ISOPERIMETRIX AND FLOATING BODIES

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ABSTRACT. It is shown that up to normalization the family of floating bodies converges to the isoperimetrix of Busemann's area.

1. Introduction

This short note is part of a project to understand the intersection body of a centered convex body, or equivalently the isoperimetrix of Busemann's area in finite dimensional normed spaces. Here we embed the isoperimetrix in a family of geometrically defined convex bodies. Namely, we show that the isoperimetrix is the limit of the family of floating bodies of the unit ball (see theorem 5.4 for a more precise statement).

This has to be compared with the fact that the family of analytically defined L_p-intersection bodies converges to the dual of the isoperimetrix. We have proved in [2] that the L_p-intersection bodies are all convex.

For self-contentness we start recalling in section 2, 3 and 4 the notions of Busemann's area, isoperimetrix and floating bodies. Then we prove the result in section 5 using a small amount of distribution theory as we did in [2].

2. Area and isoperimetrix

We briefly recall the notion of area on a vector space and give the solution to the isoperimetric problem. A more complete discussion on the subject may be found in the survey [1].

Definition 2.1. An *area* on a n-dimensional space V is a norm $\phi\colon \Lambda^{n-1}V \longrightarrow \mathbb{R}$.

Such a norm on the exterior product may be integrated on compact hypersurfaces of V using local parametrization, exactly as one integrates differential $(n-1)$-forms except that no orientation is required since the norm is symmetric.

Therefore, by the *area of a compact hypersurface* M we mean

$$Area(M) := \int_M \phi \quad .$$

Note that the notation does not refer explicitly to the norm used.

Given a volume form Ω on V, one defines a linear isomorphism $L_\Omega\colon \Lambda^{n-1} \to V^*$ by

$$L_\Omega(a) := \Omega(a \wedge \cdot) \in V^* \quad .$$

Definition 2.2. Let $\mathcal{B} \subset \Lambda^{n-1}V$ be the unit ball of an area ϕ and Ω a volume form on V. The *isoperimetrix* is the convex body

$$\mathbb{I} := (L_\Omega(\mathcal{B}))^* \subset V \quad .$$

As the name suggests, this body minimizes the area for a fixed volume:

Theorem 2.3. *Let $K \subset V$ be a convex body with the same volume as the isoperimetrix $\mathbb{I}$. Then*

$$Area(\partial K) \geq Area(\partial \mathbb{I})$$

with equality if and only if K is a translate of the isoperimetrix.

3. Busemann's area

Let $(V, \|\cdot\|)$ be a n-dimensional normed space, with unit ball B. While the length of smooth curves is defined as usual and its behavior presents little differences with the Euclidean case, there are different ways to introduce natural notions of areas. One has been defined by Busemann and is based on a construction of convex geometry. We now present it briefly and refer again to [1] for more details.

The *intersection body* of the unit ball is a body in the exterior product $\Lambda^{n-1}V$. For its construction, we need to recall the following simple fact: all $(n-1)$-vectors are simple, i.e. any $a \in \Lambda^{n-1}V$ may be written as $a = v_1 \wedge \cdots \wedge v_{n-1}$ for some choice of v_i's in V. Geometrically speaking, the boundary of the intersection body consists of all $(n-1)$-vectors $a = v_1 \wedge \cdots \wedge v_{n-1}$ with the property that the parallelotope spanned by the v_i's has the same area as the section of the unit ball by the hyperplane spanned by the v_i's:

Definition 3.1. The *intersection body*, $I(B) \subset \Lambda^{n-1}V$, contains the origin and its boundary is defined by

$$a \in \partial I(B) \Longleftrightarrow \forall \omega \in \Lambda^{n-1}V^*,\ \omega(a) = \int_{\langle a\rangle \cap B} \omega$$

where $\langle a\rangle$ is the hyperplane $\{v \in V |\ a \wedge v = 0\}$.

Theorem 3.2 (Busemann)**.** *The intersection body of a centered convex body is a centered convex body.*

This centered convex body is then the unit ball of some norm on the exterior product. Up to a constant this gives Busemann's notion of area:

Definition 3.3. Busemann's area is the norm ϕ_b on $\Lambda^{n-1}V$ whose unit ball is

$$\mathcal{B}_b := (\epsilon_{n-1})^{-1} \cdot I(B)$$

where ϵ_{n-1} is the Euclidean volume of the Euclidean unit ball of dimension $n-1$.

The main interest of this notion of area relies on the following theorem:

Theorem 3.4 (Busemann)**.** *The area of a compact hypersurface is its $(n-1)$-dimensional Hausdorff measure.*

4. Floating bodies

Let $B \subset V$ be a centered convex body. For $\frac{1}{2} \geq \alpha \geq 0$, we will call $\mathcal{H}_\alpha$ the set of closed half-spaces H (bounded by some hyperplane) with the property that

$$\mathrm{Vol}(H \cap B) = (1-\alpha) \cdot \mathrm{Vol}(B) \quad .$$

Definition 4.1. For $\frac{1}{2} \geq \alpha \geq 0$, the centered convex body

$$B_\alpha := \bigcap_{H \in \mathcal{H}_\alpha} H \subset B$$

is called the *floating body* (with parameter α).

Note that $B_0 = B$ and $B_{\frac{1}{2}} = \{0\}$.

So far, the family of floating bodies is not well understood. It is expected that if B_α is homothetic to B for some $0 < \alpha < \frac{1}{2}$ then all of them are homothetic and B is actually an ellipsoid (see [7] and [8]).

5. Distributions and result

We will use the distributions to describe the intersection body and the floating bodies of the unit ball B. The main two distributions used here will be the well-known delta distribution on $\mathbb{R}$, δ, and the translated Heaviside distribution, H_t, obtained as integration against the locally integrable function

$$H_t(x) := \begin{cases} 0 & \text{if } x < t \\ 1 & \text{if } x \geq t \end{cases}$$

Let's first quickly review the notion of *pull-back* of a distribution, specializing it for our needs (see also [4, chap.6, §1]). Assume the vector space V is equipped with a volume form Ω. It then makes sense to consider distributions on this space. Also, every non-zero linear form $\xi \in V^*$ is by definition a surjective linear map from V to $\mathbb{R}$. We will use it to pull-back to V the distributions on $\mathbb{R}$. However, to make formulas involving such pull-backs more readable, it seems adequate to us to introduce the notation $\pi_\xi : V \to \mathbb{R}$ for the linear surjection $v \mapsto \xi \cdot v$.

For every non-zero linear form ξ, there exists an $(n-1)$-form ω_ξ well defined on the hyperplanes $\xi \cdot v = cst$ and such that $\Omega = \xi \wedge \omega_\xi$. As one easily sees, the *Radon transform* of a test function $\varphi \in \mathcal{C}_0^\infty(V)$

$$\mathcal{R}_\xi \varphi(x) := \int_{\pi_\xi^{-1}(x)} \varphi \, \omega_\xi$$

is a test function on $\mathbb{R}$. We use it to define the pull-back of distributions:

Definition 5.1. Given a non-zero linear form $\xi \in V^*$, the pull-back $\pi_\xi^* f$ of a distribution f on $\mathbb{R}$ is defined by

$$(\pi_\xi^* f, \varphi) := (f, \mathcal{R}_\xi \varphi) \quad .$$

Recall also that a distribution on a n-dimensional space is said *homogeneous of degree p* if

$$\left(f, \varphi\left(\frac{\cdot}{\alpha}\right)\right) = \alpha^{p+n}(f, \varphi) \quad .$$

On $\mathbb{R}$, δ is homogeneous of degree -1 (see [3, p.10]). We will need the following proposition whose proof is an easy exercise.

Proposition 5.2. *If f is a distribution homogeneous of degree p, so is its pull-back $\pi_\xi^* f$. Moreover, the map $\xi \mapsto \pi_\xi^* f$ is also homogeneous of degree p.*

We may now use the distributions to characterize the intersection body and the floating bodies of the unit ball B. Note that we will apply distributions on the characteristic function of the unit ball. This is not a test function but a limit of test functions. Hence this has to be thought as a limit process.

Proposition 5.3. *In V^*, $L_\Omega(I(B))$ is the unit ball of the norm*

$$\xi \mapsto (\pi_\xi^* \delta, \mathbb{1}_B)^{-1}$$

while, for $0 < \alpha < \frac{1}{2}$, the hypersurface $S_\alpha := \partial(B_\alpha^*)$ is the level set at height $\alpha \cdot \mathrm{Vol}(B)$ of the function

$$\xi \mapsto (\pi_\xi^* H_1, \mathbb{1}_B) \quad .$$

Proof. The hypersurface $\partial L_\Omega(I(B))$ is easily seen to be the level set at height one of the function $(\pi_\xi^* \delta, \mathbb{1}_B)^{-1}$. The homogeneity of this function follows from the proposition 5.2 and its convexity from the convexity of the intersection body. Hence it is indeed a norm.

The geometrical interpretation of the equation

$$(\pi_\xi^* H_1, \mathbb{1}_B) = \alpha \cdot \mathrm{Vol}(B)$$

is that the volume of the set $B \cap \{v | \xi \cdot v \geq 1\}$ equals $\alpha \cdot \mathrm{Vol}(B)$. Hence, by definition of the floating body, the hyperplane $\{v | \xi \cdot v = 1\}$ either does not intersect B_α or is a support hyperplane. Since B is centered, it follows from the convexity of the flotation hypersurface (see [6] and [5, 166-176]) that this hyperplane effectively supports B_α. □

Theorem 5.4. *The family of floating bodies of a centered body converges up to normalization to the isoperimetrix of Busemann's area. More precisely*

$$\lim_{\alpha \to \frac{1}{2}} \frac{B_\alpha}{\frac{1}{2} - \alpha} = \frac{\mathrm{Vol}(B)}{\epsilon_{n-1}} \cdot \mathbb{I}_b \quad .$$

Proof. Set $t = \frac{1}{2} - \alpha$. The statement to be proved is equivalent to

$$\lim_{t \to 0} t \cdot S_{\frac{1}{2} - t} = \frac{\partial L_\Omega(I(B))}{\mathrm{Vol}(B)}$$

in the dual space V^*.

Note that the hypersurface $t \cdot S_{\frac{1}{2} - t}$ is characterized by the equation

$$(\pi_\xi^* H_t, \mathbb{1}_B) = \left(2^{-1} - t\right) \cdot \mathrm{Vol}(B)$$

which after some algebraic changes is equivalent to

$$\left(\pi_\xi^* \frac{H_0 - H_t}{t}, \mathbb{1}_B\right) = \mathrm{Vol}(B) \quad .$$

Taking the limit as t goes to zero, one gets (see e.g. [3, pp. 20 & 21])

$$(\pi_\xi^* \delta, \mathbb{1}_B) = \mathrm{Vol}(B)$$

which characterizes the hypersurface

$$\frac{\partial L_\Omega(I(B))}{\mathrm{Vol}(B)} \quad .$$

□

References

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