Multigrid algorithms for compressible laminar flows

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Plan of presentation

- **Introduction**
  - Problem: convection-diffusion
  - Implicit formulation, discretization, linearization
  - Introduction to multigrids, multigrid for convection-diffusion
  - Multigrid by agglomeration, finite volumes vs. finite elements
  - Different treatment of convective and of diffusive term

- **Petrov-Galerkin variational framework** for diffusive terms
  - Sparsity considerations
  - Essentials of convergence analysis for SPD problems

- **Finite-volume based formulation** of diffusive terms
  - Sparsity considerations
  - Essentials of convergence analysis for SPD problems
  - Finite-volume error estimate
  - Multigrid by flux-correction

- **Numerical results**

- **Conclusions, perspectives**
**Problem definition**

**Stationary problem:** find $u(x) : \Omega \mapsto \mathbb{R}$ satisfying

$$
- \text{div}(\kappa \text{grad } \bar{u}) + \text{div } F(u) = f \quad \text{in } \Omega \in \mathbb{R}^2
$$

+ suitable boundary conditions on $\partial \Omega$,

**Equivalent non-stationary problem:** find $u(x, t) : (\Omega, [0, \infty]) \mapsto \mathbb{R}$ satisfying

$$
\frac{\partial u}{\partial t} - \text{div}(\kappa \text{grad } \bar{u}) + \text{div } F(u) = f \quad \text{in } \Omega \in \mathbb{R}^2
$$

+ suitable boundary conditions on $\partial \Omega$ and suitable initial condition for $t = 0$

- convection-diffusion, Navier-Stokes
- searching just for stationary solution $u(x, t \rightarrow \infty)$ of the non-stationary problem (if this exists) $\Rightarrow$ time $t$ does not have to have a physical meaning.
Finite volume-element spatial discretization

Integrate over control volumes $C_i$:

$$
\int_{\partial C_i} \frac{\partial u}{\partial t} - \int_{\partial C_i} (\kappa \text{grad} \, \tilde{u}) \cdot \mathbf{n} d\Gamma + \int_{\partial C_i} \mathbf{F}(u) \cdot \mathbf{n} d\Gamma = \int_{C_i} f d\Omega.
$$

Replace fluxes by numerical approximations:

$$
\mu(C_i) \frac{\partial u_i}{\partial t} + \sum_{j \in N(i)} \Phi^D_h(\tilde{u}_i, \tilde{u}_j) + \sum_{j \in N(i)} \Phi^C_h(u_i, u_j, n_{ij}) = \int_{C_i} f d\Omega,
$$

with diffusive flux $\Phi^D_h$ and convective flux $\Phi^C_h$:

$$
\Phi^D_h(\tilde{u}_i, \tilde{u}_j) = (\tilde{u}_j - \tilde{u}_i) \sum_{T \in \tau_h : i, j \in T} \int_T (\bar{\kappa}(T) \text{grad} \, \varphi_j) \cdot \text{grad} \, \varphi_i d\Omega,
$$

$$
\Phi^C_h(u_i, u_j, n_{ij}) = \frac{1}{2} [\mathbf{F}(u_i) \cdot n_{ij} + \mathbf{F}(u_j) \cdot n_{ij} - |\mathbf{J}(u_{ij}) \cdot n_{ij}|(u_j - u_i)],
$$
Implicit time integration and linearization

System of nonlinear equations for \( u_i = u_i(t), \ i = 1, n_J \)

\[
\mu(C_i) \frac{\partial u_i}{\partial t} + \Upsilon_i(u) = f_i, \quad i = 1, \ldots, n_h.
\]

**Implicit time-advancing and linearization:** for \( u^n \), solve for \( \delta u^n \), \( u^{n+1} = u^n + \delta u^n \),

\[
\left( \frac{\mu(C_i)}{\delta t^n} + \left[ \frac{\partial \Upsilon_i(u^n)}{\partial u_j^n} \right] \right) \delta u^n = f_i - \Upsilon_i(u^n).
\]

- System matrix: \( A_h = T_h + D_h + C_h \).
- Jacobian: \( \Phi_h^D(\tilde{u}_i, \tilde{u}_j) = (\tilde{u}_j - \tilde{u}_i) \sum_{T \in T_h : i, j \in T} \int_T (\kappa_{ij} \text{grad } \varphi_j) \cdot \text{grad } \varphi_i d\Omega \).
- Higher order spatial discretization schemes possible for \( \Upsilon_i(u^n) \), Jacobian first-order only \( \Rightarrow \) defect-correction for second order schemes in space
- First-order convective Jacobian \( C_h \): M-matrix
General multigrid algorithm: V-cycle

Set $b_J = b_h$, $A_J = A_h$, $x_k = 0$, $k = 1, \ldots, J$.

1. Pre-smoothing: $x_k \leftarrow (I - R_k A_k)x_k + R_k b_k$, (eg. Gauss-Seidel or Jacobi iterations).

2. Coarse-grid correction: if $J > 1$,
   (a) Restrict residual to coarse-level right-hand side $b_{k-1} = I_{k-1}^k (b_k - A_k x_k)$.
   (b) Solve for $x_{k-1}$ of a coarse-level problem
       \[ A_{k-1} x_{k-1} = b_{k-1} \]
       by a recursive application of this algorithm on the coarse level $(k - 1)$.
   (c) Prolongation: correct $x_k$ by
       \[ x_k \leftarrow x_k + I_{k-1}^k x_{k-1}. \]

3. Post-smoothing: $x_k \leftarrow (I - R_k A_k)x_k + R_k b_k$.

Set $x_h = x_J$. 
Why multigrid?

- Multigrid: efficient for all frequency components of error
- Mono-grid: less efficient for low frequency components

\[
\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega, \\
u(t = 0) = 0.
\]
Classes of multigrid methods

- **Geometric MG** (re-meshing or re-triangulation of the domain, rediscretize)
  - nested coarse spaces (impossible for general unstructured meshes)
  - non-nested (needs prolongations by interpolation)

- **Algebraic MG** (finest level operator only, but usually only linear solver)
  - variational (with inherited bilinear forms): $A_{k-1} = I_{k-1}^k A_k I_{k}^{k-1}$
  - non-variational (nested or non-nested spaces)
    - by agglomeration or aggregation (nested spaces, transfer operators devised from injection)
**Algebraic multigrid by aggregation: principle**

- Take a disjoint covering of nodes (control cells) of the computational domain.

- Construct a (zero-one) prolongation (injection): \( \tilde{I}^{k-1}_k : M_{k-1} \mapsto M_k \), for \( u_{k-1} \in M_{k-1} \)

\[
\begin{align*}
\tilde{I}^{k-1}_k u_{k-1} & = u_{k-1} \\
\text{on } C^k_j & \subset C^{k-1}_i \\
\text{on } C^{k-1}_i
\end{align*}
\]

- Construct a coarse level problem based on \( \tilde{I}^{k-1}_k \) and the fine problem matrix \( A_k \).
  - variationally: \( A_{k-1} = (\tilde{I}^{k-1}_k)^T A_k \tilde{I}^{k-1}_k \),
  - otherwise.
Multigrid by aggregation of unknowns

- Two possible points of view - aggregation of nodes, or agglomeration of control cells.
- Agglomeration of control cells can be represented in finite volume framework
- Cheap setup phase. No need to calculate expensive prolongations.
- Multigrids with consistent coarse discretization with the original problem converge better than non-consistent ones (the best would be to rediscretize, but it is expensive)
- Sparsity of coarse problems.
Multigrid by aggregation: two points of view
Sparsity of coarse problems

Rediscretization: 475883 nonzeros, 100% memory

Petrov-Galerkin: $A_{k-1} = (I_k^{k-1})^T A_k (S_k I_k^{k-1})$ 621409 nonzeros, 130% memory

Ritz-Galerkin: $A_{k-1} = (S_k I_k^{k-1})^T A_k (S_k I_k^{k-1})$ 1158059 nonzeros, 243% memory
Multigrid by aggregation: coarse level construction

\[
A_{k-1} = T_{k-1} + D_{k-1} + C_{k-1}
\]

Convective term:

• \( C_{k-1} = \left( \tilde{I}_k^{k-1} \right)^T C_k \tilde{I}_k^{k-1} \)
  
  – preserves the upwind character,
  
  – can be represented in the finite volume framework as summation of fine level fluxes,
  
  \[
  \Phi^C_{k-1}(u_I, u_J, n_{IJ}) = \sum_{C_i \subset C_{k-1}^k} \Phi^C_k((\tilde{I}_k^{k-1}u)_i, (\tilde{I}_k^{k-1}u)_j, n_{ij})
  \]

  – gives consistent scheme on coarse level,

• \( C_{k-1} = \left( \tilde{I}_k^{k-1} \right)^T C_k S_k \tilde{I}_k^{k-1} \) with \( S_k \) a centered prolongation smoother does not preserve the upwinding, the coarsest level tends to be nearly centered \( \Rightarrow \) stability problems, risk of oscillations!

• Solution contains local features (discontinuities, shocks) \( \Rightarrow \) coarsening ratio 2.
Multigrid by aggregation: coarse level construction 2

Diffusive term:

- $D_{k-1} = (\tilde{I}_k^{k-1})^T D_k \tilde{I}_k^{k-1}$
  - does not give a not consistent scheme on coarse levels,
  - very poor convergence of V-cycle.
- $D_{k-1} = (S_k \tilde{I}_k^{k-1})^T D_k S_k \tilde{I}_k^{k-1}$
  - Ritz-Galerkin, $S_k$ . . . prolongation smoother,
  - large fill-in for coarsening ratio 2.
- $D_{k-1} = (\tilde{I}_k^{k-1})^T D_k S_k \tilde{I}_k^{k-1}$
  - Petrov-Galerkin,
  - reduces the fill-in
- **Non-variational formulation** of $D_{k-1}$ which is consistent.
Consistency problems for \( D_{k-1} = (\tilde{I}_{k}^{k-1})^T D_k \tilde{I}_{k}^{k-1} \)

**Finite element point of view:**
- Loss of inverse property:
  \[ A(u, u) \not\leq C h_k^2 (u, u)_0, \]
- Bad energy stability

**Finite volume point of view:**
Discrepancies in flux
\[
\Phi_{k-1}^D (u_{k-1}, n_{e_{k-1}}) = \sum_{e_{k} \in e_{k-1}} \Phi_{k}^D (I_{k}^{k-1} u_{k-1}, n_{e_{k}})
\]
Recapitulation

- We would like to use multigrid by aggregation (agglomeration)
  - Cheap setup
  - Keeps nearest neighbour stencil
  - Seems natural for finite volumes (convective term)
- Problems:
  - Prolongation by injection (non-smoothed) good for convective term, bad for diffusive term
  - Smoothed prolongation necessary for diffusive term, bad for convective term.

**Solution:** Define $C'_{k-1}$ and $D_{k-1}$ independently.
Fix $C'_{k-1} = (\tilde{I}_k^{k-1})^T C_k \tilde{I}_k^{k-1}$ and look for optimal choice for $D_{k-1}$:

- $\Rightarrow$ Petrov-Galerkin variational formulation by smoothed prolongation
- $\Rightarrow$ Finite-volume based multigrid
Petrov-Galerkin smoothed aggregation multigrid

- Coarse level diffusive part in the form

\[ D_{k-1} = (\tilde{I}_k^{k-1})^T D_k S_k \tilde{I}_k^{k-1} \]

- \( S_k \) enforces energy stability of coarse level basis, \( \varphi_i^k = S_k \psi_i^k \)

- Choose \( S_k \) to be a polynomial in \( D_k \),

\[ S_k = I - \frac{1}{\bar{\rho}(D_k)} D_k. \]

- Does not give in general a consistent scheme on coarse levels, \( S_k \) is not a linear interpolator.
Regularity-free convergence theory: assumptions

- **Ritz-Galerkin**: restriction $I_{k-1}^k$ and prolongation $I_k^{k-1}$ are adjoint with respect to $(\cdot, \cdot)_0$.
- **Smoothing condition**: compares the used pre- and post-smoother to Richardson method: there exists $C_R > 0$ such that
  \[ \|u\|_0^2 \leq C_R (R_k u, u)_0 \quad \forall u \in M_k, \]

- **Weak approximation property**: there exist linear operators $\bar{Q}_k : M_k \rightarrow M_k$, $k = 1, \ldots, J$ with $\bar{Q}_J = I$, such that
  \[ \| (\bar{Q}_k - \bar{Q}_{k-1}) u \|_0^2 \leq C \lambda_k^{-1} A(u, u), \quad k = 2, \ldots, J \]

- **Energy stability**: 
  \[ A(\bar{Q}_k u, \bar{Q}_k u) \leq C A(u, u), \quad k = 1, \ldots, J, \quad \forall u \in M_J. \]
Multigrid by smoothed aggregation: tricks

- Set $I_k^{k-1} = S_k \tilde{I}_k^{k-1}$ and separate the assumptions on $S_k$ from the assumptions on $\tilde{I}_k^{k-1}$.
- Use equivalent vector spaces instead of nested functional spaces – approximation property is easier to verify for disjoint nodal aggregates.

Discrete assumptions:

- Weak approximation property of non-smoothed space:
  \[ \| u - \tilde{I}_{J}^{k-1} Q_{k-1} u \|_{\mathbb{R}^n_{J}} \leq \frac{C^2}{\lambda_k} \| u \|_{A_j}, \quad \forall u \in M_J \]

- Smoothing property of prolongation smoother $S_k$:
  \[ \|(I - S_k^{1/2}) x\|_{\mathbb{R}^n_{k}} \leq \frac{C^2}{\varrho(A_k)} \| x \|_{A_k}, \quad \forall x \in \mathbb{R}^n_k \]

- Energy stability of coarse level:
  \[ \varrho(A_k S_k) \leq C_S^2 \lambda_k \]
Equivalent Ritz-Galerkin multigrid to Petrov-Galerkin
Abstract Convergence proof works only for Ritz-Galerkin method

Petrov-Galerkin:

1. Pre-smoothing: $x^k \leftarrow (I - R_k A_k) x^k + R_k b^k$
   Additional: $x^k \leftarrow S_k x^k + (I - S_k) A_k^{-1} b^k$

2. Coarse grid correction:
   (a) Set $b^{k-1} = (\tilde{I}_k^{-1})^T (b^k - A_k x^k)$,
   (b) if $k = 2$, solve $x^{k-1} = (A_{k-1})^{-1} b^{k-1}$, otherwise set $x^{k-1} = 0$ and repeat this algorithm for $k = k - 1$.
   (c) $x^k \leftarrow x^k + S_k \tilde{I}_k^{-1} x^{k-1}$.

3. Post-smoothing: $x^k \leftarrow (I - R_k^T A_k) x^k + R_k^T b^k$

Ritz-Galerkin:

1. Pre-smoothing: $x^k \leftarrow (I - R_k A_k) x^k + R_k b^k$
   $x^k \leftarrow S_k^{\frac{1}{2}} x^k + (I - S_k^{\frac{1}{2}}) A_k^{-1} b^k$

2. Coarse grid correction:
   (a) Set $b^{k-1} = (S_k^{\frac{1}{2}} \tilde{I}_k^{k-1})^T (b^k - A_k x^k)$,
   (b) if $k = 2$, solve $x^{k-1} = (A_{k-1})^{-1} b^{k-1}$, otherwise set $x^{k-1} = 0$ and repeat this algorithm for $k = k - 1$.
   (c) $x^k \leftarrow x^k + S_k^{\frac{1}{2}} \tilde{I}_k^{k-1} x^{k-1}$.

3. Post-smoothing: $x^k \leftarrow S_k^{\frac{1}{2}} x^k + (I - S_k^{\frac{1}{2}}) A_k^{-1} b^k$

   $x^k \leftarrow (I - R_k^T A_k) x^k + R_k^T b^k$. 
Finite volume coarse problem (diffusive term)

- The maximum number of neighbours of each cell is bounded
- The maximum number of non-zeros per line in coarse problem matrix is bounded
- One (finite-volume) framework to analyse both convective and diffusive term on coarse levels
- Finite volume framework is natural for aggregation/agglomeration technique
- If rediscretized, the coarse problem is consistent with the continuous problem

Problems:

- Shape of control cells is complicated
- Approximation of flux on complicated interfaces, might give non-symmetric coarse level matrices even for an SPD problem
Control cells on coarse level: shape
Non-nested non-inherited multigrid theory

**Smoothing property:** There exist $C_R > 1$, independent of $k$ such that

$$\frac{\|u\|^2}{\lambda_k} \leq C_R(\bar{R}_k u, u) \quad \forall u \in M_k,$$

**Regularity and approximation:** There exist constants $\alpha \in (0, 1]$ and $C_\alpha > 0$ such that

$$A_k((I - I_k^{k-1} P_{k-1}) u, u) \leq C_\alpha^2 \left( \frac{\|A_k u\|^2}{\lambda_k} \right)^\alpha A_k(u, u)^{1-\alpha} \quad \forall u \in M_k.$$

**Energy stability:**

$$A_k(I_k^{k-1} u, I_k^{k-1} u) \leq \mu A_{k-1}(u, u),$$
## Non-nested non-inherited: Abstract results

<table>
<thead>
<tr>
<th>cycle</th>
<th>A-stability of $I_{k-1}^k$</th>
<th>smoothing</th>
<th>convergence result</th>
</tr>
</thead>
<tbody>
<tr>
<td>V-cycle</td>
<td>$\mu = 1$</td>
<td>any $m$</td>
<td>$0 \leq A_k((I - B_k A_k)u, u) \leq \delta_k A_k(u, u)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>steps</td>
<td>$\delta_k \leq \frac{M_k^{(1-\alpha)/\alpha}}{M_k^{(1-\alpha)/\alpha + m\alpha}}, M &gt; 0$ const.</td>
</tr>
<tr>
<td>W-cycle</td>
<td>$\mu = 2$</td>
<td>any $m$</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>steps</td>
<td>$\delta_k \leq \frac{M}{M + m\alpha}, M &gt; 0$ const.</td>
</tr>
<tr>
<td>W-cycle</td>
<td></td>
<td>sufficiently many $m$</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td>variable</td>
<td>variable V-cycle</td>
<td>$\lambda_{\text{min}}(B_k A_k) \geq \frac{m(k)^\alpha}{M + m(k)^\alpha}$</td>
</tr>
<tr>
<td></td>
<td>$\mu = 1$</td>
<td>variable V-cycle</td>
<td>$\lambda_{\text{max}}(B_k A_k) \leq \frac{M + m(k)^\alpha}{m(k)^\alpha}, M &gt; 0$ const.</td>
</tr>
<tr>
<td>V-cycle</td>
<td>$\mu = 1 + c\lambda_k(A_k)^{-\gamma}$</td>
<td>any $m$</td>
<td>$0 \leq A_k((I - B_k A_k)u, u) \leq \delta_k A_k(u, u)$</td>
</tr>
<tr>
<td></td>
<td>$\gamma \in (0, 1]$</td>
<td>steps</td>
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</tr>
<tr>
<td></td>
<td>$\gamma \in (0, 1]$</td>
<td>steps</td>
<td>$\lambda_{\text{max}}(B_k A_k)$ independent of $k$</td>
</tr>
</tbody>
</table>

**A-stability of $I_{k-1}^k$:**

$A_k(I_{k-1}^k u, I_{k-1}^k u) \leq \mu A_{k-1}(u, u), \quad \forall u \in M_{k-1}$
Convergence analysis

Estimating largest eigenvalue: \[ |A_k(E_ku, u)| \leq \delta_k A_k(u, u). \]

- instead of analysing operator norm of \( E_k \), we find a majorant for largest eigenvalue of \( E_k \) (Rayleigh ratio).
- \( E_k \) must be symmetric.
- When stability of \( I_k^{k-1} \) with \( \mu = 1 \), \( E_k \) is positive, the lower bound of \( A_k(E_ku, u) \) is trivially 0.

Estimating condition number: \[ \eta_0 A_k(u, u) \leq A_k(B_k A_k u, u) \leq \eta_1 A_k(u, u) \]

- when we do not have lower bound of \( A_k(E_ku, u) \) or even when the multigrid diverges as iterator.
- aim: independence of the condition number of the preconditioned problem \( (\eta_1/\eta_0) \) of \( h \).
- used for variable V-cycle, where number of pre- post-smoothing sweeps increase on coarse levels.
Regularity and approximation

There exist constants $\alpha \in (0, 1]$ and $C_\alpha > 0$ such that for all $k = 1, \ldots, J$ there is

$$A_k((I - I_k^{k-1} P_{k-1})u, u) \leq C_\alpha^2 \left( \frac{\|A_k u\|^2}{\lambda_k} \right) \alpha A_k(u, u)^{1-\alpha} \quad \forall u \in M_k.$$

Existence of $\tilde{Q}_k : H_0^2(\Omega) \rightarrow M_k$, such that

1. there is $C_1 > 0$ independent of $h_k$ such that for all $w \in H_0^2(\Omega)$

$$\|(I - \tilde{Q}_k)w\|_{0,\Omega} \leq C_1 h_k |w|_{1,\Omega}$$

2. there is $C_2 > 0$ independent of $h_k$ such that $\forall u_k \in M_k$ and $w \in H_0^2(\Omega)$, solution of

$$-\Delta w = A_k u_k \quad \text{in } \Omega$$

there is

$$|\tilde{Q}_k w - u_k|_{1, h_k} \leq C_2 h_k \|w\|_{2,\Omega}.$$
Finite volume $H^1$ error estimate for coarse levels

**Admissible mesh:** orthogonality condition

- to ensure symmetry of $A_k$
- to verify coercivity condition

**Consistency:**

- Numerical flux exact for linear functions
- Bramble-Hilbert Lemma

\[
\left| \int_{e} \nabla w \cdot n_e d\Gamma - \Phi_k(Q_k w) \right| \leq C \text{diam}(e)^{d/2} |w|_{2,\Omega_{loc}}
\]
Flux-correction: remedy to loss of consistency

Numerical flux corresponding to $A_{k-1} = (\tilde{I}_k^{k-1})^T A_k \tilde{I}_k^{k-1}$, ie.

$$\Phi^D_{k-1}(u_{k-1}, n_{e_{k-1}}) = \sum_{e_k \in e_{k-1}} \Phi^D_k(\tilde{I}_k^{k-1} u_{k-1}, n_{e_k})$$

is not consistent – for linear functions $w$

$$\Phi^D_{k-1}(\tilde{Q}_k w, n_{e_{k-1}}) \neq \int_{e_{k-1}} \nabla w n d\Gamma$$

Remedy: Correct $A_{k-1} = (\tilde{I}_k^{k-1})^T A_k \tilde{I}_k^{k-1}$ so that it corresponds to

$$\Phi^D_{k-1}(u_{k-1}, n_{e_{k-1}}) = C(e_{k-1}) \sum_{e_k \in e_{k-1}} \Phi^D_k(\tilde{I}_k^{k-1} u_{k-1}, n_{e_k})$$

so that for at least one particular linear function $w$ there is

$$\Phi^D_{k-1}(\tilde{Q}_k w, n_{e_{k-1}}) = \int_{e_{k-1}} \nabla w n d\Gamma$$
Numerical experiments: Poisson equation

Poisson problem on nonstructured 2D meshes

- **Petrov-Galerkin method**
  - for isotropic meshes convergence independent of mesh-size
  - convergence depends on number of levels $J$ as $O(1 - 1/J)$
    (predicted by the convergence result)
  - computational complexity of one V-cycle is not linearly proportional to $n_J$ (large fill-in on coarse levels)

- **Flux-Correction method**
  - Very efficient compared to Petrov-Galerkin and Volume-Agglomeration multigrids
  - Computational cost of one V-cycle proportional to $n_J$
  - Convergence independent of meshsize
Unstructured 2D mesh: test1
Unstructured 2D mesh: naca0012
Unstructured 2D mesh: rae2822
test1 results

- solving $-\Delta u = 1$ with Dirichlet boundary conditions
- 5 level V-cycle, coarsening by agglomeration, coarsening factor 2
- comparison between Algebraic Petrov-Galerkin MG (green) and Volume Agglomeration with global (red) or local (blue) correction
- 3 test, pre- and postsmoothers: 3+3 Jacobi, 1+1 Gauss-Seidel, 2+2 Gauss-Seidel iterations
naca0012 results

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**rae2822 results**

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**Numerical experiments: Navier-Stokes**

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<th>Multigrid params</th>
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</thead>
<tbody>
<tr>
<td>mesh</td>
<td>Re</td>
</tr>
<tr>
<td>NACA</td>
<td>73</td>
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<tr>
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</tr>
</tbody>
</table>

- Convergence of schemes with all multigrid methods is comparable
- Volume-Agglomeration is slightly worse than Petrov-Galerkin and Flux-Correction
- Flux-Correction is far cheaper than Petrov-Galerkin
- Efficiency of multigrids compared to single-grid methods
NACA viscous transsonic flow, linear convergence
Multigrid for Navier-Stokes

Aleš Janka

NACA viscous transsonic flow, nonlinear convergence
NACA viscous transsonic flow, stationary solution
NACA viscous transsonic flow, first timestep
NACA viscous supersonic flow, nonlinear convergence
NACA viscous supersonic flow, stationary solution
NACA viscous subsonic flow, linear convergence
NACA viscous subsonic flow, nonlinear convergence
NACA viscous subsonic flow, stationary solution

NACA0012, Re=500 Ma=0.3 α=0°, stationary density

NACA0012, Re=500 Ma=0.3 α=0°, stationary Mach number
NACA viscous subsonic flow, first timestep
RAE transsonic flow, nonlinear convergence
RAE transsonic flow, stationary solution

RAE2822, Re=1000 Ma=0.7 α=3°, stationary density

RAE2822, Re=1000 Ma=0.7 α=3°, stationary Mach number
Multigrid for Navier-Stokes

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MBODY subsonic flow, nonlinear convergence

![Graphs showing convergence of MBODY simulations](image-url)
Multigrid for Navier-Stokes

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Conclusions and prospectives

• Connects engineering ideas with some analytical results
• Suitable coarse problem generation for diffusive term
• Petrov-Galerkin: gives convergence proof for SPD problems
• Flux-Correction: new point of view from finite volume framework

Future?

• Flux-Correction multigrid proof? - regularity and approximation with smoothing property
• Petrov-Galerkin multigrid: aggregation of nodes with respect to coarse level fill-in
• Turbulence models, small Mach number preconditioners, higher spatial schemes