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Abstract: We generalize the results of Kwak [16] and Bramble, Ewing, Pasciak and Shen [2] and discuss convergence proofs for several multigrid algorithms in a framework of nested multigrid spaces and noninherited quadratic forms for structured finite-volume schemes for SPD problems. Convergence proofs are stated for W-cycle with only piecewise constant transfer operators and V-cycle and variable V-cycle with either linear prolongation and restriction or bilinear prolongation and constant restriction. The theoretical conclusions on meshsize independent and number of levels independent behaviour are verified by numerical experiments on several structured but nonequidistant finite volume meshes discretizing the Poisson problem.

Key-words: multigrid, finite volumes, agglomeration

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Méthode multigrille par des volumes finis: le cas 2D structuré

Résumé : On généralise les résultats de Kwak [16] et Bramble, Ewing, Pasciak et Shen [2] et on discute la convergence de plusieurs types d'algorithmes multigrilles pour des problèmes symmétriques défini-positifs dans le cadre d'espaces grossiers bornés et des formes quadratiques non-heritiées. On presente des preuves de convergence de l'algorithme avec des opérateurs de transfert seulement constant par morceaux pour le W-cycle ou avec soit un prolongement et une restriction linéaires soit un prolongement bilinéaire et une réstriction constante pour le V-cycle et le V-cycle variable. Les conclusions théoriques sur le comportement des algorithmes par rapport a la résolution du maillage ou le nombre de niveaux sont validées par les expériences numériques sur plusieurs maillages volumes finis qui discrètisent le problème de Poisson.

Mots-clés : multigrille, volumes finis, agglomeration

1 Introduction

Although multigrid methods using a finite volume framework are only recently being analysed, they found already long time ago their place among the most efficient solvers for second order partial differential equations. Unlike for the finite element technique, convergence theory for finite volumes in H^1-seminorm has been established only recently [10, 11, 14], which has triggered the efforts for understanding the class of finite volume based (cell-centered) multigrids [2, 16].

The analysis is usually done within the "non-nested non-inherited" general convergence theory of Bramble, Pasciak and Xu [5, 1], which requires the H^2 regularity of the problem. One might use also Fourier analysis based approach [6, 12], however the Bramble's analysis is more abstract in the view of generally non-structured meshes and complicated boundary conditions.

The principal advantage of the finite volume multigrid over its finite element based competitors [18, 9, 7] is the facility of construction of nested coarse spaces by simple agglomeration of the finest level control volumes. Hence, the coarse problems might still be analysed within the finite volume framework and, moreover, the discretization gives system matrices with fairly small fill-in. For the finite element case, on the other hand, it is geometrically impossible for coarse levels to stay in general nested and finite element at the same time.

We prefer to use nested coarse spaces - from the practical point of view we thus avoid a rather expensive construction of transfer operators based on geometrical information of the underlying coarse finite element meshes which would occur when sticking to the finite element coarse levels. Limiting ourselves to nested coarse spaces, several generalizations of finite elements have been proposed: nested coarse spaces by smoothed aggregation of Vaněk, Mandel, Brezina [18], or generalizations of elements to macro-elements as in [9, 7]. The disadvantage of the former is that the smoothed coarse basis functions overlap far more than in the standard finite elements, which results in increasing fill-in of coarse problem matrices, while the latter approach requires storage of additional information such as element-to-element neighbourhood.

Rather than finite element point of view, we adopt purely finite volume layout and review the essential results of Bramble, Ewing, Pasciak and Shen [2] and Kwak [16] proposed for 2D structured equi-spaced cell-centered finite difference multigrid. These results will be then extended to 2D structured nonequidistant case. It is also interesting to observe the similarities of the proposed finite volume multigrid with the one by agglomeration by Koobus, Lallemand and Dervieux [15]. In fact, the two coincide when the mesh is equidistant.

The paper is organized as follows: in Section 2 we define the problem and tools to be used later. Section 3 gives an outline of an existing general convergence theory [5, 1] which suits our purposes. The next three Sections verify assumptions of the general convergence theory for several multigrid algorithms which differ by a choice of transfer operators. Section 4 concludes on W-cycle and variable V-cycle with only piecewise constant transfer operators, Section 5 reviews the V-cycle of Kwak [16] and Section 6 handles the case of nonsymmetric restriction and prolongation. The theoretical part is followed by numerical experiments in Section 7 whose purpose is to show mesh and number of level independent behaviour of the algorithms.

2 Preliminaries

In this presentation we design and analyse a class of volume agglomeration multigrid algorithms for cell-centered discretizations of Poisson equation. We limit ourselves to a model problem in two spatial dimensions:

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here Ω is the comutational domain, $\Omega \subset \mathbb{R}^2$, with Lipschitz continuous boundary $\partial\Omega$. The symbol Δ denotes the Laplace operator, and $f \in L^2(\Omega)$. Let us for simplicity assume that Ω is a union of rectangles.

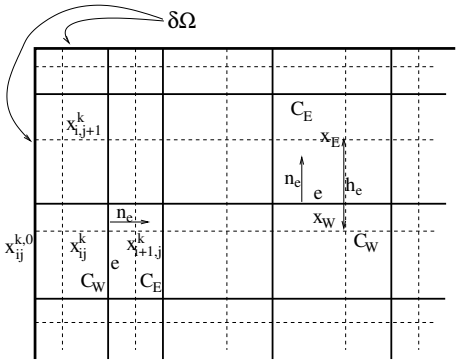

Figure 1: Finite-volume cells C_i^k on the k-th multigrid level

Definition 2.1. Let us for each multigrid level $k = 1, \ldots, J$ define the finite volume cell-centered approximation of (1) based on a structured, generally nonequidistant system of n_k rectangular finite-volume control cells C_i^k (cf. Fig. 1), $i = 1, \ldots, n_k$. We suppose that the multigrid coarsening is done by a factor of 2, ie. each coarser rectangular cell C_i^{k-1} is created by at most 2 finer-level cells in each spatial direction (cf. Fig. 2). The finest level is indexed by $k = J$, J is the total number of multigrid levels.

Let us denote by E_k a set of all cell interfaces on the k-th multigrid level,

$$E_k = \{e, e = \partial C_i^k \cap \partial C_j^k, i \neq j \text{ or } e = \partial C_i^k \cap \partial\Omega\}.$$

Let $\mathbf{n}_e$ be a unit normal vector to $e \in E_k$ with the orientation chosen apriori. Based on the orientation of the normal $\mathbf{n}_e$, we define for every cell interface e of the cell C_i^k a sign function $s_{i,e}^k$, $s_{i,e}^k = 1$ if $\mathbf{n}_e$ points out of C_i^k, $s_{i,e}^k = -1$ otherwise.

For each cell C_i^k on the k-th multigrid level let us introduce a point $x_i^k \in C_i^k$ (cf. Fig. 1), where the numerical fluxes will be evaluated. Let us choose the points x_i^k in such a way, that for each

two neighbouring cells C_i^k and C_j^k, $\partial C_i^k \cap \partial C_j^k \neq \emptyset$ the line led through x_i^k and x_j^k is perpendicular to the edge $e = \partial C_i^k \cap \partial C_j^k$.

For the boundary cells C_i^k, $\bar{C}_i^k \cap \partial\Omega \neq \emptyset$, let us also define points $x_i^{k,0} \in \partial\Omega \cap \partial C_i^k$ such that the line led through x_i^k and $x_i^{k,0}$ is perpendicular to $\partial\Omega \cap \partial C_i^k$ (cf. Fig 1).

For two neighbouring cells C_i^k and C_j^k we will use the indices W (west) and E (east) to denote x_i^k by x^W or x^E and x_j^k, by x^E or x^W such that the normal vector $\mathbf{n}_e$ of the common interface e points from west to east (cf. Fig 1).

For the boundary cell C_i^k and $e = \partial C_i^k \cap \partial\Omega$, we take $x^W = x_i^k$, $x^E = x_i^{k,0}$ if $\mathbf{n}_e$ points outside the cell, $x^W = x_i^{k,0}$, $x^E = x_i^k$ otherwise. We denote h_e the euclidean distance of x^W and x^E.

Let further M_k denote the finite-volume solution space, piecewise constant on C_i^k, $i = 1, \ldots, n_k$.

Integrating (1) over C_i^k then gives

$$-\int_{\partial C_i^k} \nabla u \, \mathbf{n} d\Gamma = \int_{C_i^k} f d\Omega. \tag{2}$$

Operating in a finite volume framework, we should in the next step approximate the continuous solution $u \in H^2(\Omega)$ by a function $u_k \in M_k$ which is discontinuous on the integration path ∂C_i^k. Therefore, we also approximate the gradient ∇u by a simple difference formula in u_k,

Let $u_k \in M_k$ be the solution of the following discrete problem

$$\sum_{e \in E_k \cap \partial C_i^k} s_{i,e} \Phi_k(u_k, \mathbf{n}_e) \mu(e) = \int_{C_i^k} f d\Omega, \tag{3}$$

where $\mu(e)$ is the measure (length) of the interface e and $\Phi_k(u_k, \mathbf{n}_e)$ is a numerical diffusive flux accross this cell interface, calculated by

$$\Phi_k(u_k, \mathbf{n}_e) = \frac{u_k^W - u_k^E}{h_e}.$$

The values of u_k^W and/or u_k^E corresponding to the boundary points $x_{i,0}^k$ are set to 0.

Remark 2.2. Note, that the mesh defined in Definition 2.1 is admissible in the sense of Definition 1.1 in [14] and the convergence theory therein applies. The assumption that the cells C_i^k are rectangular is not necessary once the mesh is admissible.

Definition 2.3. Let us for each multigrid level k define a discrete-problem operator $A_k : M_k \to M_k$ by

$$A_k u_k = \Big\{ \frac{1}{\mu(C_i^k)} \sum_{e \in E_k \cap \partial C_i^k} s_{i,e} \Phi(u_k, \mathbf{n}_e) \mu(e) \Big\}_{C_i^k},$$

for all $u_k \in M_k$. The corresponding bilinear form $A_k(\cdot,\cdot) : M_k \times M_k \to \mathbb{R}$ is defined as

$$A_k(u_k, v_k) = (A_k u_k, v_k)_{L^2(\Omega)}, \qquad \forall v_k, u_k \in M_k. \tag{4}$$

Thus, we can write equivalently to (3) that

$$A_k u_k = f_k \qquad \text{or} \qquad A_k(u_k, v_k) = (f, v) \quad \forall v_k \in M_k, \tag{5}$$

where f_k is the L^2 projection of $f \in L^2(\Omega)$ to M_k.

Remark 2.4. In the computer implementation, it is not the matrix A_k which is assembled, but a similar matrix in the sequel denoted by $\boldsymbol{A}_k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_k}$ defined by the energy form $A_k(\cdot, \cdot)$ as

$$\langle \boldsymbol{A}_k \mathbf{v}_k, \mathbf{u}_k \rangle_{l^2} = A_k(v_k, u_k),$$

where $\langle \cdot, \cdot \rangle_{l^2}$ is the standard euclidean scalar product and $\mathbf{u}_k \in \mathbb{R}^{n_k}$ is a vector of degrees of freedom, or "cell averages", of the finite-volume function $u_k \in M_k$. The same for v_k and $\mathbf{v}_k$. In terms of Definition 2.3, we can also write for all $\mathbf{u}_k \in \mathbb{R}^{n_k}$ that

$$\boldsymbol{A}_k \mathbf{u}_k = \Big\{ \sum_{e \in E_k \cap \partial C_i^k} s_{i,e} \Phi(u_k, \mathbf{n}_e) \mu(e) \Big\}_{C_i^k},$$

where u_k and $\mathbf{u}_k$ are related in the above way.

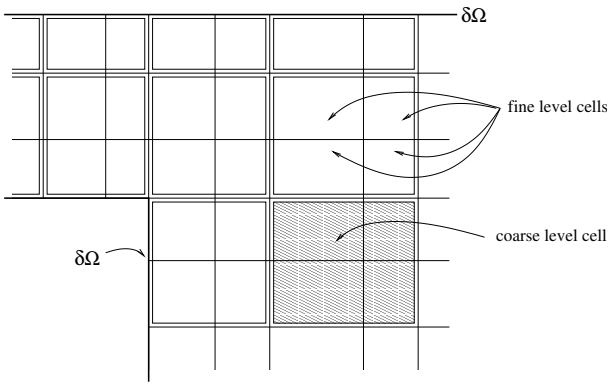

Figure 2: Volume agglomeration by coarsening ratio 2 in each direction

The first preparatory part of the algorithm of the volume agglomeration method with J multigrid levels is the following.

Algorithm 2.5. (setup phase) Let us have a set $\mathcal{A}_J$ of the finest-level finite volume cells and the finite-volume discretization based on those cells. We generate the sequence of coarse-grid problems by the following procedure:

For $k = (J–1), \ldots, 1$, step -1

1. Agglomerate volumes $\mathcal{A}_{k+1}$ on the multigrid level $k+1$ by at most 2 in each spatial direction (cf. Fig. 2) to get n_k agglomerates $\mathcal{A}_k$ on the current k-th multigrid level. Agglomerate the canonical basis $\{\chi_i^k\}_i$ of the finite-volume space M_{k+1} on the agglomerates $\mathcal{A}_k$ to get a basis of the finite-volume space M_k.
2. Construct the injection (prolongation) matrix $\tilde{\boldsymbol{I}}_{k+1}^k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_{k+1}}$ as a zero-one mapping based on finite volumes in the non-overlapping disjoint agglomerate sets $\mathcal{A}_k$ and $\mathcal{A}_{k+1}$ (volume-agglomeration). Construct a prolongation smoother $\boldsymbol{S}_k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_k}$ to get the smoothed prolongation $\boldsymbol{I}_{k+1}^k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_{k+1}}$, $\boldsymbol{I}_{k+1}^k = \boldsymbol{S}_k \tilde{\boldsymbol{I}}_{k+1}^k$.
3. Rediscretize the coarse-grid problem by finite-volume technique to get $A_k : M_k \to M_k$, $A_k(\cdot,\cdot)$, resp. $\boldsymbol{A}_k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_k}$.

End for.

Remark 2.6. Note, that in Step 3 of the above algorithm we have not yet specified the flux-evaluation points x_i^k for the coarser level. Indeed, while the flux-evaluation points are given for the finest level J, for the coarser level they are relatively free to choose, up to the limits imposed by assumptions on the admissible mesh as in [14]. We will see later, that this liberty can be used to maintain under control other conditions on the coarse-level discretisations.

Remark 2.7. It is clear from the construction of the finite volume spaces, that $M_1 \subset M_2 \subset \ldots \subset M_J$. The prolongation matrix $\tilde{\boldsymbol{I}}_{k+1}^k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_{k+1}}$ in Step 2 of the above algorithm corresponds to a prolongation by natural injection $\tilde{I}_{k+1}^k$, ie. nonsmoothed prolongation of $u_k \in M_k$ to the space M_{k+1}.

Now, we are about to give the algorithm of the multigrid V and/or W-cycle formulated by induction based on the multigrid preconditioner B_k, $k = 1, \ldots, J$.

Algorithm 2.8. (iteration phase) Let p be a positive integer cycle parameter (eg. $p = 1$ for V-cycle or $p = 2$ for W-cycle). Set $B_1 = A_1^{-1}$. Let $k > 1$ and assume that B_{k-1} has been defined. Given the right-hand side $g \in M_k$, $B_k g$ is defined as follows:

1. Set $x^0 = q^0 = 0$.
2. Define x^l for $l = 1, \ldots, m(k)$ by
$$x^l = x^{l-1} + R_k^{(l+m(k))}(g - A_k x^{l-1}).$$
3. $y^{m(k)} = x^{m(k)} + I_k^{k-1} q^p$, where q^i is defined by:
$$q^i = q^{i-1} + B_{k-1}[I_{k-1}^k(g - A_k x^{m(k)}) - A_{k-1} q^{i-1}].$$
4. Define y^l for $l = m(k)+1, \ldots, 2m(k)$ by
$$y^l = y^{l-1} + R_k^{(l+m(k))}(g - A_k y^{l-1}).$$

5. Set $B_k g = y^{2m(k)}$.

Here $R_k^{(l)} = R_k$ when l is odd and $R_k^{(l)} = R_k^T$ when l is even. R_k is the preconditioner of a simple smoothing method applied $m(k)$-times as pre- and post-smoothing on a multigrid level k. $I_{k-1}^k : M_k \to M_{k-1}$ is an abstract restriction operator and $I_k^{k-1} : M_{k-1} \to M_k$ is a prolongation operator.

It is clear from Algorithm 2.5, Step 3, that the volume-agglomeration multigrid is a multigrid with nested coarse spaces and non-inherited forms - the coarse problems are not devised by a variational technique from the same continuous quadratic form. This calls for use of an appropriate general convergence theorem, such as eg. in [5] and [1], Chapter 4.

Let us review the assumptions needed for the abstract convergence theory of Bramble, Pasciak, and Xu [5]. Assuming we are given a certain prolongation operator $I_k^{k-1} : M_{k-1} \to M_k$ we define a restriction operator $I_{k-1}^k : M_k \to M_{k-1}$ as its adjoint with respect to the L^2 inner product $(\cdot, \cdot)$:

$$(I_{k-1}^k u, v) = (u, I_k^{k-1} v) \qquad \forall u \in M_k, v \in M_{k-1}. \tag{6}$$

Since the finite-volume function space M_k can be viewed as a vector space $\mathbb{R}^{n_k}$, we use the same symbols for linear operators in M_k and for their corresponding matrix representations in $\mathbb{R}^{n_k}$. Matrices are put in bold characters, eg. $L_k : M_k \to M_k$ and its corresponding matrix $\boldsymbol{L}_k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_k}$.

3 MG analysis

The abstract multigrid convergence theory in [1, 5] rests on three assumptions, each of them controlling the three blocks of any multigrid algorithm: pre- and post-smoothers, definition of the coarse-grid problem and the transfer operators. We name these assumptions, respectively, the *smoothing property*, the *approximation property* (elliptic regularity pickup) of the coarse-space problem and the *energy stability* of the prolongations I_k^{k-1}.

Smoothing property: The following smoothing property judges the behaviour of the pre- and post-smoother by comparison to the simplest Richardson iteration.

Assumption 3.1. (smoothing property) There exist $C_R > 1$, independent of k such that

$$\frac{\|u\|^2}{\lambda_k} \leq C_R(\bar{R}_k u, u) \qquad \forall u \in M_k,$$

where $\bar{R}_k = (I - (I - R_k^T A_k)(I - R_k A_k))A_k^{-1}$ and $\|\cdot\|$ is the L^2 norm.

We can use, for example, point Jacobi or Gauss-Seidel smoothing procedures to define R_k. The smoothing properties are then a consequence of the general smoothing theory in [3]. In the following, we consider Assumption 3.1 to be satisfied.

cycle type	$A_k(I_k^{k-1}u, I_k^{k-1}u) \le \mu A_{k-1}(u,u)$, $\forall u \in M_{k-1}$	smoothing steps $m(k)$	convergence result
V-cycle	$\mu = 1$	any $m(k) = m$	$0 \le A_k((I - B_kA_k)u,u) \le \delta_k A_k(u,u)$ $\delta_k \le \frac{M}{Mk^{(1-\alpha)/\alpha}+m^\alpha}$, $M > 0$ const.
W-cycle	$\mu = 2$	any $m(k) = m$	$\lvert A_k((I - B_kA_k)u,u)\rvert \le \delta_k A_k(u,u)$ $\delta_k \le \frac{M}{M+m^\alpha}$, $M > 0$ const.
W-cycle		sufficiently many $m(k) = m$	$\lvert A_k((I - B_kA_k)u,u)\rvert \le \delta_k A_k(u,u)$ $\delta_k \le \frac{M}{M+m^\alpha}$, $M > 0$ const.
variable V-cycle		variable V-cycle	$\lambda_{\min}(B_kA_k) \ge \frac{m(k)^\alpha}{M+m(k)^\alpha}$ $\lambda_{\max}(B_kA_k) \le \frac{m(k)^\alpha}{M+m(k)^\alpha}$, $M > 0$ const.
variable V-cycle	$\mu = 1$	variable V-cycle	$0 \le A_k((I - B_kA_k)u,u) \le \delta_k A_k(u,u)$ $\delta_k \le \frac{M}{M+m(k)^\alpha}$, $M > 0$ const.
V-cycle	$\mu = 1 + c\lambda_k(A_k)^{-\gamma}$, $\gamma \in (0,1]$	any $m(k) = m$	$\lambda_{\min}(B_kA_k) \ge \frac{m^\alpha}{Mk^{(1-\alpha)/\alpha}1+m^\alpha}$ $\lambda_{\max}(B_kA_k)$ independent of k

Table 1: Available convergence results from Bramble, Pasciak, Xu [5]

Regularity and approximation property: Next, we need to verify the so-called regularity and approximation property. Let us first define elliptic projections $P_{k-1} : M_k \to M_{k-1}$ and $\tilde{P}_{k-1} : M_k \to M_{k-1}$ by

$$A_{k-1}(P_{k-1}u, v) = A_k(u, I_k^{k-1}v) \qquad \forall u \in M_k, v \in M_{k-1}, \tag{7}$$

$$A_{k-1}(\tilde{P}_{k-1}u, v) = A_k(u, v) \qquad \forall u \in M_k, v \in M_{k-1} \subset M_k. \tag{8}$$

Then the crucial regularity and approximation property can be written in the following way

Assumption 3.2. (regularity and approximation) There exist constants $\alpha \in (0, 1]$ and $C_\alpha > 0$ such that for all $k = 1, \ldots, J$ there is

$$A_k((I - I_k^{k-1}P_{k-1})u, u) \leq C_\alpha^2 \left(\frac{\|A_k u\|^2}{\lambda_k} \right)^\alpha A_k(u, u)^{1-\alpha} \qquad \forall u \in M_k.$$

Here $\|\cdot\| = \|\cdot\|_{0,\Omega}$ is the $L^2(\Omega)$ norm and λ_k is the largest eigenvalue of A_k.

Remark 3.3. Unlike for the variational algebraic V-cycle theory [4, 18] where the coarse problems do not need to be consistent rediscretizations of the same continuous problems, it seems that for multigrids (both V and W cycles) with non-inherited quadratic forms the regularity and approximation property is indispensable.

Already with the approximation property and smoothing property we can state first convergence results concerning the W-cycle, as can be seen in Table 3.

Energy stability of the transfer operators: The last component of the non-inherited convergence theory of Bramble and Pasciak is the energy stability of the prolongation I_k^{k-1}. We are free to choose among various assumptions of different strength to get convergence results for different multigrid algorithms. In Table 3 we give an overview of the available convergence results.

Seen from the prospective of frequency analysis [6], the energy property coincides with an assumption that the prolongation operator I_k^{k-1} of a coarse-grid solution to finer grid does not re-introduce very high frequencies of errors onto the finer grid.

Assumption 3.4. (energy stability) There exist a constant $\mu > 0$ bounded by values in Table 3 (according to the desired convergence result) for which there is

$$A_k(I_k^{k-1}u, I_k^{k-1}u) \leq \mu A_{k-1}(u, u), \tag{9}$$

for all $u \in M_{k-1}$.

4 Prolongation I_k^{k-1} by natural injection

Let us suppose within this section, that I_k^{k-1} is identity, ie. the prolongation from M_{k-1} to M_k is done by natural injection. We are going to generalize results of Bramble, Ewing, Pasciak and

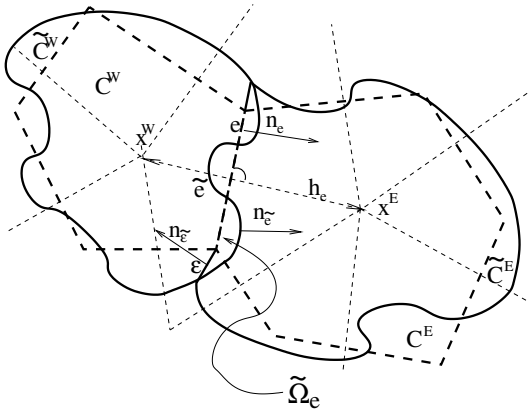

Figure 3: General admissible mesh cells as in [14]

Shen [2] for volume agglomeration W-cycle for multigrid levels with structured nonequidistant and even non-structured finite volumes, once the mesh is still admissible in the general sense of [14], see Fig. 3.

Error estimates for cell centred finite volumes or finite differences on *regular structured* meshes are well known [10, 8]. Let Q_k denote the $L^2(\Omega)$ orthogonal projection onto M_k. If u solves (1) and $u_k \in M_k$ solves (5) then

$$A_k(u_k - Q_k u, u_k - Q_k u) \leq C h_k^2 \|f\|_{0,\Omega}^2. \tag{10}$$

Here, h_k is the meshsize corresponding to M_k.

This estimate depends on H^2 regularity of the solution of (1). It is an essential tool in the verification of Assumption 3.2 as it has been shown for the regular structured case by Bramble, Ewing, Pasciak and Shen [2] in W-cycle context with prolongation I_k^{k-1} by natural injection.

However, we do not need for our purposes Q_k in (10) to be exactly the L^2 projection. Note, that the choice of $\tilde{Q}_k$ (replacing Q_k) and the coordinates of the flux-evaluation points $x_i^k \in C_i^k$ must be related. Indeed, for all linear functions u we must have $\tilde{Q}_k u|_{C_i} = u(x_i^k)$. Thus, for any $x_i^k \in C_i^k$ we can construct a mapping $\tilde{Q}_k : H^2(\Omega) \to M_k$ by

$$\tilde{Q}_k u = \sum_{j=1}^{N_i^k} \frac{w_{i,j}}{\mu(C_{i,j}^k)} \int_{C_{i,j}^k} u d\Omega \qquad \text{on } C_i^k, \tag{11}$$

where N_i^k is number of neighbours of the cell C_i^k, and $C_i^k = \bigcup_{j=1}^{N_i} C_{i,j}^k$, and the weights $w_{i,j}$ are chosen in such a manner that for linear functions u we have $\tilde{Q}_k u|_{C_i} = u(x_i^k)$, for more refer to [14]. For such a weighted L^2 projection $\tilde{Q}_k$ it has been shown even for unstructured generalized

admissible meshes [14] that for u, solution of (1), and $u_k \in M_k$, solution of (5), we have the error estimate

$$A_k(u_k - \tilde{Q}_k u, u_k - \tilde{Q}_k u) \leq C h_k^2 \|f\|_{0,\Omega}^2. \tag{12}$$

Moreover, the following proposition is also verified.

Proposition 4.1. The $\tilde{Q}_k$ from (11) verifies the so-called weak approximation property: there exist a constant $C > 0$ independent of h_k duch that for all $u \in H^1(\Omega)$ there is

$$\|u - \tilde{Q}_k u\|_{0,\Omega} \leq C h_k |u|_{1,\Omega},$$

where $\|\cdot\|_{0,\Omega}$ is the standard L^2 norm and $|\cdot|_{1,\Omega}$ is the H^1 seminorm.

Proof . From the construction of $\tilde{Q}_k$ it follows that $\tilde{Q}_k$ is stable in L^2 and that it preserves piecewise constant functions on cells C_i^k. Let us choose $c_i = \int_{C_i^k} u d\Omega$. Then using scaled Poincaré inequality locally on C_i^k, and the fact that $\mathrm{diam}(C_i^k) \leq c h_k$, we have

$$\begin{aligned}
\|(I - \tilde{Q}_k)u\|_{0,\Omega}^2 &= \sum_i \|(I - \tilde{Q}_k)u\|_{0,C_i^k}^2 \\
&= \sum_i \|(u - c_i) - \tilde{Q}_k(u - c_i)\|_{0,C_i^k}^2 \\
&\leq C \sum_i \|u - c_i\|_{0,C_i^k}^2 \\
&\leq C h_k^2 \sum_i |u|_{1,C_i^k}^2 \leq C h_k^2 |u|_{1,\Omega}^2,
\end{aligned}$$

which is the statement of the proposition.

■

Proposition 4.2. Let the discrete operator $A_k : M_k \to M_k$ be as in Definition 2.3. Then there exists a constant $C > 0$ independent of h_k such that for all $w \in H^2(\Omega)$ there is

$$A_k(Q_k w, Q_k w) \leq C A(w, w),$$

where Q_k is the L^2 projection and $A(\cdot,\cdot) : H^2 \times L^2 \to \mathbb{R}$ is the continuous bilinear form

$$A(u, v) = (-\Delta u, v)_{L^2(\Omega)},$$

$u \in H^2(\Omega)$, $v \in L^2(\Omega)$.

Proof . Following the methodology of [14] we use scaled Bramble-Hilbert lemma locally for every edge $e \in E_k$ for a linear functional $R_e(w) = (Q_k w)^E - (Q_k w)^W$, which vanishes for all $w \in H^2(\Omega)$ constant, to get $R_e(w) \leq C h_k^{2-d} |w|_{1,C^E \cup C^W}$. Hence, we get

$$\begin{aligned}
A_k(Q_k w, Q_k w) &= \sum_{e \in E_k} \frac{[(Q_k w)^E - (Q_k w)^W]^2}{h_e} \mu(e) \\
&\leq C \sum_{e \in E_k} |w|_{1,C^E \cup C^W}^2 \leq C |w|_{1,\Omega}^2 = C A(w, w),
\end{aligned}$$

which finishes the proof

■

Proposition 4.3. Let the symmetric positive definite discrete operator $A_k : M_k \to M_k$ be as in Definition 2.3. Then its spectral radius $\varrho(A_k)$ is bounded by ch_k^{-2}, ie.

$$A_k(u_k, u_k) \leq ch_k^{-2}\|u_k\|^2_{0,\Omega}.$$

Proof . Simple calculus gives, using $\mu(C_i^k) \geq ch_k^2$,

$$\begin{aligned} A_k(u_k, u_k) &= \sum_{e\in E_k} \frac{(u_k^W - u_k^E)^2}{h_e}\mu(e) \\ &\leq 2\sum_{e\in E_k} \frac{\mu(e)}{h_e}\left[(u_k^W)^2 + (u_k^E)^2\right] \\ &\leq 2N\sum_i (u_k)^2_{C_i^k} \leq 2N\sum_i \frac{1}{\mu(C_i^k)}\int_{C_i^k}(u_k)^2 d\Omega \\ &\leq Ch_k^{-2}\sum_i \|u_k\|^2_{0,C_i^k} \leq Ch_k^{-2}\|u_k\|_{0,\Omega}, \end{aligned}$$

which is the statement of the proposition.

■

Following Bramble, Ewing, Pasciak and Shen [2], we can proceed to the verification of the importatnt regularity and approximation property for the case of I_k^{k-1} being natural injection.

Lemma 4.4. (elliptic regularity pickup) Let the operator A_k be defined as in (5) and $\tilde{P}_{k-1}$ be defined by (8). Then there is a constant $C_\alpha > 0$ not depending on J nor on h_k such that Assumption 3.2

$$A_k((I - \tilde{P}_{k-1})u, u) \leq C_\alpha^2 \left(\frac{\|A_k u\|^2}{\lambda_k}\right)^\alpha A_k(u,u)^{1-\alpha} \qquad \forall u \in M_k, \quad k = 1, \ldots, J$$

holds with $\alpha = 1/2$.

Proof . (cf. [2]) Fix k and let $v \in M_k$ be arbitrary. Let w be the solution of the following boundary value problem:

$$\begin{aligned} -\Delta w &= A_k v \qquad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{13}$$

We first note that for any $\tilde{Q}_k : H^2 \to M_k$ and $\tilde{Q}_{k-1} : H^2 \to M_{k-1}$ we can do the following splitting

$$\begin{aligned} |A_k((I - \tilde{P}_{k-1})v, v)| \leq |A_k(v - \tilde{Q}_k w, v)| + |A_k(\tilde{Q}_k w - \tilde{Q}_{k-1} w, v)| \\ + |A_k(\tilde{Q}_{k-1} w - \tilde{P}_{k-1} v, v)|. \end{aligned} \tag{14}$$

Note that v is the finite volume approximation to w in M_k. For the first term in (14), we apply the Schwarz inequality and the error estimate (12) to get

$$\begin{aligned} |A_k(v - \tilde{Q}_k w, v)| &\leq A_k(v - \tilde{Q}_k w, v - \tilde{Q}_k w)^{\frac{1}{2}} A_k(v, v)^{\frac{1}{2}} \\ &\leq C h_k \|A_k v\| A_k(v, v)^{\frac{1}{2}}. \end{aligned}$$

Similarly, $\tilde{P}_{k-1} v$ is the cell centered approximation of w in M_{k-1}. Thus, repeating the above argument and using (9) gives

$$\begin{aligned} &|A_k(\tilde{Q}_{k-1}\ w\ - \tilde{P}_{k-1} v, v)| \\ &\qquad \leq C A_{k-1}(\tilde{Q}_{k-1} w - \tilde{P}_{k-1} v, \tilde{Q}_{k-1} w - \tilde{P}_{k-1} v)^{\frac{1}{2}} A_k(v, v)^{\frac{1}{2}} \\ &\qquad \leq C h_k \|A_k v\| A_k(v, v)^{\frac{1}{2}} \end{aligned} \tag{15}$$

To complete the proof, we need only to estimate the middle term of (14). We clearly have

$$\begin{aligned} |A_k(\tilde{Q}_k w - \tilde{Q}_{k-1} w, v)| &= |(\tilde{Q}_k w - \tilde{Q}_{k-1} w, A_k v)| \\ &\leq \|\tilde{Q}_k w - \tilde{Q}_{k-1} w\| \|A_k v\| \leq C h_k \|w\|_1 \|A_k v\|. \end{aligned} \tag{16}$$

Here $\|\cdot\|_1$ denotes the norm in $H^1(\Omega)$. The last inequality followed from Proposition 4.1 for $\tilde{Q}_k$, $\tilde{Q}_{k-1}$ and by obvious manipulations.

To complete the proof we need a bound for $\|w\|_1$. Let $A(\cdot, \cdot)$ denote a bilinear form defined by

$$A(u, v) = \int_\Omega -\Delta u v d\Omega.$$

By the definition (13) of w, we clearly have that

$$A(w, w) = (A_k v, w).$$

Thus

$$A(w, w) = (A_k v, Q_k w) \leq A_k(v, v)^{\frac{1}{2}} A_k(Q_k w, Q_k w)^{\frac{1}{2}},$$

here Q_k denotes the L^2 projection onto M_k. Continuing by Proposition 4.2 gives

$$A_k(Q_k w, Q_k, w) \leq C A(w, w)$$

and hence by cancelling $A(w, w)^{\frac{1}{2}}$ from both sides of the inequality gives

$$A(w, w) \leq C A_k(v, v).$$

The above estimates show that

$$|A_k(\tilde{Q}_k w - \tilde{Q}_{k-1} w, v)| \leq C h_k \|A_k v\| A_k(v, v)^{\frac{1}{2}}. \tag{17}$$

Combining (14) - (17) gives

$$|A_k((I - \tilde{P}_{k-1}) v, v)| \leq C h_k \|A_k v\| A_k(v, v)^{\frac{1}{2}}. \tag{18}$$

The regularity and approximation condition with $\alpha = 1/2$ follows from (18) and from Gershgorin's estimate of Proposition 4.3 of the spectral radius of A_k, $\lambda_k \leq c h_k^{-2}$.

■

Thus, Assumptions 3.1 and 3.2 are verified. Even without the energy stability of the prolongation I_k^{k-1} (Assumption 3.4) we can state the following convergence results.

Theorem 4.5. (convergence of W-cycle, sufficiently many smoothings) Let B_J be a W-cycle preconditioner as defined in Algorithm 2.8 with $p = 2$ and let the pre- and post-smoothers satisfy Assumption 3.1. Then for sufficiently many pre- and post-smoothings $m(k) = m$ the W-cycle iteration applied to problem (1) converges with a rate independent of the meshsize,

$$|A_J((I - B_J A_J)u, u)| \leq \delta_J A_J(u, u), \qquad \forall u \in M_J,$$

where $\delta_J \leq \frac{M}{M+m^{\frac{1}{2}}}$, for some constant $M > 0$, independent of J and h_k.

Proof . The proof follows from an abstract convergence Theorem 4.3 in Bramble [1], where all its assumptions are verified by Assumption 3.1 and Lemma 4.4.

■

Theorem 4.6. (variable V-cycle preconditioner) Let B_J be a variable V-cycle preconditioner as defined by Algorithm 2.8 with $p = 1$ and $\beta_0 m(k) \leq m(k-1) \leq \beta_1 m(k)$, for some $1 < \beta_0 \leq \beta_1$ (eg. $\beta_0 = \beta_1 = 2$). Then the multigrid algorithm provides a meshsize independent preconditioner for solving the problem (1),

$$\lambda_{\min}(B_k A_k) \geq \frac{m(k)^{\frac{1}{2}}}{M + m(k)^{\frac{1}{2}}}, \quad \lambda_{\max}(B_k A_k) \leq \frac{M + m(k)^{\frac{1}{2}}}{m(k)^{\frac{1}{2}}},$$

where $M > 0$ is constant.

Proof . The proof follows from an abstract convergence Theorem 4.6 in Bramble [1], where all its assumptions are verified by Assumption 3.1 and Lemma 4.4.

■

Remark 4.7. The convergence achieved by this algorithm is contrary to the popular belief that the sum of the orders of the prolongation and restriction operators should be greater than the order of the differential operator, ie.

$$m_p + m_r > 2, \tag{19}$$

where m_p and m_r denote respectively the order of the prolongation and restriction operators. In our case, the order of both the prolongation and restriction is one. The analysis given in [12] requires that (19) holds.

Let us now focus on the energy stability. For any $u \in M_{k-1}$ we have

$$A_k(u, u) = \sum_{e_k \in E_k} \frac{(u^W - u^E)^2}{h_{e_k}} \mu(e_k).$$

As $u \in M_{k-1} \subset M_k$ we can continue by

$$\sum_{e_k \in E_k} \frac{(u^W - u^E)^2}{h_{e_k}} \mu(e_k) = \sum_{e_{k-1} \in E_{k-1}} (u^W - u^E)^2 [\sum_{e_k \in E_k \cap e_{k-1}} \frac{\mu(e_k)}{h_{e_k}}]$$
$$\leq \sum_{e_{k-1} \in E_{k-1}} \max_{\substack{e_{k-1} \\ e_k \in e_{k-1} \cap E_k}} \left(\frac{\mu(e_k)}{h_{e_k}} \frac{h_{e_{k-1}}}{\mu(e_{k-1})} \right) \frac{(u^W - u^E)^2}{h_{e_{k-1}}} \mu(e_{k-1}).$$

So finally, we get for every $u \in M_k$

$$A_k(u,u) \leq \max_{\substack{e_{k-1} \\ e_k \in e_{k-1} \cap E_k}} \left(\frac{\mu(e_k)}{h_{e_k}} \frac{h_{e_{k-1}}}{\mu(e_{k-1})} \right) A_{k-1}(u,u),$$

and thus

$$\mu = \max_{\substack{e_{k-1} \\ e_k \in e_{k-1} \cap E_k}} \left(\frac{\mu(e_k)}{h_{e_k}} \frac{h_{e_{k-1}}}{\mu(e_{k-1})} \right).$$

Note in Table 3 that the W-cycle might converge with *any* number of pre- and post-smoothings if the energy stability property is verified with $\mu = 2$.

This property is not satisfied in general, but we can enforce it by violating on some interfaces the construction of the coarser level by using $h_{e_{k-1}}$ which makes satisfy the energy stability, but which is no longer the real distance of the corresponding flux-evaluation points x^W_{k-1} and x^E_{k-1}. This will be reflected locally by the loss of consistency (cf. final Remark in [14]) of the underlying coarser-level finite volume discretization, which can be compensated by the use of Il'in inequality, if the perturbation by such a violation is limited to at most K, K bounded independently of h_k, bands of width ch_k accross the computational domain Ω.

Using Il'in inequality will diminish the order of the coarse level approximation from $O(h)$ to $O(h^{\frac{1}{2}})$, which will diminish α in the regularity and approximation Assumption 3.2 to $\alpha = 1/4$. However, the influence of α on the final convergence rate is small (when eg. the number m of pre- and post-smoothing relaxations is small).

For such cases, the W-cycle of Theorem 4.5 might diverge for small number of smoothings, whereas W-cycle with the above correction might still work by the following theorem.

Theorem 4.8. (convergence of W-cycle, any number of smoothings) Let B_J be a W-cycle preconditioner as defined in Algorithm 2.8 with $p = 2$ and let the pre- and post-smoothers satisfy Assumption 3.1. Then for any $m(k) = m$ the W-cycle iteration with the above correction for fluxes applied to problem (1) converges with a rate independent of the meshsize,

$$|A_J((I - B_J A_J)u, u)| \leq \delta_J A_J(u,u), \qquad \forall u \in M_J,$$

where $\delta_J \leq \frac{M}{M + m^{\frac{1}{2}}}$, for some constant $M > 0$.

Proof . The proof follows from an abstract convergence Theorem 4.2 in Bramble [1], where all its assumptions are verified by Assumption 3.1, Lemma 4.4 and (9).

■

5 Linear prolongation and restriction

We devote this section to the analysis of a V-cycle algorithm (cf. Algorithm 2.8, with $p = 1$) for an equidistant structured 2D mesh. The V-cycle algorithms require generally better energy stability properties of the prolongation I_k^{k-1}. With this in mind, let us choose I_k^{k-1} to be natural injection followed by smoothing by a prolongator smoother S_k, whose matrix $\boldsymbol{S}_k$ is given by its stencil

$$\frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{20}$$

which corresponds to the choice $S_k = I - 2\bar{\varrho}(A_k)^{-1}A_k$, where $\bar{\varrho}(A_k) = \|A_k\|_\infty$.

We will proceed like in the previous section, by verifying the assumptions of the abstract convergence theorem in Bramble [1]. Let us begin with Kwak's [16] verification of the elliptic regularity pickup.

Lemma 5.1. Let the operator A_k be defined as in (5) and P_{k-1} be defined by (7). Then there is a constant C not depending on J such that Assumption 3.2

$$A_k((I - I_k^{k-1}P_{k-1})u, u) \le C_\alpha^2 \left(\frac{\|A_k u\|^2}{\lambda_k}\right)^\alpha A_k(u,u)^{1-\alpha} \qquad \forall u \in M_k, \quad k = 1, \dots J$$

holds with $\alpha = 1/2$.

Proof . (cf. [16]) Like in the proof of Lemma 4.4, we build on the fact that for the boundary value problem (13) on coarser levels we have (12):

$$A_k(u_k - \tilde{Q}_k u, u_k - \tilde{Q}_k u) \le C h_k^2 \|f\|_0^2. \tag{21}$$

By definition of P_{k-1} and by adding and subtracting the terms with $\tilde{Q}_k w$ and $\tilde{Q}_{k-1} w$ we can write

$$\begin{aligned} A_k((I - I_k^{k-1}P_{k-1})u, u) &= A_k(u,u) - A_{k-1}(P_{k-1}u, P_{k-1}u) \\ &= A_k(u - \tilde{Q}_k w, u) \\ &\quad + A_{k-1}(\tilde{Q}_{k-1}w - P_{k-1}u, P_{k-1}u) \\ &\quad + A_k(\tilde{Q}_k w, u) - A_{k-1}(\tilde{Q}_{k-1}w, P_{k-1}u). \end{aligned} \tag{22}$$

Applying the Cauchy-Schwarz inequality to the first term and by (21) we have

$$\begin{aligned} |A_k(u - \tilde{Q}_k w, u)| &\le A_k(u - \tilde{Q}_k w, u - \tilde{Q}_k w)^{\frac{1}{2}} A_k(u,u)^{\frac{1}{2}} \\ &\le C h_k \|A_k u\| A_k(u,u)^{\frac{1}{2}}. \end{aligned} \tag{23}$$

The second term can be estimated similarly by noting that $\tilde{P}_{k-1}u$ is the cell-centered approximation of w in M_{k-1}. By the definition of the elliptic projections P_k and $\tilde{P}_k$ we have

$$\begin{aligned} A_{k-1}(\tilde{Q}_{k-1}w - P_{k-1}u, P_{k-1}u) &= A_{k-1}(\tilde{Q}_{k-1}w - \tilde{P}_{k-1}u, P_{k-1}u) \\ &\quad + A_{k-1}(\tilde{P}_{k-1}u - P_{k-1}u, P_{k-1}u) \\ &= A_{k-1}(\tilde{Q}_{k-1}w - \tilde{P}_{k-1}u, P_{k-1}u) \\ &\quad + A_k(u, (I - I_k^{k-1})P_{k-1}u). \end{aligned} \tag{24}$$

By similar arguments as above we see that

$$|A_{k-1}(\tilde{Q}_{k-1}w - \tilde{P}_{k-1}u, P_{k-1}u)| \leq Ch_k\|A_k u\|A_{k-1}(P_{k-1}u, P_{k-1}u)^{\frac{1}{2}}. \tag{25}$$

Since I_k^{k-1} preserves piecewise constant functions and is stable in L^2, we have

$$\begin{aligned} |A_k(u, (I - I_k^{k-1})P_{k-1}u)| &\leq \|A_k u\| \cdot \|(I - I_k^{k-1})P_{k-1}\| \\ &\leq Ch_k A_{k-1}(P_{k-1}u, P_{k-1}u)^{\frac{1}{2}}\|A_k u\|. \end{aligned} \tag{26}$$

Substituting (25) and (26) into (24) we see that the second term in (22) satisfies

$$\begin{aligned} |A_{k-1}(\tilde{Q}_{k-1}w - P_{k-1}u, P_{k-1}u)| &\leq Ch_k\|A_k u\|A_{k-1}(P_{k-1}u, P_{k-1}u)^{\frac{1}{2}} \\ &\leq Ch_k\|A_k u\|A_k(u, u)^{\frac{1}{2}}. \end{aligned} \tag{27}$$

Now we estimate the third and fourth term of (22). Since $I_k^{k-1}\tilde{Q}_{k-1}$ preserves piecewise constant functions, we have

$$\begin{aligned} |A_k(\tilde{Q}_k w, u) - A_{k-1}(\tilde{Q}_{k-1}w, P_{k-1}u)| &= |A_k(\tilde{Q}_k w, u) - A_k(I_k^{k-1}\tilde{Q}_{k-1}w, u)| \\ &\leq \|(I - I_k^{k-1}\tilde{Q}_{k-1})w\| \cdot \|A_k u\| \\ &\leq Ch_k\|w\|_1\|A_k u\|, \end{aligned}$$

where $\|\cdot\|_1$ is the $H^1(\Omega)$ norm. The remaining bound of $\|w\|_1$ by $A_k(u, u)^{\frac{1}{2}}$ is done exactly like in the proof of Lemma 4.4. Thus we have

$$|A_k(\tilde{Q}_k w, u) - A_{k-1}(\tilde{Q}_{k-1}w, P_{k-1}u)| \leq Ch_k\|A_k u\|A_k(u, u)^{\frac{1}{2}}. \tag{28}$$

Combining the estimates (23), (27) and (28) together with Gershgorin's estimate $\lambda_k \leq Ch_k^{-2}$ completes the proof.

■

Before we state the convergence theorem for the V-cycle algorithm, it remains to show the energy stability of the prolongation

Lemma 5.2. (energy stability of prolongation) Let all the cells be rectangular with dimensions h_x and h_y which do not change accross the domain. Further, let the flux-evaluation points x_i^k coincide with barycenters. Then we have for all $u \in M_k$

$$A_k(I_k^{k-1}u, I_k^{k-1}u) \leq A_{k-1}(u, u).$$

Proof . The proof is done by direct computation for rectangular domains, for more details see [16].

■

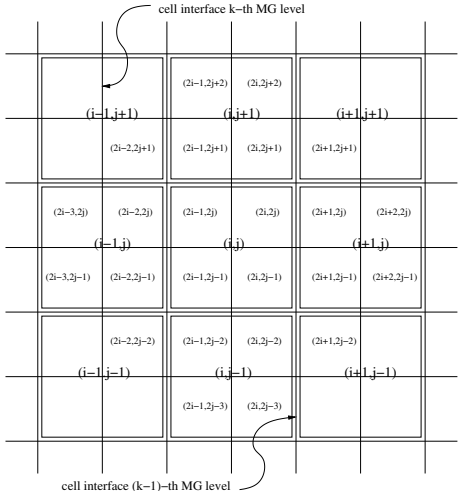

Figure 4: Numbering of cells in a square domain

Theorem 5.3. (convergence of V-cycle) Let B_J be a V-cycle multigrid preconditioner as defined by Algorithm 2.8, ($p = 1$) with the prolongation and the restriction smoothed by the smoother S_k as in (20). Let the pre- and post-smoothers satisfy Assumption 3.1. Then for any $m(k) = m$ the V-cycle applied to the problem (1) converges independently of the meshsize, with the rate δ_J,

$$0 \leq A_J((I - B_J A_J)u, u) \leq \delta_J A_J(u, u), \qquad \forall u \in M_J,$$

where $\delta_J = \frac{MJ}{MJ+m^{\frac{1}{2}}}$, with some constant $M > 0$.

Proof . The proof follows from an abstract convergence Theorem 4.1 in Bramble [1], where all its assumptions are verified by Assumption 3.1, Lemma 5.1 and Lemma 5.2.

■

6 Non-symmetric prolongation and restriction

Up to this point we have investigated multigrids with symmetric restrictions and prolongations as required by (6). Let us now consider the case, when the restriction $I_{k-1}^k : M_k \to M_{k-1}$ is an L^2 projection to M_{k-1} and the prolongation $I_k^{k-1} : M_{k-1} \to M_k$ is chosen as a simple injection $\tilde{I}_k^{k-1}$ followed by a prolongation smoother S_k.

Although the abstract convergence results of Table 3 require the symmetric (Ritz-Galerkin) type of prolongation and restriction, we can also analyse an algorithm with non-symmetric (Petrov-Galerkin) type of transfer operators using the trick of Vaněk, Janka, Guillard [17], this time in the context of 'non-inherited quadratic forms'.

We will, in fact, analyse a slight modification of Algorithm 2.8 - after the Step 2. we will do one additional presmoothing step using the matrix S_k as the iteration matrix. In the computer implementation, however, we skip this auxiliary step, which corresponds to leaving out one Jacobi-like iteration. Numerical results show that this does not effect much the convergence rate.

Algorithm 6.1. (iteration phase, Petrov-Galerkin) Let p be a positive integer cycle parameter (eg. $p = 1$ or $p = 2$). Set $B_1 = A_1^{-1}$. Let $k > 1$ and assume that B_{k-1} has been defined. Given the right-hand side $g \in M_k$, $B_k g$ is defined as follows:

1. Set $x^0 = q^0 = 0$.
2. Define x^l for $l = 1, \ldots, m(k)$ by
$$x^l = x^{l-1} + R_k^{(l+m(k))}(g - A_k x^{l-1}).$$
2a. Make one additional presmoothing step (for theoretical purposes only)
$$x^l \leftarrow S_k x^l + (I - S_k)A_k^{-1} g.$$
3. $y^{m(k)} = x^{m(k)} + S_k \tilde{I}_k^{k-1} q^p$, where q^i is defined by:
$$q^i = q^{i-1} + B_{k-1}[\tilde{I}_{k-1}^k (g - A_k x^{m(k)}) - A_{k-1} q^{i-1}].$$
4. Define y^l for $l = m(k) + 1, \ldots, 2m(k)$ by
$$y^l = y^{l-1} + R_k^{(l+m(k))}(g - A_k y^{l-1}).$$
5. Set $B_k g = y^{2m(k)}$.

Lemma 6.2. Let us have the Petrov-Galerkin Algorithm 6.1 with the additional presmoothing Step 2a. and assume that

1. S_k is a positive semi-definite symmetric operator,
2. S_k commutes with A_k.

Then the operator $S_k^{\frac{1}{2}}$ is well defined as the square root of S_k and the Petrov-Galerkin algorithm is equivalent to a Ritz-Galerkin multigrid with the prolongation $\bar{I}_k^{k-1} = S_k^{\frac{1}{2}} P_{k+1}^k$, and the symmetric restriction $\bar{I}_{k-1}^k$ which verifies (6).

Proof . (cf. [17])

■

Remark 6.3. Note, that while S_k needs to be sparse (for algorithmic and memory reasons), the square root of it is generally full matrix, ie. one coarse-grid nodal value is prolongated by $S_k^{\frac{1}{2}} \tilde{I}_k^{k-1}$ to the whole fine-level domain.

Definition 6.4. Let us decompose the operator $A_k : M_k \to M_k$ to the components A_k^x and A_k^y, $A_k = A_k^x + A_k^y$, coming from discretization of the terms $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$, respectively. Let us define the prolongation smoother $S_k : M_k \to M_k$ in terms of A_k^x and A_k^y,

$$S_k = \left(I - \frac{1}{\varrho(A_k^x)} A_k^x\right)\left(I - \frac{1}{\varrho(A_k^y)} A_k^y\right),$$

where $\varrho(A_k^x)$ and $\varrho(A_k^y)$ are respectively the spectral radii of A_k^x and A_k^y.

From Definition 6.4 it clearly follows, that S_k is symmetric, as A_k^x and A_k^y are symmetric, and that it is positive semi-definite.

Thus it has only real non-negative eigenvalues and $S_k^{\frac{1}{2}}$ is well defined. Note that linear interpolation smoothing operator of Kwak [16] in the previous Section is indefinite.

Hence, we need to prove just the commutativity of S_k and A_k to be able to pass from the Petrov-Galerkin non-symmetric prolongation-restriction algorithm to the symmetric case of Bramble [5].

Proposition 6.5. Let the finite volume cells be rectangular, with sides h_k^x and h_k^y, different in general but constant for all cells and let the flux evaluation points $x_i^k \in C_i^k$ be chosen as the barycenters. Then the operators A_k and S_k commute, ie. $A_k S_k = S_k A_k$.

Proof . From the definition of S_k we have

$$\begin{aligned}
S_k A_k &= [I - \varrho(A_k^x)^{-1} A_k^x][I - \varrho(A_k^y)^{-1} A_k^y](A_k^x + A_k^y) \\
&= [I - \varrho(A_k^x)^{-1} A_k^x - \varrho(A_k^y)^{-1} A_k^y + \varrho(A_k^x)^{-1}\varrho(A_k^y)^{-1} A_k^x A_k^y](A_k^x + A_k^y) \\
&= (A_k^x + A_k^y) - \varrho(A_k^x)^{-1}(A_k^x A_k^x + A_k^x A_k^y) - \varrho(A_k^y)^{-1}(A_k^y A_k^x + A_k^y A_k^y) \\
&\quad + \varrho(A_k^x)^{-1}\varrho(A_k^y)^{-1}(A_k^x A_k^y A_k^x + A_k^x A_k^y A_k^y).
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
A_k S_k = (A_k^x + A_k^y) &- \varrho(A_k^x)^{-1}(A_k^x A_k^x + A_k^y A_k^x) - \varrho(A_k^y)^{-1}(A_k^x A_k^y + A_k^y A_k^y) \\
&+ \varrho(A_k^x)^{-1}\varrho(A_k^y)^{-1}(A_k^x A_k^x A_k^y + A_k^y A_k^x A_k^y).
\end{aligned}$$

Hence, it suffices to show that A_k^x and A_k^y commutes to get the commutativity of A_k and S_k. Moreover, because of the symmtery of A_k^x and A_k^y we have $A_k^x A_k^y = (A_k^y A_k^x)^T$ and the whole proof reduces to verification of symmetry of $A_k^x A_k^y$. More precisely, we are going to verify by direct calculation the commutativity of the corresponding *matrices* $\boldsymbol{A}_k^x$ and $\boldsymbol{A}_k^y$. Let us denote $\mathbf{y}_{(i,j)} \in \mathbb{R}^{n_k}$ a vector composed of zeros with 1 at the position corresponding to the cell $C_{i,j}^k$ (cf. Fig. 4). Let us denote $p_{i,j+\frac{1}{2}} = h_y/h_x$ and $p_{i+\frac{1}{2},j} = h_x/h_y$.

$$\boldsymbol{A}_k^y \mathbf{x}_{(i,j)} = \mathbf{y}_{(i,j)}$$

$$\begin{aligned}
\mathbf{y}_{(i,j)(i,j)} &= p_{i,j+\frac{1}{2}} + p_{i,j-\frac{1}{2}} \\
\mathbf{y}_{(i,j)(i,j+1)} &= -p_{i,j+\frac{1}{2}} \\
\mathbf{y}_{(i,j)(i,j-1)} &= -p_{i,j-\frac{1}{2}}
\end{aligned}$$

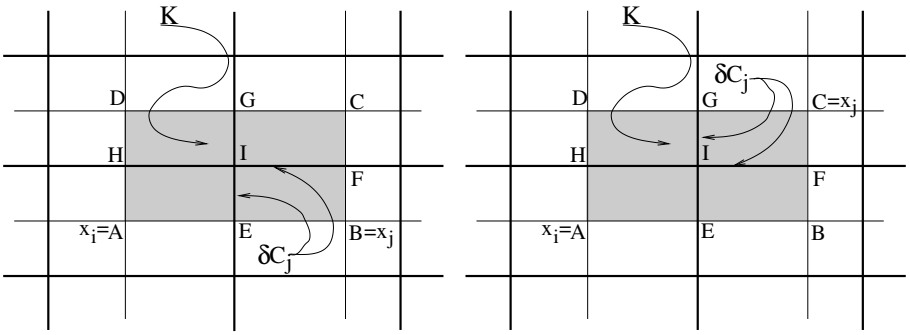

Figure 5: Finite element-finite volume Q1 element K and dual cells C_j

$$\boldsymbol{A}_k^x \mathbf{y}_{(i,j)} = \mathbf{z}_{(i,j)}$$

$$\begin{aligned}
\mathbf{z}_{(i,j)(i,j)} &= (p_{i+\frac{1}{2},j} + p_{i-\frac{1}{2},j})(p_{i,j+\frac{1}{2}} + p_{i,j-\frac{1}{2}}) \\
\mathbf{z}_{(i,j)(i,j+1)} &= (p_{i+\frac{1}{2},j+1} + p_{i-\frac{1}{2},j+1})(-p_{i,j+\frac{1}{2}}) \\
\mathbf{z}_{(i,j)(i,j-1)} &= (p_{i+\frac{1}{2},j-1} + p_{i-\frac{1}{2},j-1})(-p_{i,j-\frac{1}{2}}) \\
\mathbf{z}_{(i,j)(i-1,j)} &= (-p_{i-\frac{1}{2},j})(p_{i,j+\frac{1}{2}} + p_{i,j-\frac{1}{2}}) \\
\mathbf{z}_{(i,j)(i-1,j+1)} &= (-p_{i-\frac{1}{2},j+1})(-p_{i,j+\frac{1}{2}}) \\
\mathbf{z}_{(i,j)(i-1,j-1)} &= (-p_{i-\frac{1}{2},j-1})(-p_{i,j-\frac{1}{2}}) \\
\mathbf{z}_{(i,j)(i+1,j)} &= (-p_{i+\frac{1}{2},j})(p_{i,j+\frac{1}{2}} + p_{i,j-\frac{1}{2}}) \\
\mathbf{z}_{(i,j)(i+1,j+1)} &= (-p_{i+\frac{1}{2},j+1})(-p_{i,j+\frac{1}{2}}) \\
\mathbf{z}_{(i,j)(i+1,j-1)} &= (-p_{i+\frac{1}{2},j-1})(-p_{i,j-\frac{1}{2}})
\end{aligned}$$

Now, the commutativity of $\boldsymbol{A}_k^x$ and $\boldsymbol{A}_k^y$ follows from the fact that

$$\mathbf{z}_{(i,j)(n,m)} = \mathbf{z}_{(n,m)(i,j)},$$

which is true, if $p_{i\pm\frac{1}{2},j}$ is constant for all possible i and j, ie. if h_x and h_y are constant over the whole domain. The same for $p_{j\pm\frac{1}{2},i}$.

■

Corollary 6.6. A finite volume-element discretization with bilinear gradient reconstruction and a finite element Q1 discretization of a 2D elliptic problem

$$\begin{aligned}
-\nabla \cdot (\bar{A} \nabla u) &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}$$

with $\bar{A}$ a (2×2) diagonal positive matrix with piecewise constant entries on each finite element give on rectangular elements $\{K_n\}_{\forall n}$ with Voronoi-based dual finite volume control cells $\{C_m\}_{\forall m}$ (cf. Fig. 5) equivalent discrete problems: for each two Q1 finite element basis functions ϕ_i, ϕ_j there is

$$-\sum_{\forall m}\left(\int_{\partial C_m}(\bar{A}\nabla\phi_i)\mathbf{n}_{\partial C_m}d\Gamma\right)\phi_j(x_m)=\int_\Omega(\bar{A}\nabla\phi_i)\nabla\phi_j d\Omega.$$

Proof . Inspired by [13], the statement of the corollary will be verified when

$$-\int_{\partial C_j}(\bar{A}\nabla\phi_i)\mathbf{n}_{\partial C_j}d\Gamma=\int_\Omega(\bar{A}\nabla\phi_i)\nabla\phi_j d\Omega. \tag{29}$$

The identity (29) will be proved, if we show that for any finite element K there is

$$-\int_{\partial C_j\cap K}(\bar{A}\nabla\phi_i)\mathbf{n}_{\partial C_j}d\Gamma=\int_K(\bar{A}\nabla\phi_i)\nabla\phi_j d\Omega. \tag{30}$$

From Green's formula we get for the left-hand side of (30)

$$\begin{aligned}-\int_{\partial C_j\cap K}(\bar{A}\nabla\phi_i)\mathbf{n}_{\partial C_j}d\Gamma&=-\int_{C_j\cap K}\nabla\cdot(\bar{A}\nabla\phi_i)d\Omega+\int_{C_j\cap\partial K}(\bar{A}\nabla\phi_i)\mathbf{n}d\Gamma\\&=\int_{C_j\cap\partial K}(\bar{A}\nabla\phi_i)\mathbf{n}d\Gamma,\end{aligned}$$

where the first term on the rigth-hand side vanishes, because $\bar{A}$ is diagonal and ϕ_i is bilinear. Expressing the integration path for the two possible configurations of ϕ_i and ϕ_j (cf. Fig. 5) gives

$$\int_{C_j\cap\partial K}(\bar{A}\nabla\phi_i)\mathbf{n}d\Gamma=\begin{cases}\int_{|EB|}(\bar{A}\nabla\phi_i)\mathbf{n}_{EB}d\Gamma+\int_{|BF|}(\bar{A}\nabla\phi_i)\mathbf{n}_{BF}d\Gamma & \text{Fig. 5 left,}\\ \int_{|FC|}(\bar{A}\nabla\phi_i)\mathbf{n}_{FC}d\Gamma+\int_{|CG|}(\bar{A}\nabla\phi_i)\mathbf{n}_{CG}d\Gamma & \text{Fig. 5 right.}\end{cases} \tag{31}$$

For the right-hand side of (30) we have by integration by parts

$$\begin{aligned}\int_K(\bar{A}\nabla\phi_i)\nabla\phi_j d\Omega&=-\int_K\nabla\cdot(\bar{A}\nabla\phi_i)\phi_j d\Omega+\int_{\partial K}(\bar{A}\nabla\phi_i)\mathbf{n}\phi_j d\Gamma\\&=\int_{\partial K}(\bar{A}\nabla\phi_i)\mathbf{n}\phi_j d\Gamma.\end{aligned}$$

Also here, the first term on the right-hand side vanishes. Expressing the integration path for the two possible configurations of ϕ_i and ϕ_j (cf. Fig. 5) and considering that ϕ_j vanishes on some parts of the integration path and that $(\bar{A}\nabla\phi_i)\mathbf{n}$ is constant along each of the sides of the rectangle $ABCD$,

$$\int_{\partial K}(\bar{A}\nabla\phi_i)\mathbf{n}\phi_j d\Gamma=\begin{cases}\frac{1}{2}\left(\int_{|AB|}(\bar{A}\nabla\phi_i)\mathbf{n}_{AB}d\Gamma+\int_{|BC|}(\bar{A}\nabla\phi_i)\mathbf{n}_{BC}d\Gamma\right) & \text{Fig. 5 left,}\\ \frac{1}{2}\left(\int_{|BC|}(\bar{A}\nabla\phi_i)\mathbf{n}_{BC}d\Gamma+\int_{|CD|}(\bar{A}\nabla\phi_i)\mathbf{n}_{CD}d\Gamma\right) & \text{Fig. 5 right.}\end{cases} \tag{32}$$

Comparing (31) to (32) and realizing that $|AB| = 2|EB|$, $|BC| = 2|BF|$, $|CD| = 2|CG|$ verifies the statement of the corollary.

■

To be able to prove the convergence of the V-cycle, we have to show, that the smoothed prolongation $\bar{I}_k^{k-1} : M_k \to M_{k-1}$ of the equivalent Ritz-Galerkin algorithm is stable in energy.

Lemma 6.7. (energy stability of prolongation) For all $u \in M_{k-1}$ there is

$$A_k(\bar{I}_k^{k-1}u, \bar{I}_k^{k-1}u) \leq A_{k-1}(u, u).$$

Proof . First we note, that $A_k(\bar{I}_k^{k-1}u, \bar{I}_k^{k-1}u) = A_k(I_k^{k-1}u, u)$, because $S_k^{\frac{1}{2}}$ commutes with A_k.
The proof will be done in two steps: first, we show either by Corollary 6.6 or by direct computation that for all $u \in M_{k-1}$ there is

$$A_k(I_k^{k-1}u, u) = A_{k-1}^{Q1}(u, u),$$

here $A_{k-1}^{Q1}(\cdot, \cdot)$ is a quadratic form

$$A_{k-1}^{Q1}(u, v) = \sum_{ij} u_i v_j A(\phi_i, \phi_j),$$

where ϕ_i, ϕ_j are the standard Q1 finite element basis functions and u_i is the value of $u \in M_{k-1}$ on the cell C_i^{k-1}. For a structured mesh with square elements the corresponding $(n_{k-1} \times n_{k-1})$ matrix $\boldsymbol{A}_{k-1}^{Q1}$ defined for $v = \sum_i \mathbf{v}_i \chi_i^{k-1}$, $u = \sum_i \mathbf{u}_i \chi_i^{k-1}$,

$$\langle \boldsymbol{A}_{k-1}^{Q1}\mathbf{u}, \mathbf{v}\rangle = A_{k-1}^{Q1}(u, v) = \sum_{ij} u_i v_j A(\phi_i, \phi_j),$$

is equivalent to a nine-point stencil discretization A_{k-1}^{9pt} of Δ, with the stencil

$$\frac{1}{4}\begin{bmatrix} -1 & -2 & -1 \\ -2 & 12 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

Thus, by the Definition 6.4, S_k has the following nine-point stencil

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \tag{33}$$

in the interior. The typical stencils on edges and in corners are

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad , \quad \frac{1}{16}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

respectively.

Then, in the second step we will show that for all $u \in M_{k-1}$ there is

$$A_{k-1}^{9pt}(u,u) \le A_{k-1}(u,u).$$

Let us denote $\mathbf{v}^k \in \mathbb{R}^{n_k}$ be a vector from the k-th multigrid level with the entries $v_{i,j}^k$, where we use the indexing as for a grid function in 2D. Also, set $n_k^1 = \sqrt{n_k}$.

Now, let us estimate $A_k(I_k^{k-1}u, u)$. From the definition (4) of the quadratic form $A_k(\cdot,\cdot)$ we have

$$A_k(v,u) = \sum_{i,j=1}^{2n_{k-1}^1, 2n_{k-1}^1} u_{i,j}^k \left[(v_{i,j}^k - v_{i+1,j}^k) + (v_{i,j}^k - v_{i-1,j}^k) + (v_{i,j}^k - v_{i,j+1}^k) + (v_{i,j}^k - v_{i,j-1}^k)\right].$$

Using the fact, that if $\mathbf{u}^k = P_k^{k-1}\mathbf{u}^{k-1}$ then $u_{i,j}^{k-1} = u_{2i-1,2j-1}^k = u_{2i,2j-1}^k = u_{2i-1,2j}^k = u_{2i,2j}^k$. Thus,

$$\begin{aligned} A_k(v,u) = \sum_{i,j=1}^{n_{k-1}^1, n_{k-1}^1} u_{i,j}^{k-1}[&(v_{2i-1,2j-1}^k - v_{2i-2,2j-1}^k) + (v_{2i-1,2j-1}^k - v_{2i-1,2j-2}^k) + \\ &(v_{2i,2j-1}^k - v_{2i+1,2j-1}^k) + (v_{2i,2j-1}^k - v_{2i,2j-2}^k) + \\ &(v_{2i-1,2j}^k - v_{2i-2,2j}^k) + (v_{2i-1,2j}^k - v_{2i-1,2j+1}^k) + \\ &(v_{2i,2j}^k - v_{2i+1,2j}^k) + (v_{2i,2j}^k - v_{2i,2j+1}^k)]. \end{aligned}$$

Now, let $\mathbf{v}^k = S_k P_k^{k-1}\mathbf{u}^{k-1}$, S_k is given by (33).

$$\begin{aligned} &A_k(I_k^{k-1}u, u) = \\ &= \frac{1}{8}\sum_{i,j=1}^{n_{k-1}^1, n_{k-1}^1} u_{i,j}^{k-1}[(3u_{i,j}^{k-1} + u_{i,j-1}^{k-1} - 3u_{i-1,j}^{k-1} - u_{i-1,j-1}^{k-1}) + (3u_{i,j}^{k-1} + u_{i-1,j}^{k-1} - 3u_{i,j-1}^{k-1} - u_{i-1,j-1}^{k-1}) \\ &\qquad +(3u_{i,j}^{k-1} + u_{i,j-1}^{k-1} - 3u_{i+1,j}^{k-1} - u_{i+1,j-1}^{k-1}) + (3u_{i,j}^{k-1} + u_{i+1,j}^{k-1} - 3u_{i,j-1}^{k-1} - u_{i+1,j-1}^{k-1}) \\ &\qquad +(3u_{i,j}^{k-1} + u_{i,j+1}^{k-1} - 3u_{i-1,j}^{k-1} - u_{i-1,j+1}^{k-1}) + (3u_{i,j}^{k-1} + u_{i-1,j}^{k-1} - 3u_{i,j+1}^{k-1} - u_{i-1,j+1}^{k-1}) \\ &\qquad +(3u_{i,j}^{k-1} + u_{i,j+1}^{k-1} - 3u_{i+1,j}^{k-1} - u_{i+1,j+1}^{k-1}) + (3u_{i,j}^{k-1} + u_{i+1,j}^{k-1} - 3u_{i,j+1}^{k-1} - u_{i+1,j+1}^{k-1})] \\ &= \frac{1}{4}\sum_{i,j=1}^{n_{k-1}^1, n_{k-1}^1} u_{i,j}^{k-1}[2(u_{i,j}^{k-1} - u_{i,j+1}^{k-1}) + 2(u_{i,j}^{k-1} - u_{i,j-1}^{k-1}) + 2(u_{i,j}^{k-1} - u_{i+1,j}^{k-1}) + 2(u_{i,j}^{k-1} - u_{i-1,j}^{k-1}) \\ &\qquad +(u_{i,j}^{k-1} - u_{i+1,j+1}^{k-1}) + (u_{i,j}^{k-1} - u_{i-1,j-1}^{k-1}) + (u_{i,j}^{k-1} - u_{i+1,j-1}^{k-1}) + (u_{i,j}^{k-1} - u_{i-1,j+1}^{k-1})] \\ &= A_{k-1}^{9pt}(u,u). \end{aligned}$$

Now, it remains to show, that for all $u \in M_k$ there is

$$A_k^{9pt}(u,u) \le A_k(u,u), \tag{34}$$

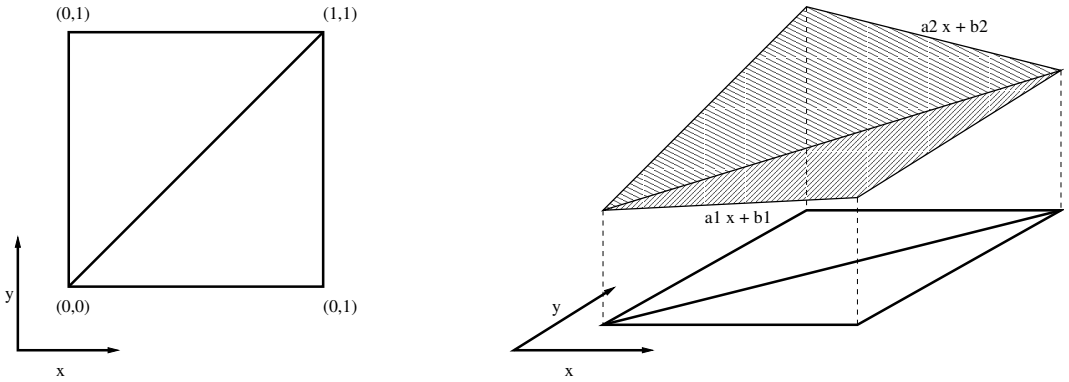

Figure 6: Reference Q1 element (left), P1 function of the reference element (right)

Let us have $\mathbf{v} \in \mathbb{R}^{n_k}$ and denote respectively u^{Q1} and u^{P1} the Q1 and P1 finite element functions corresponding to the vector $\mathbf{u}$. Then it suffices to show that

$$\int_\Omega \left[\left(\frac{\partial u^{Q1}}{\partial x} \right)^2 + \left(\frac{\partial u^{Q1}}{\partial y} \right)^2 \right] d\Omega \leq \int_\Omega \left[\left(\frac{\partial u^{P1}}{\partial x} \right)^2 + \left(\frac{\partial u^{P1}}{\partial y} \right)^2 \right] d\Omega. \tag{35}$$

Let us consider $K = [0,1] \times [0,1]$ to be the Q1 reference element (ie. two P1 reference elements, cf. Fig. 6, left). Let us have $u(0,0) = b_1$, $u(0,1) = a_1 + b_1$, $u(1,0) = b_2$, $u(1,1) = a_2 + b_2$, for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$ (cf. Fig. 6, right). Now, we will focus on the $\partial/\partial x$ part of the inequality (35), the $\partial/\partial y$ part follows by similarity.

For u^{P1}, the gradient is constant on each simplex element, thus

$$\frac{\partial u^{P1}}{\partial x}|_{(x,y)} = \begin{cases} a_2 & x \in [0,y], \\ a_1 & x \in [y,1]. \end{cases}$$

For u^{Q1}, the gradient varies linearly, ie.

$$\frac{\partial u^{Q1}}{\partial x}|_{(x,y)} = (1-y)a_1 + ya_2.$$

Now,

$$\begin{aligned} \int_K \left(\frac{\partial u^{P1}}{\partial x} \right)^2 &= \int_0^1 \left[\int_0^y a_2^2 dx + \int_y^1 a_1^2 dx \right] dy \\ &= \int_0^1 [a_2^2 y + a_1^2 - a_1^2 y] dy = \left[\frac{1}{2} a_2^2 y^2 + a_1^2 y - \frac{1}{2} a_1^2 y^2 \right]_0^1 = \frac{1}{2}(a_1^2 + a_2^2), \end{aligned}$$

and

$$\int_K \left(\frac{\partial u^{Q1}}{\partial x}\right)^2 = \int_0^1 \int_0^1 [(1-y)a_1 + ya_2]^2 dxdy$$
$$= \int_0^1 \left[a_1^2 y + a_1(a_2-a_1)y^2 + \frac{1}{3}(a_2-a_1)^2 y^3\right]_0^1 dx = \frac{1}{3}(a_1^2 + a_1a_2 + a_2^2).$$

The estimate (34) will follow from the fact that

$$\frac{1}{3}(a_1^2 + a_1a_2 + a_2^2) \le \frac{1}{2}(a_1^2 + a_2^2), \qquad \forall a_1, a_2 \in \mathbb{R}.$$

Indeed, by elementary operations,

$$2a_1^2 + 2a_1a_2 + 2a_2^2 \le 3a_1^2 + 3a_2^2$$
$$0 \le a_1^2 - 2a_1a_2 + a_2^2 = (a_1 - a_2)^2.$$

This completes the proof.

■

Now, we should turn our attention to an analogy of Lemma 5.1 for the equivalent Ritz-Galerkin algorithm of Lemma 6.2. Similarly to (7) we define $\bar{P}_{k-1} : M_k \to M_{k-1}$ such that

$$A_{k-1}(\bar{P}_{k-1}u, v) = A_k(u, \bar{I}_k^{k-1}v), \qquad \forall u \in M_k, \quad \forall v \in M_{k-1}. \tag{36}$$

Lemma 6.8. Let $S_k^{\frac{1}{2}}$, A_k be defined by Definitions 6.4 and 2.3. Then there exists a constant $C > 0$ independent of h_k such that for all $u \in M_k$ there is

$$\|(I - S_k^{\frac{1}{2}})u\|_{0,\Omega}^2 \le Ch_k^2 |u|_{1,h_k}^2.$$

Here, $|\cdot|_{1,h_k}$ is the usual H^1 discrete seminorm on M_k.

Proof . Using the fact that $S_k^{\frac{1}{2}}$ is positive semi-definite, we have

$$\|u\|_{0,\Omega}^2 \le \|(I + S_k)u\|_{0,\Omega}^2.$$

As S_k preserves the constant and is stable in L^2 we get by discrete scaled Poincaré's inequality (cf. [11], Lemma 3.7) locally on each of the control-volume agglomerates

$$\|(I - S_k^{\frac{1}{2}})u\|_{0,\Omega}^2 \le \|(I + S_k^{\frac{1}{2}})(I - S_k^{\frac{1}{2}})u\|_{0,\Omega}^2 = \|(I - S_k)u\|_{0,\Omega}^2 \le Ch_k^2 |u|_{1,h_k}^2$$

which ends the proof.

■

Now, we are about to verify the regularity assumption for the equivalent Ritz-Galerkin algorithm of Lemma 6.2.

Lemma 6.9. Let the operator A_k be defined as in (5) and $\bar{P}_{k-1}$ be defined by (36). Then there is a constant C not depending on J such that Assumption 3.2 modified for the equivalent Ritz-Galerkin algorithm from Lemma 6.2, ie.

$$A_k((I - \bar{I}_k^{k-1}\bar{P}_{k-1})u, u) \leq C_\alpha^2 \left(\frac{\|A_k u\|^2}{\lambda_k}\right)^\alpha A_k(u,u)^{1-\alpha} \qquad \forall u \in M_k, \quad k = 1, \dots J$$

holds with $\alpha = 1/2$.

Proof . This proof takes heavily from the Kwak's proof of Lemma 5.1. We just need to realize from Lemma 6.8, from the fact that S_l preserves constant functions apart from the boundary, and Lemma 6.7, that for all $u \in M_{k-1}$ there is

$$\begin{aligned}
\|(I - \bar{I}_k^{k-1})u\| &= \|(I - S_k^{\frac{1}{2}}\bar{I}_k^{k-1} + (S_k^{\frac{1}{2}} - I)\bar{I}_k^{k-1})u\| \\
&\leq \|(I - S_k^{\frac{1}{2}}\bar{I}_k^{k-1})u\| + \|(I - S_k^{\frac{1}{2}})\bar{I}_k^{k-1}u\| \\
&\leq \|(I - I_k^{k-1})u\| + Ch_k A_k(\bar{I}_k^{k-1}u, \bar{I}_k^{k-1}u)^{\frac{1}{2}} \\
&\leq Ch_k A_k(u,u)^{\frac{1}{2}}.
\end{aligned}$$

Otherwise we just formally replace P_k by $\bar{P}_k$ and I_k^{k-1} by $\bar{I}_k^{k-1}$ in the proof of Lemma 5.1. This completes the proof.

■

Theorem 6.10. (convergence of Petrov-Galerkin V-cycle) Let B_J be a V-cycle multigrid preconditioner (cf. Algorithm 2.8, $p = 1$) with the smoothed prolongation $I_k^{k-1} = S_k P_k^{k-1}$ and non-smoothed restriction I_{k-1}^k. Then the multigrid algorithm with the additional presmoothing step by S_k (cf. Lemma 6.2) converges independently of the mesh size, with the rate δ_J,

$$0 \leq A_J((I - B_J A_J)u, u) \leq \delta_J A_J(u,u), \qquad \forall u \in M_J,$$

where $\delta_J = \frac{MJ}{MJ + m^{\frac{1}{2}}}$, with some constant $M > 0$.

Proof . In Lemma 6.2 we stated, that the Petrov-Galerkin V-cycle with the additional presmoothing step is equivalent with one Ritz-Galerkin V-cycle.

The proof for the equivalent Ritz-Galerkin V-cycle follows from an abstract convergence Theorem 4.1 in Bramble [1], whose assumptions are verified by Assumption 3.1, Lemma 6.9 and Lemma 6.7.

■

7 Numerical experiments

Here, we are going to give some numerical results for the multigrid algorithms described in this presentation (ie. W-cycle, Ritz-Galerkin V-cycle, Petrov-Galerkin V-cycle and the variable V-cycles of the latter type) applied to structured (possibly non-regular) cell-centered discretization

of the following problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{37}$$

for several Ω, $\Omega \subset [0,1] \times [0,1]$. We present the converegence rates in Tables below.

We employ in all cases the multigrid cycle with J levels and as the pre- and post-smoothers we use one iteration of a simple damped Richardson method, $R_l = 1.6/\varrho(A_l)$.

Each time, we record as a function of the mesh parameter h_J and the numer of MG levels J the estimate of minimal and maximal eigenvalues $\lambda_{\min}(C_J)$ and $\lambda_{\max}(C_J)$ of the preconditioned problem $C_J = B_J A_J$ (B_J is the multigrid preconditioner), the condition number of the preconditioned problem $\mathcal{K}(C_J)$, the estimated worst convergence rate δ based on the eigenvalues of the preconditioned problem, $\delta = \max(1 - \lambda_{\min}, \lambda_{\max} - 1)$ and one particular convergence rate $\bar{\delta}$ based on the $H1$ semi-norm of error for the case when the right-hand side of the problem f is constant over the domain Ω.

The estimates of $\lambda_{\max}$ and $\lambda_{\min}$ were obtained by 100 iterations of power-method algorithm for the original and the shifted operators, respectively.

7.1 Regular structured square domain

Let us now concentrate on Table 2 describing the convergence properties of the tested multigrid algorithms for a regular structured discretization of Laplacian over a unit square, ie. when $\Omega = [0,1] \times [0,1]$. Table 2 on the top shows the dependency of the convergence properties on the finest-level mesh-size h_J. We have fixed the number of MG levels J to 6 and progressively increase the number of nodes on the finest level, from $(128)^2$=16384 to $(640)^2$=409600. We observe, that all five methods behave more or less independently of h_J, which matches the theoretical conclusions. Moreover, we find out that Petrov-Galerkin V-cycle with constant restriction and bi-linear prolongation is suprisingly better then Ritz-Galerkin V-cycle with a linear restriction and prolongation. Also notice, that the maximal eigenvalue $\lambda_{\max}$ of the preconditioned problem is always less or equal to one for all V-cycles. This corresponds to the proven fact, that the error reduction operator of a multigrid iteration $E_J = I - B_J A_J$ is positive definite. Last but not least, we notice that Petrov-Galerkin V-cycles behave according to the theoretical results independently of h_J, even without the additional pre-smoothing Step 2a. in Algorithm 6.1.

Table 2 on the bottom shows the dependency of the convergence properties of the five algorithms with respect to the number of MG level J. We fixed the number of finest-level nodes to $(512)^2$=262144 and we increase the number of MG levels from 4 to 8. Observe, that while the convergence properties of W-cycle and the variable V-cycles do not depend on J, while they do for simple V-cycles. This fact is also predicted by the theoretical results in Table 3.

7.2 Regular structured L-shaped domain

Let us now choose Ω to be an L-shaped domain (cf. Fig. 7). Table 3 show the convergence properties for this case. This case is just to illustrate the advantage of the theoretical approach

			W-cycle				
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$
128	4	6	0.7964	1.2187	1.5303	0.2187	0.1445
192	6	6	0.7962	1.2186	1.5305	0.2186	0.1389
256	8	6	0.7962	1.2185	1.5304	0.2185	0.1350
320	10	6	0.7961	1.2185	1.5306	0.2185	0.1320
384	12	6	0.7961	1.2185	1.5306	0.2185	0.1297
448	14	6	0.7961	1.2185	1.5306	0.2185	0.1277
512	16	6	0.7961	1.2185	1.5306	0.2185	0.1260
576	18	6	0.7961	1.2185	1.5306	0.2185	0.1245
640	20	6	0.7960	1.2185	1.5308	0.2185	0.1232

			Ritz-Galerkin-V-cycle					Petrov-Galerkin-V-cycle				
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$
128	4	6	0.5711	0.9961	1.7442	0.4289	0.3014	0.6604	0.9961	1.5083	0.3396	0.2676
192	6	6	0.5733	0.9957	1.7368	0.4267	0.3461	0.6618	0.9971	1.5066	0.3382	0.2448
256	8	6	0.5686	0.9752	1.7151	0.4314	0.3342	0.6623	0.9984	1.5075	0.3377	0.2623
320	10	6	0.5693	0.9610	1.6880	0.4307	0.1848	0.6623	1.0002	1.5102	0.3377	0.2143
384	12	6	0.5689	0.9723	1.7091	0.4311	0.2214	0.6622	1.0019	1.5130	0.3378	0.2252
448	14	6	0.5690	0.9795	1.7214	0.4310	0.2785	0.6621	1.0032	1.5152	0.3379	0.2638
512	16	6	0.5690	0.9842	1.7297	0.4310	0.3319	0.6621	1.0042	1.5167	0.3379	0.2833
576	18	6	0.5691	0.9875	1.7352	0.4309	0.3498	0.6620	1.0050	1.5181	0.3380	0.2749
640	20	6	0.5691	0.9898	1.7392	0.4309	0.3772	0.6619	1.0055	1.5191	0.3381	0.2882

			variable-RG-V-cycle					variable-PG-V-cycle				
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$
128	4	6	0.6465	0.9933	1.5364	0.3535	NaN	0.6537	0.9925	1.5183	0.3463	NaN
192	6	6	0.6475	0.9689	1.4964	0.3525	NaN	0.6539	0.9926	1.5180	0.3461	NaN
256	8	6	0.6481	0.9598	1.4809	0.3519	NaN	0.6540	0.9934	1.5190	0.3460	NaN
320	10	6	0.6486	0.9693	1.4944	0.3514	NaN	0.6540	0.9949	1.5213	0.3460	NaN
384	12	6	0.6490	0.9767	1.5049	0.3510	NaN	0.6540	0.9965	1.5237	0.3460	NaN
448	14	6	0.6493	0.9819	1.5122	0.3507	NaN	0.6540	0.9979	1.5258	0.3460	NaN
512	16	6	0.6495	0.9857	1.5176	0.3505	NaN	0.6540	0.9990	1.5275	0.3460	NaN
576	18	6	0.6497	0.9884	1.5213	0.3503	NaN	0.6541	0.9999	1.5287	0.3459	NaN
640	20	6	0.6499	0.9904	1.5239	0.3501	NaN	0.6541	1.0005	1.5296	0.3459	NaN

			W-cycle				
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$
512	64	4	0.7960	1.2185	1.5308	0.2185	NaN
512	32	5	0.7961	1.2185	1.5306	0.2185	NaN
512	16	6	0.7961	1.2185	1.5306	0.2185	NaN
512	8	7	0.7961	1.2185	1.5306	0.2185	NaN
512	4	8	0.7961	1.2185	1.5306	0.2185	NaN

			RG-V-cycle					PG-V-cycle				
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$
512	64	4	0.5651	0.9985	1.7669	0.4349	NaN	0.6459	1.0054	1.5566	0.3541	NaN
512	32	5	0.5670	0.9958	1.7563	0.4330	NaN	0.6557	1.0061	1.5344	0.3443	NaN
512	16	6	0.5690	0.9842	1.7297	0.4310	NaN	0.6621	1.0042	1.5167	0.3379	NaN
512	8	7	0.5699	0.9569	1.6791	0.4301	NaN	0.6545	0.9993	1.5268	0.3455	NaN
512	4	8	0.5746	0.9937	1.7294	0.4254	NaN	0.6464	0.9984	1.5446	0.3536	NaN

			variable-RG-V-cycle					variable-PG-V-cycle				
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\bar{\delta}$
512	64	4	0.6495	0.9985	1.5373	0.3505	NaN	0.6541	1.0038	1.5346	0.3459	NaN
512	32	5	0.6495	0.9958	1.5332	0.3505	NaN	0.6540	1.0026	1.5330	0.3460	NaN
512	16	6	0.6495	0.9857	1.5176	0.3505	NaN	0.6540	0.9990	1.5275	0.3460	NaN
512	8	7	0.6495	0.9707	1.4945	0.3505	NaN	0.6540	0.9952	1.5217	0.3460	NaN
512	4	8	0.6495	0.9706	1.4944	0.3505	NaN	0.6540	0.9952	1.5217	0.3460	NaN

Table 2: square: dependency on h_J (top) and on number of MG levels J (bottom)

of Bramble et al. over the standard Fourier analysis of a multigrid algorithm. Indeed, while the approach of Bramble used in this paper still works, the geometry of Ω, though simple, does not allow to make the Fourier analysis. However, all the methods still behave very well, which is well predicted by Bramble's theory.

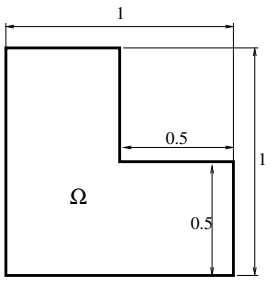

Figure 7: L-shaped computational domain Ω

All conclusions made in the previous section for the square computational domain holds also here. Note, that suprisingly, the simple Petrov-Galerkin V-cycle is slightly better that Petrov-Galerkin variable V-cycle.

7.3 Locally non-quasiuniform mesh

Let us now consider Ω to be a unite square discretized by a locally non-quasiuniform mesh (as in Fig. 8). Table 4 presents the numerical results. Here, we choose the flux evaluation points x_i^k so that the energy stability property is verified for piecewise constant prolongation with $\mu = 2$. If such points do not exist, we produce coarse grid problems which still satisfy the energy stability condition, but which are not consistent with the finest level problem.

First, let us discuss the convergence results as a function of meshsize h_J, as listed on the top of Table 4. For the W-cycle for which we moved the flux-evaluation points and which thus satisfies the energy stability we see, that it converges and behaves well and independently of h_J (nb. it depends on the configuration of cells along the boundary of the locally non-uniform mesh). On the top-right of Table 4 we give an example of another W-cycle for the same finest-level problem, for which we did not shift the flux-evaluation points (here defined to be barycenters of the coarse cells) and which hence does not satisfy the energy stability. The effect is evident - the method diverges.

The results of simple V-cycles are given just for comparison; as we do not have the stability property with $\mu = 1$, neither of the developed theorems apply for this case, however, we observe still quite a favourable behaviour. In the case of the variable V-cycles, we know that even without the energy stability the algorithms will give h_J independent preconditioners.

The bottom part of Table 4 sheds some light on the dependency on number of multigrid levels J. We fix the number of the finest level nodes to $(512)^2$=262144 and we increase the number of

			W-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.7965	1.2187	1.5301	0.2187
192	6	6	0.7963	1.2186	1.5303	0.2186
256	8	6	0.7962	1.2186	1.5305	0.2186
320	10	6	0.7962	1.2185	1.5304	0.2185
384	12	6	0.7961	1.2185	1.5306	0.2185
448	14	6	0.7961	1.2185	1.5306	0.2185
512	16	6	0.7961	1.2185	1.5306	0.2185
576	18	6	0.7961	1.2185	1.5306	0.2185
640	20	6	0.7961	1.2185	1.5306	0.2185

			Ritz-Galerkin-V-cycle				Petrov-Galerkin-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.5730	1.0000	1.7452	0.4270	0.6654	0.9997	1.5024	0.3346
192	6	6	0.5713	1.0000	1.7504	0.4287	0.6618	0.9986	1.5089	0.3382
256	8	6	0.5712	1.0000	1.7507	0.4288	0.6607	0.9978	1.5102	0.3393
320	10	6	0.5684	1.0000	1.7593	0.4316	0.6631	0.9975	1.5043	0.3369
384	12	6	0.5704	1.0000	1.7532	0.4296	0.6619	0.9979	1.5076	0.3381
448	14	6	0.5700	1.0000	1.7544	0.4300	0.6623	0.9988	1.5081	0.3377
512	16	6	0.5693	1.0000	1.7565	0.4307	0.6621	0.9998	1.5100	0.3379
576	18	6	0.5691	0.9999	1.7570	0.4309	0.6621	1.0009	1.5117	0.3379
640	20	6	0.5695	0.9998	1.7556	0.4305	0.6620	1.0018	1.5133	0.3380

			variable-RG-V-cycle				variable-PG-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.6460	1.0000	1.5480	0.3540	0.6492	0.9948	1.5323	0.3508
192	6	6	0.6469	1.0000	1.5458	0.3531	0.6506	0.9941	1.5280	0.3494
256	8	6	0.6475	1.0000	1.5444	0.3525	0.6514	0.9940	1.5259	0.3486
320	10	6	0.6480	1.0000	1.5432	0.3520	0.6519	0.9937	1.5243	0.3481
384	12	6	0.6484	1.0000	1.5423	0.3516	0.6522	0.9936	1.5235	0.3478
448	14	6	0.6488	1.0000	1.5413	0.3512	0.6525	0.9939	1.5232	0.3475
512	16	6	0.6490	0.9999	1.5407	0.3510	0.6527	0.9946	1.5238	0.3473
576	18	6	0.6492	0.9999	1.5402	0.3508	0.6528	0.9954	1.5248	0.3472
640	20	6	0.6494	0.9997	1.5394	0.3506	0.6529	0.9962	1.5258	0.3471

			W-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.7961	1.2185	1.5306	0.2185
512	32	5	0.7961	1.2185	1.5306	0.2185
512	16	6	0.7961	1.2185	1.5306	0.2185
512	8	7	0.7961	1.2185	1.5306	0.2185
512	4	8	0.7961	1.2185	1.5306	0.2185

			RG-V-cycle				PG-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.5650	0.9983	1.7669	0.4350	0.6456	1.0053	1.5572	0.3544
512	32	5	0.5671	0.9981	1.7600	0.4329	0.6534	1.0042	1.5369	0.3466
512	16	6	0.5693	1.0000	1.7565	0.4307	0.6621	0.9998	1.5100	0.3379
512	8	7	0.5727	1.0000	1.7461	0.4273	0.6549	0.9977	1.5234	0.3451
512	4	8	0.5748	1.0000	1.7397	0.4252	0.6450	0.9975	1.5465	0.3550

			variable-RG-V-cycle				variable-PG-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.6491	0.9983	1.5380	0.3509	0.6527	1.0029	1.5365	0.3473
512	32	5	0.6490	0.9982	1.5381	0.3510	0.6527	1.0001	1.5323	0.3473
512	16	6	0.6490	0.9999	1.5407	0.3510	0.6527	0.9946	1.5238	0.3473
512	8	7	0.6490	1.0000	1.5408	0.3510	0.6527	0.9935	1.5221	0.3473
512	4	8	0.6490	1.0000	1.5408	0.3510	0.6527	0.9939	1.5228	0.3473

Table 3: L-shaped domain: dependency on h_J and on number of MG levels J

			W-cycle				W-cycle not stable in energy			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.3365	1.0024	2.9789	0.6635	-0.4523	0.9858	n.a.	diverges
192	6	6	0.4444	1.0004	2.2511	0.5556	-0.4532	1.0004	n.a.	diverges
256	8	6	0.4154	0.9980	2.4025	0.5846	-0.4727	1.0103	n.a.	diverges
320	10	6	0.3640	1.0022	2.7533	0.6360	-0.4555	0.9854	n.a.	diverges
384	12	6	0.4631	1.0008	2.1611	0.5369	-0.4554	1.0004	n.a.	diverges
448	14	6	0.4165	0.9984	2.3971	0.5835	-0.4745	1.0110	n.a.	diverges
512	16	6	0.3815	1.0022	2.6270	0.6185	-0.4560	0.9850	n.a.	diverges
576	18	6	0.4702	1.0010	2.1289	0.5298	-0.4560	1.0004	n.a.	diverges
640	20	6	0.4170	0.9985	2.3945	0.5830	-0.4751	1.0112	n.a.	diverges

			Ritz-Galerkin-V-cycle				Petrov-Galerkin-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.2313	0.9968	4.3096	0.7687	0.2400	1.3151	5.4796	0.7600
192	6	6	0.2330	1.2318	5.2867	0.7670	0.2418	1.5446	6.3879	0.7582
256	8	6	0.2720	1.1688	4.2971	0.7280	0.2465	1.5237	6.1813	0.7535
320	10	6	0.2325	1.1783	5.0680	0.7675	0.2388	1.5345	6.4259	0.7612
384	12	6	0.2338	1.3871	5.9328	0.7662	0.2436	1.7040	6.9951	0.7564
448	14	6	0.2745	1.2334	4.4933	0.7255	0.2493	1.5750	6.3177	0.7507
512	16	6	0.2332	1.2312	5.2796	0.7668	0.2407	1.5742	6.5401	0.7593
576	18	6	0.2346	1.4294	6.0929	0.7654	0.2447	1.7475	7.1414	0.7553
640	20	6	0.2759	1.2494	4.5285	0.7241	0.2521	1.5869	6.2947	0.7479

			variable-RG-V-cycle				variable-PG-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.2394	1.1651	4.8668	0.7606	0.2328	1.2898	5.5404	0.7672
192	6	6	0.2437	1.3557	5.5630	0.7563	0.2437	1.5557	6.3837	0.7563
256	8	6	0.2400	1.2044	5.0183	0.7600	0.2353	1.3308	5.6558	0.7647
320	10	6	0.2399	1.2102	5.0446	0.7601	0.2336	1.3389	5.7316	0.7664
384	12	6	0.2458	1.4332	5.8308	0.7542	0.2449	1.6392	6.6933	0.7551
448	14	6	0.2403	1.2187	5.0716	0.7597	0.2369	1.3487	5.6931	0.7631
512	16	6	0.2402	1.2177	5.0695	0.7598	0.2343	1.3519	5.7700	0.7657
576	18	6	0.2474	1.4588	5.8965	0.7526	0.2455	1.6664	6.7878	0.7545
640	20	6	0.2405	1.2227	5.0840	0.7595	0.2384	1.3550	5.6837	0.7616

			W-cycle				W-cycle not stable in energy			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.3776	1.0012	2.6515	0.6224	-0.4552	0.9844	n.a.	diverges
512	32	5	0.3816	1.0021	2.6260	0.6184	-0.4557	0.9850	n.a.	diverges
512	16	6	0.3815	1.0022	2.6270	0.6185	-0.4559	0.9850	n.a.	diverges
512	8	7	0.3815	1.0022	2.6270	0.6185	-0.4559	0.9850	n.a.	diverges
512	4	8	0.3815	1.0022	2.6270	0.6185	-0.4559	0.9850	n.a.	diverges

			Ritz-Galerkin-V-cycle				Petrov-Galerkin-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.2331	1.1990	5.1437	0.7669	0.2282	1.3275	5.8173	0.7718
512	32	5	0.2347	1.2270	5.2280	0.7653	0.2395	1.4612	6.1010	0.7605
512	16	6	0.2332	1.2312	5.2796	0.7668	0.2407	1.5742	6.5401	0.7593
512	8	7	0.2341	1.1823	5.0504	0.7659	0.2846	1.6627	5.8422	0.7154
512	4	8	0.2701	1.0171	3.7656	0.7299	0.2825	1.5960	5.6496	0.7175

			variable-RG-V-cycle				variable-PG-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.2398	1.2074	5.0350	0.7602	0.2324	1.2888	5.5456	0.7676
512	32	5	0.2402	1.2182	5.0716	0.7598	0.2352	1.3287	5.6492	0.7648
512	16	6	0.2402	1.2177	5.0695	0.7598	0.2343	1.3519	5.7700	0.7657
512	8	7	0.2403	1.2241	5.0940	0.7597	0.2360	1.3529	5.7326	0.7640
512	4	8	0.2403	1.2241	5.0940	0.7597	0.2360	1.3529	5.7326	0.7640

Table 4: non-quasiuniform: dependency on h_J (top) and number of MG levels J (bottom)

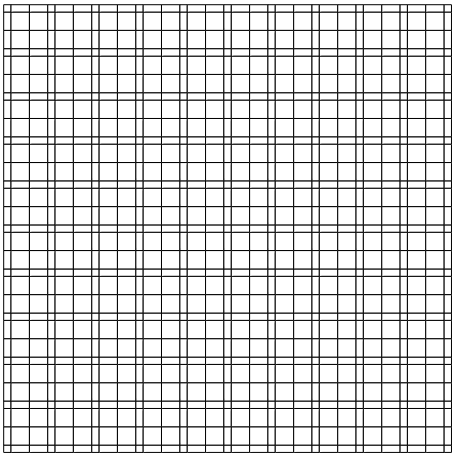

Figure 8: non-quasiuniform mesh

grids from 4 to 8. Again, the W-cycle with flux-evaluation points in barycenters diverges, otherwise the data for W-cycle and variable V-cycles witness independence of number of levels, as predicted by the theoretical part of this paper.

7.4 Linearly varied spacing

Now, we choose Ω to be a unit square with a system of control cells whose spacing varies linearly from the finest space-step on the border to the largest in the center of the domain, see Fig. 9, so that the largest ratio of sides of the rectangular control cells is 4.

Like in the precedent numerical test, we observe first the behaviour of the methods in terms of meshsize parameter h_J. Here, the convergence theory presented predicts mesh-independent behaviour of W-cycle iterator and variable V-cycle preconditioner. Possible small fluctuations of the presented eigenvalues in Table 5 might be caused by not too well converged power method used for their estimation. Observe, that either of the W-cycles (stable in energy but non-consistent and consistent but not stable) behave well. No theoretical results apply for simple V-cycles.

The deterioration of convergence rate compared to regular structured square mesh is caused by increasing values of constants in the convergence estimate (12) reflecting the higher cell stretching ratio.

The bottom part of Table 5 clearly shows that the convergence of W-cycle and variable V-cycles for the same finest level problem does not depend on number of multigrid levels J.

			W-cycle				W-cycle not stable in energy			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.4897	1.2114	2.4738	0.5103	0.4897	1.2114	2.4738	0.5103
192	6	6	0.4715	1.2136	2.5739	0.5285	0.4715	1.2136	2.5739	0.5285
256	8	6	0.4616	1.2150	2.6321	0.5384	0.4616	1.2150	2.6321	0.5384
320	10	6	0.4555	1.2160	2.6696	0.5445	0.4555	1.2160	2.6696	0.5445
384	12	6	0.4516	1.2167	2.6942	0.5484	0.4516	1.2167	2.6942	0.5484
448	14	6	0.4491	1.2173	2.7105	0.5509	0.4491	1.2173	2.7105	0.5509
512	16	6	0.4474	1.2178	2.7219	0.5526	0.4475	1.2178	2.7213	0.5525
576	18	6	0.4465	1.2182	2.7283	0.5535	0.4465	1.2182	2.7283	0.5535
640	20	6	0.4460	1.2186	2.7323	0.5540	0.4460	1.2186	2.7323	0.5540
			Ritz-Galerkin-V-cycle				Petrov-Galerkin-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.3925	0.9974	2.5411	0.6075	0.3930	0.9945	2.5305	0.6070
192	6	6	0.3661	0.9974	2.7244	0.6339	0.3839	0.9934	2.5877	0.6161
256	8	6	0.3633	0.9973	2.7451	0.6367	0.3771	0.9936	2.6348	0.6229
320	10	6	0.3537	0.9890	2.7962	0.6463	0.3734	0.9958	2.6668	0.6266
384	12	6	0.3522	0.9753	2.7692	0.6478	0.3706	0.9978	2.6924	0.6294
448	14	6	0.3468	0.9786	2.8218	0.6532	0.3684	0.9993	2.7125	0.6316
512	16	6	0.3454	0.9831	2.8463	0.6546	0.3666	1.0004	2.7289	0.6334
576	18	6	0.3429	0.9865	2.8769	0.6571	0.3650	1.0011	2.7427	0.6350
640	20	6	0.3414	0.9891	2.8972	0.6586	0.3635	1.0016	2.7554	0.6365
			variable-RG-V-cycle				variable-PG-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
128	4	6	0.3678	0.9971	2.7110	0.6322	0.3663	1.0081	2.7521	0.6337
192	6	6	0.3526	0.9955	2.8233	0.6474	0.3514	1.0076	2.8674	0.6486
256	8	6	0.3445	0.9863	2.8630	0.6555	0.3435	1.0076	2.9333	0.6565
320	10	6	0.3395	0.9757	2.8739	0.6605	0.3385	1.0075	2.9764	0.6615
384	12	6	0.3360	0.9773	2.9086	0.6640	0.3351	1.0073	3.0060	0.6649
448	14	6	0.3335	0.9815	2.9430	0.6665	0.3326	1.0071	3.0280	0.6674
512	16	6	0.3316	0.9851	2.9707	0.6684	0.3307	1.0069	3.0448	0.6693
576	18	6	0.3302	0.9878	2.9915	0.6698	0.3292	1.0066	3.0577	0.6708
640	20	6	0.3289	0.9899	3.0097	0.6711	0.3280	1.0062	3.0677	0.6720

			W-cycle				W-cycle not stable in energy			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.4452	1.2178	2.7354	0.5548	0.4452	1.2178	2.7354	0.5548
512	32	5	0.4474	1.2178	2.7219	0.5526	0.4474	1.2178	2.7219	0.5526
512	16	6	0.4474	1.2178	2.7219	0.5526	0.4475	1.2178	2.7213	0.5525
512	8	7	0.4474	1.2178	2.7219	0.5526	0.4475	1.2178	2.7213	0.5525
512	4	8	0.4474	1.2178	2.7219	0.5526	0.4475	1.2178	2.7213	0.5525
			Ritz-Galerkin-V-cycle				Petrov-Galerkin-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.3333	0.9986	2.9961	0.6667	0.3440	1.0021	2.9131	0.6560
512	32	5	0.3397	0.9957	2.9311	0.6603	0.3567	0.9991	2.8010	0.6433
512	16	6	0.3454	0.9831	2.8463	0.6546	0.3666	1.0004	2.7289	0.6334
512	8	7	0.3534	0.9972	2.8217	0.6466	0.3703	0.9942	2.6849	0.6297
512	4	8	0.3648	0.9974	2.7341	0.6352	0.3695	0.9963	2.6963	0.6305
			variable-RG-V-cycle				variable-PG-V-cycle			
h_J^{-1}	h_1^{-1}	J	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ	$\lambda_{\min}(C_J)$	$\lambda_{\max}(C_J)$	$\mathcal{K}(C_J)$	δ
512	64	4	0.3317	0.9986	3.0106	0.6683	0.3309	1.0030	3.0311	0.6691
512	32	5	0.3317	0.9957	3.0018	0.6683	0.3307	1.0059	3.0417	0.6693
512	16	6	0.3316	0.9851	2.9707	0.6684	0.3307	1.0069	3.0448	0.6693
512	8	7	0.3316	0.9749	2.9400	0.6684	0.3307	1.0042	3.0366	0.6693
512	4	8	0.3316	0.9749	2.9400	0.6684	0.3307	1.0042	3.0366	0.6693

Table 5: linearly-varied h: dependency on h_J (top) and on number of MG levels J (bottom)

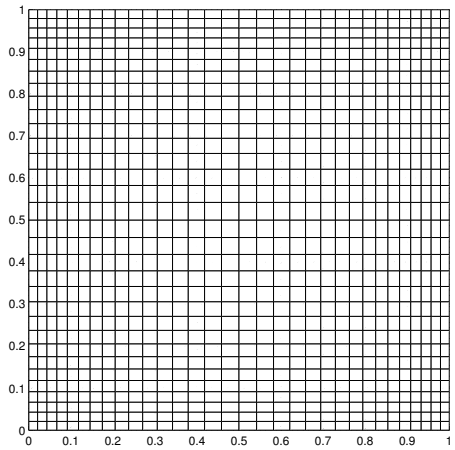

Figure 9: mesh with lineraly varied h

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