will assume that ∂K is smooth, the general result will follow by approximation. According to Lemma 2.1, we have to prove that for $\nu \in T_\xi \partial \mathbb{I}K$ one has

$$(6) \quad \frac{d^2}{dt^2} (\|\xi + t\nu\|_{\mathbb{I}K})|_{t=0} \geq 0 \quad .$$

Since $\|\cdot\|_{\mathbb{I}K}$ is homogeneous of degree one,

$$\xi^t \cdot (D_\xi^2 \|\cdot\|_{\mathbb{I}K}) \cdot \eta = 0 \quad \forall \eta \in V^* \quad .$$

Hence, for a fixed $\lambda \in \mathbb{R}$ and for $\eta := \nu + \lambda\xi$,

$$(7) \quad \frac{d^2}{dt^2} (\|\xi + t\eta\|_{\mathbb{I}K}^{-1})|_{t=0} = 2\lambda^2 - \frac{d^2}{dt^2} (\|\xi + t\nu\|_{\mathbb{I}K})|_{t=0} \quad .$$

By Proposition 4.2, one also has

$$\frac{d^2}{dt^2} (\|\xi + t\eta\|_{\mathbb{I}K}^{-1})|_{t=0} = (\pi_\xi^* \delta'', \mathbf{1}_K \cdot \eta^2)$$

which is nothing else than the second order derivative of the second symmetric moments for η of sections of K by hyperplanes parallel to $\xi = 0$.

Note that the set $\{\nu + \lambda\xi | \lambda \in \mathbb{R}\}$ is precisely the set of linear forms whose restrictions to $\ker \xi$ coincide with the restriction of ν . It then follows from Theorem 3.5 that there exists such a λ for which (7) is negative. Hence (6) holds. □

Theorem 4.9. *The L_p -intersection bodies of a centered convex body are convex bodies.*

Proof. The proof mimics the one of Theorem 4.8, replacing the distribution δ by s_+^p . We will assume that ∂K is smooth and prove that for $\nu \in T_\xi \partial \mathbb{I}_p K$

$$(8) \quad \frac{d^2}{dt^2} (\|\xi + t\nu\|_{\mathbb{I}_p K})|_{t=0} \geq 0 \quad .$$

By homogeneity of $\|\cdot\|_{\mathbb{I}_p K}$, we have for $\eta = \nu + \lambda\xi$

$$(9) \quad \frac{d^2}{dt^2} (\|\xi + t\eta\|_{\mathbb{I}_p K}^p)|_{t=0} = p(p-1)\lambda^2 + p \frac{d^2}{dt^2} (\|\xi + t\nu\|_{\mathbb{I}_p K})|_{t=0} \quad ,$$

which by Proposition 4.2 is also equal to

$$(\pi_\xi^* (s_+^p)'', \mathbf{1}_K \cdot \eta^2) \quad .$$

Assume first that $p > 1$. Then, using equations (1) and (2),

$$(\pi_\xi^* (s_+^p)'', \mathbf{1}_K \cdot \eta^2) = p(p-1) \int_{x \in K_\eta^+} (\eta \cdot x)^{p-2} \Omega$$

which is positive for all λ , hence (8) holds.

The derivative involved in (9) is continuous with respect to p . Therefore for $p = 1$, one obtains

$$((s_+^p)'' , \mathcal{R}_\xi(\mathbf{1}_K \cdot \eta^2))|_{p=1} = (\delta, \mathcal{R}_\xi(\mathbf{1}_K \cdot \eta^2)) \quad .$$

Hence (9) is positive, so (8) holds.

If $0 < p < 1$, then

$$((s_+^p)'' , \varphi) = p(p-1) \int_0^\infty s^{p-2}(\varphi(s) - \varphi(0))ds \quad .$$

For $\varphi = \mathcal{R}_\xi(\mathbf{1}_K \cdot \eta^2)$, Theorem 3.5 implies that we can choose η such that $0 \leq \varphi(s) \leq \varphi(0)$. Hence (9) is positive, so (8) holds.

Finally, if $-1 < p < 0$, then

$$((s_+^p)'' , \varphi) = p(p-1) \int_0^\infty s^{p-2}(\varphi(s) - \varphi(0) - s\varphi'(0))ds \quad .$$

For $\varphi = \mathcal{R}_\xi(\mathbf{1}_K \cdot \eta^2)$, the symmetry of K implies that $\varphi'(0) = 0$ for all η . Moreover, Theorem 3.5 implies that we can choose η such that $0 \leq \varphi(s) \leq \varphi(0)$. Hence (9) is negative, so (8) holds. \square

It remains to prove one lemma.

Proof of Lemma 3.6. Assume $\eta \in V^*$ is any fixed extension of η_0 . Note that any other extension of η_0 is given by $\eta + \lambda\xi$ for some $\lambda \in \mathbb{R}$. Then for $p \geq 0$,

$$\frac{d}{dt} \mathcal{M}_p^+|_{t=0} = -(\pi_\xi^* \delta' \cdot \pi_\eta^* s_+^p, \mathbf{1}_K) \quad .$$

Moreover

$$(10) \quad \frac{d}{d\lambda} (\pi_\xi^* \delta' \cdot \pi_{\eta+\lambda\xi}^* s_+^p, \mathbf{1}_K) = p(\pi_\xi^* \delta' \cdot \xi \cdot \pi_{\eta+\lambda\xi}^* s_+^{p-1}, \mathbf{1}_K) \quad .$$

Since $\xi \cdot \pi_\xi^* \delta' = -\pi_\xi^* \delta$ (see (4)), relation (10) is also equal to

$$-p(\pi_\xi^* \delta \cdot \pi_{\eta+\lambda\xi}^* s_+^{p-1}, \mathbf{1}_K) \quad .$$

This last expression does not depend on λ but only on η_0 and is not zero. The map $\lambda \mapsto (\pi_\xi^* \delta' \cdot \pi_{\eta+\lambda\xi}^* s_+^p, \mathbf{1}_K)$ is then affine and not constant, therefore it vanishes for some $\lambda \in \mathbb{R}$. \square

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