Numerical methods for ferromagnetic plates *


1 Introduction

We compare some numerical methods for the simulation of ferromagnetic phenomenons in a metallic plate, with or without holes.

First we briefly recall the physical model we use for describing the ferromagnetic phenomenon. Next we present the discretization methods we want to compare. We then apply these methods on the simple test-case of a thin ferromagnetic plate placed in front of a rectilinear electric conductor. We compare the various obtained results: magnetic field on a line perpendicular to the plate and relative permeability on a given line in the plate; we also compare memory requirements for each method.

2 Modeling of ferromagnetism

Let $\Lambda \subset \mathbb{R}^3$ be a domain with boundary $\partial\Lambda$ occupied by a ferromagnetic material with relative magnetic permeability $\mu_r \left( \| \vec{H} \| \right) \geq 1$ depending on the euclidean norm of the magnetic field $\vec{H}$, denoted by $\| \vec{H} \|$. In the following, we suppose that $\Lambda$ is a bounded open, possibly non simply connected set, surrounded by known stationary electric currents denoted by $\vec{j}_0$. We denote by $\vec{n}$ the unit normal on $\partial\Lambda$, external to $\Lambda$. Moreover, we assume that all the external currents are not modified by the presence of the ferromagnetic material and no electric current flows in the domain $\Lambda$. Without the ferromagnetic material, it is possible to explicit the magnetic induction field $\vec{B}_0$ due to $\vec{j}_0$ by using Biot-Savart law:

$$\vec{B}_0(\vec{x}) = \mu_0 \int_{\mathbb{R}^3} \vec{\nabla}_x \cdot G(\vec{x}, \vec{y}) \wedge \vec{j}_0(\vec{y}) \, dy, \quad \forall \vec{x} \in \mathbb{R}^3,$$

where $\mu_0$ is the magnetic permeability of the void, $G(\vec{x}, \vec{y})$ is the Green kernel given by

$$G(\vec{x}, \vec{y}) = \frac{1}{4\pi} \frac{1}{\| \vec{x} - \vec{y} \|} \quad \text{with} \quad \vec{x}, \vec{y} \in \mathbb{R}^3, \vec{x} \neq \vec{y},$$

and $\vec{\nabla}_x$ denotes the gradient with respect to the variable $\vec{x}$.

Let us remark that if $\vec{H}_0$ is the magnetic field corresponding to $\vec{B}_0$, we have in the whole space $\mathbb{R}^3$ without ferromagnetic materials

$$\vec{B}_0 = \mu_0 \vec{H}_0,$$
$$\text{div} \vec{B}_0 = 0,$$
$$\text{curl} \vec{H}_0 = \vec{j}_0.$$

Due to the presence of the ferromagnetic domain $\Lambda$, the magnetic field $\vec{H}$ and the induction field $\vec{B}$ cannot be explicitly given in function of $\vec{j}_0$, but they are governed by the following relationships,

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true in the whole space $\mathbb{R}^3$:
\[
\vec{B} = \mu_0 \mu_r \vec{H}, \tag{6}
\]
\[
\text{div} \vec{B} = 0, \tag{7}
\]
\[
\text{curl} \vec{H} = \vec{J}_0. \tag{8}
\]

We note that outside the domain $\Lambda$ we have $\mu_r = 1$. Since the magnetization field $\vec{M}$ is defined by $\vec{M} = \mu_0 (\mu_r - 1) \vec{H}$, we will be able to compute $\vec{M}$ if we are able to calculate $\vec{H}$. In the following, we are looking for the field $\vec{H}$.

### 2.1 Scalar potential model

By subtracting (5) and (8) we obtain the existence of a continuous function $\psi$ satisfying
\[
\vec{H}(x) - \vec{H}_0(x) = -\nabla \psi(x) \quad \forall x \in \mathbb{R}^3. \tag{9}
\]

By using equalities (3), (4) and (6), (7) together with (9), we easily verify that
\[
-\text{div} \left( \mu_r \nabla \psi \right) = -\text{div} (\mu_r - 1) \vec{H}_0 \quad \text{in} \, \mathbb{R}^3. \tag{10}
\]

In order to obtain a finite energy, we assume that
\[
\psi(x) = O \left( \frac{1}{\|x\|} \right) \quad \text{when} \, \|x\| \to \infty. \tag{11}
\]

Let $\Lambda'$ be the exterior open domain $\Lambda' = \mathbb{R}^3 \setminus \overline{\Lambda}$. Since $\mu_r = 1$ in $\Lambda'$, we obtain
\[
\Delta \psi = 0 \quad \text{in} \, \Lambda', \tag{12}
\]
where $\Delta$ is the laplacian operator.

In fact, equation (10) is non linear and it is necessary to precise what is $\mu_r$, which is a discontinuous function since $\mu_r = 1$ in $\Lambda'$ and $\mu_r = \mu_r(\|\vec{H}\|)$ in $\Lambda$. In order to write correctly the model, we define the mapping $\bar{\mu} : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^+$ by
\[
\bar{\mu}(x,s) = \begin{cases} 
1 & \text{if } x \notin \Lambda', s \in \mathbb{R}^+,

\mu_r(s) & \text{if } x \in \Lambda', s \in \mathbb{R}^+,
\end{cases} \tag{13}
\]
where $\mu_r(s)$ is the relative magnetic permeability of the ferromagnetic plates occupying $\Lambda$ given in function of $s = \|\vec{H}\|$. Since $\vec{H} = \vec{H}_0 - \nabla \psi$, it follows that the model consists to find $\psi : \mathbb{R}^3 \to \mathbb{R}$ satisfying
\[
-\text{div} \left( \bar{\mu}(\cdot, \|\vec{H}_0 - \nabla \psi\|) \nabla \psi \right) = -\text{div} \left( \bar{\mu}(\cdot, \|\vec{H}_0 - \nabla \psi\|) - 1 \right) \vec{H}_0 \quad \text{in} \, \mathbb{R}^3, \tag{14}
\]
\[
\psi(x) = O \left( \frac{1}{\|x\|} \right) \quad \text{when} \, \|x\| \to \infty. \tag{15}
\]

In order to simplify the notation, we will leave out in the following the argument of the mapping $\bar{\mu}$, knowing that it depends on $x \in \mathbb{R}^3$ and $\|\vec{H}_0 - \nabla \psi\|$, in order to write
\[
\begin{cases} 
-\text{div} (\bar{\mu} \nabla \psi) = -\text{div} (\bar{\mu} - 1) \vec{H}_0 & \text{in} \, \mathbb{R}^3, \\
\psi(x) = O \left( \frac{1}{\|x\|} \right) & \text{when} \, \|x\| \to \infty.
\end{cases} \tag{16}
\]

Remark that $\vec{H}_0$ need not be known in $\mathbb{R}^3$ but only on $\Lambda$ because $\bar{\mu} = 1$ outside $\Lambda$. 

\[2\]
2.2 Vector potential model

From (4) and (7) we obtain \( \text{div}(\vec{B} - \vec{B}_0) = 0 \) in \( \mathbb{R}^3 \). Thus, there exists a vector potential \( \vec{A} : \mathbb{R}^3 \to \mathbb{R}^3 \) satisfying Coulomb’s gauge such that

\[
\vec{B} - \vec{B}_0 = \text{curl} \vec{A} \quad \text{in} \ \mathbb{R}^3 \tag{17}
\]

with

\[
\text{div} \vec{A} = 0 \quad \text{in} \ \mathbb{R}^3. \tag{18}
\]

By subtracting (5) and (8) we have

\[
\text{curl}(\vec{H} - \vec{H}_0) = 0 \quad \text{in} \ \mathbb{R}^3, \tag{19}
\]

and, using (3), (6) and (19), we get

\[
\text{curl} \left( \frac{1}{\mu_r} \text{curl} \vec{A} \right) = \text{curl} \left( \left( 1 - \frac{1}{\mu_r} \right) \vec{B}_0 \right) \quad \text{in} \ \mathbb{R}^3. \tag{20}
\]

Let us remark that outside \( \Lambda \) we have \( \mu_r = 1 \) and consequently \( \text{curl} \left( \text{curl} \vec{A} \right) = 0 \). Together with (18) we obtain

\[
\Delta \vec{A} = 0 \quad \text{in} \ \Lambda'. \tag{21}
\]

To assure the physical meaning of finite magnitude energy, we impose the following condition at infinity:

\[
\| \vec{A}(\vec{x}) \| = \mathcal{O} \left( \frac{1}{\|\vec{x}\|} \right) \quad \text{when} \ \|\vec{x}\| \text{ tends to infinity .} \tag{22}
\]

Here, the model consists to find \( \vec{A} : \mathbb{R}^3 \to \mathbb{R}^3 \) satisfying (18),(20) and (22). In (20), \( \mu_r = 1 \) outside \( \Lambda \) and \( \mu_r \) is a function of \( \|\vec{H}\| \) inside \( \Lambda \). In fact, Equation (20) is non linear. Let us precise this point. Using definition (13), the meaning of \( \mu_r \) in Equation (20) is in fact \( \bar{\mu}(x,\|\vec{H}\|) \) when we adopt this notation. In the applications, the mapping \( s \to \mu_r(s) \) is smooth, bounded, strictly decreasing and satisfying

\[
\mu_r(s) + s \frac{d}{ds} \mu_r(s) \geq 1, \quad \forall s \in \mathbb{R}^+. \tag{23}
\]

It follows that

\[
\frac{d}{ds} \left( s \mu_0 \mu_r(s) \right) \geq \mu_0 > 0, \quad \forall s \in \mathbb{R}^+ \tag{24}
\]

and we can consider the reciprocal function \( \nu : \mathbb{R}^+ \to \mathbb{R}^+ \) of \( s \to s \mu_0 \mu_r(s) \) satisfying

\[
\nu(s \mu_0 \mu_r(s)) = s, \quad \forall s \in \mathbb{R}^+. \tag{25}
\]

Since \( s = \|\vec{H}\| \) in the plate, we have \( \|\vec{H}\| = \nu \left( \mu_0 \mu_r(\|\vec{H}\|) \|\vec{H}\| \right) = \nu(\|\vec{B}\|) \) and consequently

\[
\mu_r(\|\vec{H}\|) = \frac{1}{\mu_0} \frac{\|\vec{B}\|}{\|\vec{H}\|} = \frac{1}{\mu_0} \frac{\|\vec{B}\|}{\nu(\|\vec{B}\|)} \tag{26}
\]

in the plates. Remark that the function \( \nu_r \) defined by

\[
\nu_r(s) = \mu_0 \frac{\nu(s)}{s}, \quad s \in \mathbb{R}^+ \tag{27}
\]
satisfies \( \frac{1}{\mu_r(\|\vec{H}\|)} = \nu_r(\|\vec{B}\|) \). The function \( \nu_r \) is called “relative reluctivity”.

By using (17) we obtain in the ferromagnetic material:

\[
\mu_r(\|\vec{H}\|) = \frac{1}{\nu_r(\|\vec{B}\|)} = \frac{1}{\nu_r(\|\vec{B}_0 + \text{curl} \vec{A}\|)}.
\]

(28)

Now, by defining \( \bar{\nu} : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^+ \) by

\[
\bar{\nu}(\vec{x}, s) = \begin{cases} 
1 & \text{if } \vec{x} \in \Lambda', s \in \mathbb{R}^+, \\
\nu_r(s) & \text{if } \vec{x} \in \Lambda, s \in \mathbb{R}^+, 
\end{cases}
\]

(29)

we have to find a field \( \vec{A} : \mathbb{R}^3 \to \mathbb{R}^3 \) satisfying the nonlinear problem:

\[
\begin{align*}
\text{curl} \left( \bar{\nu} \cdot (\|\vec{B}_0 + \text{curl} \vec{A}\|) \text{ curl} \vec{A} \right) &= \text{curl} \left( \left(1 - \bar{\nu} \cdot (\|\vec{B}_0 + \text{curl} \vec{A}\|)\right) \vec{B}_0 \right) & \text{in } \mathbb{R}^3, \\
\text{div} \vec{A} &= 0 & \text{in } \mathbb{R}^3, \\
\|\vec{A}(\vec{x})\| &= \mathcal{O}(\frac{1}{\|\vec{x}\|}) & \text{when } \|\vec{x}\| \to \infty.
\end{align*}
\]

(30)

In order to simplify the notations, we leave out the argument of the mapping \( \bar{\nu} \) knowing it depends on \( \vec{x} \in \mathbb{R}^3 \) and \( \|\vec{B}_0 + \text{curl} \vec{A}\| \).

Birò et al. [4, 5] treated this problem by inserting (31) in (30) in order to obtain an equivalent form:

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{curl} \left( \bar{\nu} \text{ curl} \vec{A} \right) - \vec{\nabla} \left( \bar{\nu} \text{ div} \vec{A} \right) = \text{curl} \left( \left(1 - \bar{\nu} \cdot (\|\vec{B}_0 + \text{curl} \vec{A}\|)\right) \vec{B}_0 \right) & \text{in } \mathbb{R}^3, \\
\|\vec{A}(\vec{x})\| &= \mathcal{O}(\frac{1}{\|\vec{x}\|}) & \text{when } \|\vec{x}\| \to \infty.
\end{array} \right.
\]

(33)

Clearly speaking, if \( \vec{A} \) is a solution of (33) we obtain by taking the divergence of (33):

\[
\Delta \left( \bar{\nu} \text{ div} \vec{A} \right) = 0 & \text{in } \mathbb{R}^3.
\]

(34)

Since \( \vec{A} \) is harmonic outside \( \Lambda \) (see (21)) and if \( \|\vec{A}(\vec{x})\| \) tends to zero when \( \|\vec{x}\| \) tends to infinity, we have

\[
\text{div} \vec{A} = \mathcal{O}(\frac{1}{\|\vec{x}\|^2}) & \text{when } \|\vec{x}\| \to \infty.
\]

(35)

We conclude with (34) and (35) that \( \text{div} \vec{A} = 0 \) in \( \mathbb{R}^3 \). Remark that in formulation (33), \( \vec{B}_0 \) need not be known in \( \mathbb{R}^3 \) but only in \( \Lambda \) because \( \bar{\nu} = 1 \) outside \( \Lambda \).

3 Formulations of the scalar and vector potential problems

Let us now focus on the weak formulations of the scalar or vector potential models. The main difficulty with problems (16) resp. (33), is that we seek a function \( \psi \) or \( \vec{A} \) defined in the whole space \( \mathbb{R}^3 \).

We will use two different ways to overcome this problem: the first one uses an integral formulation on \( \partial \Lambda \) to replace the so-called “exterior” problem by a relation valid on the boundary of \( \Lambda \); the numerical approximation leads to a big non sparse matrix to “invert”.

The other way formulates the problem only on a bounded domain, stretching from the ferromagnetic object \( \Lambda \) to the nearby artificial boundary. The behaviour of the solution in the far-field (“exterior” problem) is simulated through a “harmonic” boundary condition given by the Poisson representation formula.
3.1 Formulations for the scalar potential model

We have seen that the scalar potential model leads to find a mapping $\psi$ satisfying (16). Since $\text{div} \, \vec{H}_0 = 0$ in $\mathbb{R}^3$, we can write this problem in the form

$$\text{div} \left( \vec{\mu} (\vec{H}_0 - \vec{\nabla} \psi) \right) = 0 \quad \text{in } \mathbb{R}^3$$

(36)

with $\psi(\vec{x}) = O \left( \frac{1}{|\vec{x}|^3} \right)$ when $|\vec{x}| \to \infty$. If $W^1(\mathbb{R}^3)$ is the Beppo-Levi space given by

$$W^1(\mathbb{R}^3) = \{ v : \mathbb{R}^3 \to \mathbb{R} : \frac{v(\vec{x})}{1 + |\vec{x}|} \in L^2(\mathbb{R}^3), \vec{\nabla} v \in L^2(\mathbb{R}^3) \},$$

(37)

it is proven in [12] that there exists a unique $\psi \in W^1(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} \vec{\mu} \left( \vec{H}_0 - \vec{\nabla} \psi \right) \cdot \vec{\nabla} \varphi \, dx = 0, \quad \forall \varphi \in W^1(\mathbb{R}^3).$$

(38)

It follows that Problem (16) possesses a unique weak solution $\psi \in W^1(\mathbb{R}^3)$.

We now present three different approaches to compute the scalar potential $\psi$.

3.1.1 Boundary integral formulation of the scalar potential model

It is known [9], that if $v$ is a harmonic function in $\Lambda$ and in $\Lambda'$ satisfying $v(\vec{x}) = O \left( |\vec{x}|^{-1} \right)$ when $|\vec{x}| \to \infty$, and sufficiently regular (say $C^1$ in $\overline{\Lambda}$ and in $\overline{\Lambda}'$), then we have for $\vec{x} \in \partial \Lambda$:

$$\frac{1}{2} \left( v^E(\vec{x}) + v^I(\vec{x}) \right) = - \int_{\partial \Lambda} \frac{\partial v}{\partial n}(\vec{y}) \, d\partial \Lambda \, G(\vec{x}, \vec{y}) \, ds(\vec{y}) + \int_{\partial \Lambda} [v(\vec{y})]_{\partial \Lambda} \frac{\partial G(\vec{y}, \vec{y})}{\partial n(\vec{y})} \, ds(\vec{y}),$$

(39)

where $v^E$ is the restriction of $v$ to $\Lambda'$, $v^I$ is the restriction of $v$ to $\Lambda$ and $[v]_{\partial \Lambda} = v^E - v^I$ is the jump of $v$ through the boundary $\partial \Lambda$ of $\Lambda$.

If $\psi$ is the solution of (36), let $w$ be a harmonic function in $\Lambda \cup \Lambda'$ satisfying $w = \psi$ on $\partial \Lambda$, $w(\vec{x}) = O \left( |\vec{x}|^{-1} \right)$ when $|\vec{x}| \to \infty$. Clearly, because $\psi$ is harmonic in $\Lambda'$, we have $w = \psi$ in $\overline{\Lambda}$. Moreover, by using relationship (39) with $v = w$ in $\Lambda$ and $v = \psi$ in $\Lambda'$, we obtain for $x \in \partial \Lambda$

$$\psi(\vec{x}) = - \int_{\partial \Lambda} \left( \frac{\partial \psi^E}{\partial n}(\vec{y}) - \frac{\partial w^I}{\partial n}(\vec{y}) \right) G(\vec{x}, \vec{y}) \, ds(\vec{y}),$$

(40)

which is equivalent to

$$\int_{\partial \Lambda} \psi \eta \, ds = - \int_{\partial \Lambda} \eta(\vec{x}) \int_{\partial \Lambda} \left( \frac{\partial \psi^E}{\partial n}(\vec{y}) - \frac{\partial w^I}{\partial n}(\vec{y}) \right) G(\vec{x}, \vec{y}) \, ds(\vec{y}),$$

(41)

for all $\eta \in H^{-1/2}(\partial \Lambda)$.

By using the fact that $\text{div} \, \vec{H}_0 = 0$ in $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \vec{H}_0 \cdot \vec{\nabla} \varphi \, dx = 0, \quad \forall \varphi \in W^1(\mathbb{R}^3),$$

(42)

and with the weak formulation (38):

$$\int_{\mathbb{R}^3} \vec{\mu} \vec{\nabla} \psi \cdot \vec{\nabla} \varphi \, dx = \int_{\mathbb{R}^3} (\vec{\mu} - 1) \vec{H}_0 \cdot \vec{\nabla} \varphi \, dx, \quad \forall \varphi \in W^1(\mathbb{R}^3).$$

(43)

We note by $H^1(\Lambda)$ the classical Sobolev space of order 1, $H^{1/2}(\partial \Lambda)$ the space of traces on $\partial \Lambda$ of mappings belonging to $H^1(\Lambda)$ and $H^{-1/2}(\partial \Lambda)$ its dual space.
Since \( \bar{\mu} = 1 \) outside \( \Lambda \) we obtain

\[
\int_{\Lambda} \bar{\mu} \nabla \phi \cdot \nabla \varphi dx + \int_{\partial \Lambda} \bar{\nabla} \phi \cdot \nabla \varphi dx = \int_{\Lambda} (\bar{\mu} - 1) \bar{H}_0 \cdot \nabla \varphi dx, \quad \forall \varphi \in W^1(\mathbb{R}^3),
\]

and, by integrating by parts (\( \bar{n} \) is pointing inside \( \Lambda' \) and \( \Delta \psi = 0 \) in \( \Lambda' \)):

\[
\int_{\Lambda} \bar{\mu} \nabla \phi \cdot \nabla \varphi dx - \int_{\partial \Lambda} \frac{\partial \psi E}{\partial n} \varphi ds = \int_{\Lambda} (\bar{\mu} - 1) \bar{H}_0 \cdot \nabla \varphi dx, \quad \forall \varphi \in W^1(\mathbb{R}^3).
\]

Since \( w \) is harmonic in \( \Lambda \), we have \( w \in H^1(\Lambda) \) satisfying \( w = \psi \) on \( \partial \Lambda \) and \( \int_{\Lambda} \nabla w \cdot \nabla v dx = 0, \forall v \in H^1_0(\Lambda) \).

By using the definition (13) of \( \bar{\mu} \), we can replace \( \bar{\mu} \) in (45) by \( \bar{\mu} = \mu_r(\|\bar{H}_0 - \nabla \psi\|) \) and by setting \( \lambda = \frac{\partial w}{\partial n} \) (external Steklov-Poincaré operator) we obtain the non linear problem:

\[
\text{find } \psi \in H^1(\Lambda), w \in H^1(\Lambda) \text{ and } \lambda \in H^{-\frac{1}{2}}(\partial \Lambda) \text{ satisfying } w = \psi \text{ on } \partial \Lambda \text{ and:}
\]

\[
\int_{\Lambda} \mu_r \nabla \psi \cdot \nabla \varphi dx - \int_{\partial \Lambda} \lambda \varphi ds = \int_{\Lambda} (\mu_r - 1) \bar{H}_0 \cdot \nabla \varphi dx, \quad \forall \varphi \in H^1(\Lambda) \quad (46)
\]

\[
\int_{\Lambda} \nabla w \cdot \nabla v dx = 0, \quad \forall v \in H^1_0(\Lambda) \quad (47)
\]

\[
\int_{\partial \Lambda} \psi \eta ds = -\int_{\partial \Lambda} \eta(\bar{x}) ds(\bar{x}) \int_{\partial \Lambda} \left( \lambda(\bar{g}) - \frac{\partial w}{\partial n}(\bar{g}) \right) G(\bar{x}, \bar{g}) ds(\bar{g}), \quad \forall \eta \in H^{-\frac{1}{2}}(\partial \Lambda); \quad (48)
\]

where \( \mu_r = \mu_r(\|\bar{H}_0 - \nabla \psi\|) \).

### 3.1.2 Scalar potential problem with Poisson formula boundary condition

Let us now introduce another way to reduce the "exterior" problem to a problem expressed in a bounded domain.

Let \( B_R \) be the ball of radius \( R > 0 \) and \( B_r \) be the ball of radius \( r \) with \( R > r > 0 \), both centered at the origin, and such that \( \Lambda \subset B_r \).

We use the following Poisson formula which says that since \( \psi \) is a harmonic function outside the ball \( B_r \) and radially decreasing at infinity (11), then for each point \( \bar{x} \) outside the ball \( B_r \) we have

\[
\psi(\bar{x}) = \frac{1}{2\pi} \int_{\partial B_r} \frac{\psi(\bar{g})}{\|\bar{g} - \bar{x}\|^3} ds(\bar{g}).
\]

By using this formula for \( \bar{x} \in \partial B_R \) we obtain the formulation: find \( \psi \in H^1(B_R) \) satisfying for all \( \varphi \in H^1_0(B_R) \),

\[
\int_{B_R} \bar{\mu} \nabla \psi \cdot \nabla \varphi dx = \int_{\Lambda} (\bar{\mu} - 1) \bar{H}_0 \cdot \nabla \varphi dx, \quad (50)
\]

\[
\psi(\bar{x}) = \frac{R^2 - r^2}{2\pi} \int_{\partial B_r} \frac{\psi(\bar{g})}{\|\bar{g} - \bar{x}\|^3} ds(\bar{g}),
\]

for all points \( \bar{x} \in \partial B_R \); here \( \bar{\mu} = \bar{\mu}(\cdot, \|\bar{H}_0 - \nabla \psi\|) \).
3.1.3 Mixed formulation of the scalar potential problem with Poisson formula boundary condition

We finally present a variant formulation of the previous one, inspired by the fact that we seek a good (numerical) approximation of $\tilde{H}$, not of $\psi$ which is just an auxiliary potential. To this aim, we remark that equation (10) just expresses that the divergence of $\vec{p}$ vanishes, where $\vec{p} = \mu_0 \vec{A} - (\mu_0 - 1)\vec{H}_0$; moreover we have $\mu_0 \vec{p} = \vec{B}_0 - \vec{B}$ which corresponds to the physical field we are interested in.

At this point we remark by using (17) that $\vec{p} = -\frac{1}{\mu_0} \text{curl} \vec{A}$ and by using (28),(29) we have

$$\mu(\cdot, \|\tilde{H}_0 - \nabla \psi\|)^{-1} = \nu(\cdot, \|\tilde{B}_0 - \mu_0 \vec{p}\|).$$

(52)

So, for the system:

$$\frac{1}{\mu} \vec{p} = \nabla \psi - (1 - \frac{1}{\mu}) \tilde{H}_0,$$

in $\mathcal{B}_R$, (53)

$$\text{div} \vec{p} = 0,$$

in $\mathcal{B}_R$, (54)

$$\psi = \psi_0,$$

on $\partial \mathcal{B}_R$, (55)

when $\psi_0$ is given on the boundary of $\mathcal{B}_R$ (say $\psi_0 \in H^{1/2}(\partial \mathcal{B}_R)$), we obtain with (52) the natural formulation:

find $\psi \in L^2(\mathcal{B}_R)$, $\vec{p} \in H(\text{div}, \mathcal{B}_R)$ such that

$$\int_{\mathcal{B}_R} \nu \vec{p} \cdot \vec{q} \, dx + \int_{\mathcal{B}_n} \psi \text{div} \vec{q} \, ds - \int_{\Lambda} (1 - \nu) \tilde{H}_0 \cdot \vec{q} \, dx,$$

(56)

$$= \int_{\mathcal{B}_n} \varphi \text{div} \vec{p} \, dx,$$

(57)

for all $\vec{q} \in H(\text{div}, \mathcal{B}_R)$ and for all $\varphi \in L^2(\mathcal{B}_R)$. Here, $H(\text{div}, \mathcal{B}_R) = \{ \vec{v} \in L^2(\mathcal{B}_R)^3; \text{div} \vec{v} \in L^2(\mathcal{B}_R) \}$ and $\nu = \nu(\cdot, \|\tilde{B}_0 - \mu_0 \vec{p}\|)$.

Now we would like to replace in (56) $\psi_0$ on the boundary $\partial \mathcal{B}_R$ by the Poisson formula (49), i.e.:

$$\psi_0(\vec{x}) = \frac{R^2 - r^2}{4\pi r} \int_{\partial \mathcal{B}_R} \frac{\psi(\vec{y})}{\|\vec{y} - \vec{x}\|^3} \, ds(\vec{y}), \quad \forall \vec{x} \in \partial \mathcal{B}_R.$$

Strictly taken, this formulation leads to a difficulty since $\psi$ is a priori found in $L^2(\mathcal{B}_R)$ and the trace of $\psi$ on $\partial \mathcal{B}_R$ does not exist a priori. For this reason, we are obliged to introduce a regularizing operator $R_\epsilon$ defined by

$$R_\epsilon(f)(\vec{x}) = \int_{\mathcal{B}_R} J_\epsilon(\vec{x} - \vec{y}) f(\vec{y}) \, dy, \quad \forall f \in L^2(\mathcal{B}_R), \quad \forall \vec{x} \in \mathcal{B}_R,$$

(58)

where $J_\epsilon$ is a mollifier, and we add to (56),(57) :

$$\psi_0(\vec{x}) = \frac{R^2 - r^2}{4\pi r} \int_{\partial \mathcal{B}_R} \frac{R_\epsilon \psi(\vec{y})}{\|\vec{y} - \vec{x}\|^3} \, ds(\vec{y}), \quad \forall \vec{x} \in \partial \mathcal{B}_R.$$

(59)

Clearly speaking, a solution $(\vec{p}, \psi)$ of Problem (56),(57),(59) depends on $\epsilon$.

3.2 Formulations for the vector potential model

Let us come now to the vector potential model. In this case we will use only the second way to treat the exterior problem, i.e., boundary condition on $\partial \mathcal{B}_R$ obtained by Poisson’s formula.
In order to establish a weak formulation of Problem (33) we suppose the geometric setting \( \Lambda \subset \mathcal{B}_r \subset \mathcal{B}_R \) of Section 3.1.2 and we formally multiply the first equation of (33) by \( \bar{w} \), with \( \bar{w} = 0 \) on \( \partial \mathcal{B}_R \), for obtaining, after integrating by parts:

\[
\int_{\mathcal{B}_n} \bar{\nu} \, \text{curl} \, \bar{A} \cdot \text{curl} \, \bar{w} \, dx + \int_{\mathcal{B}_n} \bar{\nu} \, \text{div} \, \bar{A} \, \text{div} \, \bar{w} \, dx = \int_{\Lambda} (1 - \bar{\nu}) \, \bar{B}_0 \cdot \text{curl} \, \bar{w} \, dx. \tag{60}
\]

As \( \bar{A} \) has to be harmonic in \( \mathcal{B}_r' \equiv \mathbb{R}^3 \setminus \bar{\mathcal{B}}_r \subset \Lambda' \), we can write the Poisson representation formula for the exterior problem in \( \mathcal{B}_r' \),

\[
\bar{A} (\bar{x}) = \frac{\| \bar{x} \|^2 - r^2}{4\pi r} \int_{\partial \mathcal{B}_r} \frac{\bar{A} (\bar{y})}{\| \bar{y} - \bar{x} \|^3} \, ds (\bar{y}), \quad \text{for} \quad \bar{x} \in \mathcal{B}_r'. \tag{61}
\]

The weak formulation of (33) restricted only on \( \mathcal{B}_R \subset \mathbb{R}^3 \), with (61) in the role of boundary condition for all \( \bar{x} \in \partial \mathcal{B}_R \), gives the following problem: find \( \bar{A} \in H (\text{curl}, \mathcal{B}_R) \cap H (\text{div}, \mathcal{B}_R) \) such that for all \( \bar{w} \in H (\text{curl}, \mathcal{B}_R) \cap H (\text{div}, \mathcal{B}_R) \), \( \bar{w} = 0 \) on \( \partial \mathcal{B}_R \), we have

\[
\int_{\mathcal{B}_n} \bar{\nu} \, \text{curl} \, \bar{A} \cdot \text{curl} \, \bar{w} \, dx + \int_{\mathcal{B}_n} \bar{\nu} \, \text{div} \, \bar{A} \, \text{div} \, \bar{w} \, dx = \int_{\Lambda} (1 - \bar{\nu}) \, \bar{B}_0 \cdot \text{curl} \, \bar{w} \, dx, \tag{62}
\]

\[
\bar{A} (\bar{x}) = \frac{R^2 - r^2}{4\pi r} \int_{\partial \mathcal{B}_r} \frac{\bar{A} (\bar{y})}{\| \bar{y} - \bar{x} \|^3} \, ds (\bar{y}), \tag{63}
\]

for all boundary points \( \bar{x} \in \partial \mathcal{B}_R \). Here, \( H (\text{curl}, \mathcal{B}_R) = \{ \bar{w} \in L^2 (\mathcal{B}_R)^3 ; \text{curl} \, \bar{w} \in L^2 (\mathcal{B}_R)^3 \} \) and \( H (\text{div}, \mathcal{B}_R) = \{ \bar{w} \in L^2 (\mathcal{B}_R)^3 ; \text{div} \, \bar{w} \in L^2 (\mathcal{B}_R) \} \) and \( \bar{\nu} = \nu (\cdot, \| \bar{B}_0 + \text{curl} \, \bar{A} \|) \).

## 4 Discretization and numerical methods

Before we describe the methods we derived from the above formulations, let us introduce some notations of discretization spaces, common for all of them.

Let us assume that the ferromagnetic domain \( \Lambda \) is a polyhedron. Let us also consider polyhedra \( \mathcal{B}_{Rh} \) and \( \mathcal{B}_{rh} \) approximating the big and the small balls \( \mathcal{B}_R \), resp. \( \mathcal{B}_r \).

**Definition 1** (interior and boundary mesh). Let \( \Omega \) be a polyhedral domain.

1. Let us denote \( \tau_h (\Omega) \) the **tetrahedral mesh of** \( \Omega \) with conforming tetrahedra, in the finite-element sense. The tetrahedron \( K \in \tau_h (\Omega) \) is understood as a closed tetrahedron.
2. The set of all internal and external faces of the tetrahedral mesh \( \tau_h (\Omega) \) is denoted \( \mathcal{F}_h (\Omega) \).
3. Let us denote \( \tau_h (\partial \Omega) \) the **trace of the mesh** \( \tau_h (\Omega) \) on the boundary \( \partial \Omega \). The **boundary mesh** \( \tau_h (\partial \Omega) \) is composed of all triangular faces \( F \in \mathcal{F}_h (\Omega) \) such that \( F \subset \partial \Omega \).

**Definition 2** (finite element spaces). Let \( \Omega \) be a polyhedral domain, \( \tau_h (\Omega) \) its tetrahedral mesh and \( \tau_h (\partial \Omega) \) the corresponding boundary mesh.

1. On the tetrahedral mesh \( \tau_h (\Omega) \), we define:
   
   (a) The **finite element space** \( \mathcal{P}_{1h} (\Omega) \) of continuous functions which are piecewise polynomial of degree 1 on \( K \in \tau_h (\Omega) \),
   \[
   \mathcal{P}_{1h} (\Omega) = \{ w \in C^0 (\Omega), \forall K \in \tau_h (\Omega), \exists a \in \mathbb{R}, \bar{b} \in \mathbb{R}^3 \text{ s.t. } w (\bar{x}) = a + \bar{b} \cdot \bar{x}, \forall \bar{x} \in K \}. \]
   (b) The **finite element space** \( \mathcal{P}_{1h} (\partial \Omega) \) of functions \( w \in \mathcal{P}_{1h} (\Omega) \) such that \( w = 0 \) on \( \partial \Omega \).
   (c) The **finite element space** \( \mathcal{P}_{0h} (\Omega) \) of piecewise constant functions on \( K \in \tau_h (\Omega) \),
   \[
   \mathcal{P}_{0h} (\Omega) = \{ w \in L^2 (\Omega), \forall K \in \tau_h (\Omega), \exists a \in \mathbb{R} \text{ s.t. } w (\bar{x}) = a, \forall \bar{x} \in K \}. \]
(d) The finite element space $\mathcal{RT}_h(\Omega)$, the lowest-order Raviart-Thomas space of piecewise linear vector-functions on $K \in \tau_h(\Omega)$, with normal component continuous through each internal face $F$ belonging to $\mathcal{F}_h(\Omega)$,

$$\mathcal{RT}_h(\Omega) = \{ \vec{w} \in L^2(\Omega)^3, \forall K \in \tau_h(\Omega) \exists \vec{a} \in \mathbb{R}^3, b \in \mathbb{R} \text{ s.t. } \vec{w}(\vec{x}) = \vec{a} + b \vec{n}, \forall \vec{x} \in K \text{ and } [\vec{w} \cdot \vec{n}]_F = 0, \forall F \in \mathcal{F}_h(\Omega), F \not\subset \partial\Omega \}. $$

2. On the boundary mesh $\tau_h(\partial \Omega)$ we define

(a) The finite element space $\mathcal{P}_0h(\partial \Omega)$ of piecewise constant functions on the faces $F \in \tau_h(\partial \Omega)$,

$$\mathcal{P}_0h(\partial \Omega) = \{ w \in L^2(\partial \Omega), \forall F \in \tau_h(\partial \Omega) \exists a \in \mathbb{R} \text{ s.t. } w(\vec{x}) = a, \forall \vec{x} \in F \}. $$

(b) The finite element space $\mathcal{P}_1h(\partial \Omega)$ of continuous functions which are piecewise polynomial of degree 1 on each face $F \in \tau_h(\partial \Omega)$,

$$\mathcal{P}_1h(\partial \Omega) = \{ w \in C^0(\partial \Omega), \forall F \in \tau_h(\partial \Omega) \exists a \in \mathbb{R}, \vec{b} \in \mathbb{R}^3 \text{ s.t. } w(\vec{x}) = a + \vec{b} \cdot \vec{x}, \forall \vec{x} \in F \}. $$

4.1 Discretization of scalar potential formulations

4.1.1 Boundary integral method for the scalar potential model

Let us approximate the spaces $H^1(\Lambda)$, resp. $H^{-\frac{1}{2}}(\partial \Lambda)$ by the space $\mathcal{P}_1h(\Lambda)$ of piecewise linear functions, resp. by the space $\mathcal{P}_0h(\Lambda)$ of piecewise constant functions on the boundary mesh. We can write the discrete formulation corresponding to the problem (46)-(48):

find $(\psi_h, \lambda_h, w_h) \in \mathcal{P}_1h(\Lambda) \times \mathcal{P}_0h(\partial \Lambda) \times \mathcal{P}_1h(\Lambda)$, $w_h = \psi_h$ on $\partial \Lambda$ such that

$$\int_{\Lambda} \bar{\mu} \nabla \psi_h \cdot \nabla \varphi_h \, dx - \int_{\partial \Lambda} \lambda_h \varphi_h \, ds = \int_{\Lambda} (\bar{\mu} - 1) \vec{H}_0 \cdot \nabla \varphi_h \, dx, \quad (64)$$

$$\int_{\partial \Lambda} \psi_h \eta_h \, ds + \int_{\partial \Lambda} \eta_h(\vec{x}) \, ds(\vec{x}) \int_{\partial \Lambda} \left( \lambda_h(\vec{y}) - \frac{\partial w_h}{\partial \vec{y}}(\vec{y}) \right) G(\vec{x}, \vec{y}) \, ds(\vec{y}) = 0, \quad (65)$$

$$\int_{\Lambda} \vec{v}_w \cdot \nabla \psi_h \, dx = 0, \quad (66)$$

for all $(\varphi_h, \eta_h, v_h) \in \mathcal{P}_1h(\Lambda) \times \mathcal{P}_0h(\partial \Lambda) \times \mathcal{P}_1h(\Lambda)$. The function $\bar{\mu} \in \mathcal{P}_0h(\Lambda)$ is the approximation of $\mu_r$ in $\Lambda$ defined by

$$\bar{\mu} = \mu_r \left( \| Q_h \vec{H}_0 - \nabla \psi_h \| \right), \quad (67)$$

where $Q_h : L^2(\Lambda)^3 \to \mathcal{P}_0h(\Lambda)^3$ is the $L^2$-orthogonal projection.

Note that all integrals are done exactly except for the one involving the Green’s kernel $G$ which must be numerically approximated.

The nonlinear problem (64)-(66) is solved using a standard fixed point method, cf. Algorithm 1. The convergence of this fixed point method is proven in [12].

Algorithm 1 (Boundary integral method for scalar potential, fixed point).

Let us set $\psi_0^h \in \mathcal{P}_1h(\Lambda)$, $\psi_0^h = 0$.

For $k = 1, \ldots, J$
do

1. Evaluate $\bar{\mu}_k = \mu_r \left( \| Q_h \vec{H}_0 - \nabla \psi_{k-1}^h \| \right)$. 

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2.) Find \((\psi_k^h, \lambda_k^h, w_k^h) \in \mathcal{P}_{1h}(\Lambda) \times \mathcal{P}_{0h}(\partial\Lambda) \times \mathcal{P}_{1h}(\Lambda), \ w_k^h = \psi_k^h\) on \(\partial\Lambda\) such that

\[
\int_{\Lambda} \bar{\mu}^h \vec{\nabla} \psi_k^h \cdot \vec{\nabla} \varphi_h \, dx - \int_{\partial\Lambda} \lambda_k^h \varphi_h \, ds = \int_{\Lambda} (\bar{\mu}^k - 1) \bar{H}_0 \cdot \vec{\nabla} \varphi_h \, dx,
\]

\[
\int_{\partial\Lambda} \psi_k^h \eta_h \, ds - \int_{\partial\Lambda} \eta_h(\bar{x}) \, ds(\bar{x}) \int_{\partial\Lambda} \left( \lambda_k^h(\bar{y}) - \frac{\partial w_k^h}{\partial n}(\bar{y}) \right) G(\bar{x}, \bar{y}) \, ds(\bar{y}) = 0,
\]

\[
\int_{\Lambda} \vec{\nabla} w_k^h \cdot \vec{\nabla} v_h \, dx = 0,
\]

for all \((\varphi_h, \eta_h, v_h) \in \mathcal{P}_{1h}(\Lambda) \times \mathcal{P}_{0h}(\partial\Lambda) \times \mathcal{P}_{1h0}(\Lambda)\).

until estimated convergence.

In the applications, the ferromagnetic structures are often thin and their discretizations contain
a lot of triangles on their surfaces. The main drawback arising from formulation (64)-(66) is that
the second term of equation (65) leads in this case to a full matrix with a big order. Consequently,
formulation (64)-(66) imposes an important restriction on the mesh, which is not operational in a
lot of applications on the presented form.

4.1.2 Scalar potential problem with Poisson formula boundary condition

Let us now discretize the formulation (50)-(51). We are approximating the functional space
\(H^1(B_R)\) by the space \(\mathcal{P}_{1h}(B_{Rh})\) by assuming that \(\partial\Lambda\) and \(\partial B_{Rh}\) are made of faces of \(\mathcal{F}_h(B_{Rh})\).

The discrete formulation for (50)-(51) reads: find \(\psi_h^k \in \mathcal{P}_{1h}(B_{Rh})\) such that for all \(\varphi_h \in \mathcal{P}_{1h0}(B_{Rh})\) we have

\[
\int_{B_{Rh}} \bar{\mu} \vec{\nabla} \psi_h \cdot \vec{\nabla} \varphi_h \, dx = \int_{\Lambda} (\bar{\mu} - 1) \bar{H}_0 \cdot \vec{\nabla} \varphi_h \, dx,
\]

\[
\psi_h(\bar{x}_i) = \frac{R^2 - r^2}{4\pi r} \int_{\partial B_{Rh}} \frac{\psi_h(\bar{y})}{\|\bar{y} - \bar{x}_i\|^3} \, ds(\bar{y}),
\]

pointwise for all vertices \(\bar{x}_i\) of the mesh \(\tau_h(\partial B_{Rh})\). Here, \(\bar{\mu} \in \mathcal{P}_{0h}(B_{Rh})\) is the piecewise-constant
approximation of \(\bar{\mu}\) defined by

\[
\bar{\mu} = \begin{cases} 
\mu_r \left(\|Q_h \bar{H}_0 - \vec{\nabla} \psi^h\|\right) & \text{in } \Lambda, \\
1 & \text{otherwise},
\end{cases}
\]

where \(Q_h : L^2(\Lambda)^3 \to \mathcal{P}_{0h}(\Lambda)^3\) is the \(L^2\)-orthogonal projection.

Note that all integrals are done exactly except for the one on \(\partial B_{Rh}\). In that case, first recall
that the sphere is approximated by a triangular mesh, second on each of these triangles we use a
simple Gauss integration scheme.

The problem (68)-(70) is nonlinear. Moreover, the coupling of (68) and (69) is non-local, thus
filling the matrix of the underlying linear system with full blocks. This is why we propose in
Algorithm 2 a fixed point iteration, combined with a multiplicative Dirichlet-Dirichlet domain
decomposition between the meshed interior and the exterior, represented by the Poisson
representation formula (69) (see [12] for the convergence).

Algorithm 2 (Domain decomposition for scalar potential, fixed point).

Let us set \(\psi_h^0 \in \mathcal{P}_{1h}(B_{Rh}), \psi_h^0 = 0\).

For \(k = 1, \ldots, J\) do
1. Define \( \psi_h^{k-\frac{1}{2}} \in P_{1h}(\partial B_{Rh}) \) such that

\[
\psi_h^{k-\frac{1}{2}}(\vec{x}_i) = \frac{R^2 - r^2}{4\pi r} \int_{\partial B_{sh}} \frac{\psi_h^{k-1}(\vec{y})}{||\vec{y} - \vec{x}_i||^3} \, ds(\vec{y}),
\]

pointwise on each vertex \( \vec{x}_i \) of the mesh \( \tau_h(\partial B_{Rh}) \).

2. Evaluate \( \bar{\mu}^k \in P_{0h}(B_{Rh}) \) by

\[
\bar{\mu}^k = \begin{cases} \mu_r \left( ||Q_h \bar{H}_0 - \nabla \psi_h^{k-1}|| \right) & \text{in } \Lambda, \\ 1 & \text{otherwise.} \end{cases}
\]

3. Find \( \psi_h^k \in P_{1h}(B_{Rh}) \), \( \psi_h^k = \psi_h^{k-\frac{1}{2}} \) on \( \partial B_{Rh} \), such that for all \( \varphi_h \in P_{1h0}(B_{Rh}) \) there is

\[
\int_{B_{Rh}} \bar{\mu}^k \nabla \psi_h^k \cdot \nabla \varphi_h \, dx = \int_{\Lambda} \left( (\bar{\mu}^k - 1) \bar{H}_0 \cdot \nabla \varphi_h \right) dx
\]

until estimated convergence.

To solve (72) approximatively, we use several iterations of an algebraic multigrid AMG [11].

4.1.3 Mixed \( P_{1h} - P_{1h} \) method with Poisson boundary conditions

Consider the problem (56), (57), (59) for finding \((\bar{p}, \psi) \in H(\text{div}, B_R) \times L^2(B_R)\), with the regularizing operator \( R \) of (58).

If we approximate the involved functional spaces \( H(\text{div}, B_R) \), \( L^2(B_R) \) with spaces \( P_{1h}(B_{Rh})^3 \), \( P_{1h}(B_{Rh}) \) of piecewise-linear functions, we have to add a stabilisation term to the weak formulation.

We choose the GLS stabilisation (see [8]) with the stabilization parameter \( \alpha = O(1) \) and write the discretized problem: find \( (\bar{p}_h, \psi_h) \in P_{1h}(B_{Rh})^3 \times P_{1h}(B_{Rh}) \) such that for all \( (\bar{q}_h, \varphi_h) \in P_{1h}(B_{Rh})^3 \times P_{1h}(B_{Rh}) \) there is

\[
\int_{B_{Rh}} \bar{\nu} \bar{p}_h \cdot \bar{q}_h \, dx + \int_{B_{Rh}} (\psi_h \text{div } \bar{q}_h + \varphi_h \text{div } \bar{p}_h) \, dx + \alpha \int_{B_{Rh}} \left( \bar{\nu} \bar{p}_h - \nabla \psi_h \right) \left( \bar{\nu} \bar{q}_h - \nabla \varphi_h \right) \, dx \]

\[
= \int_{\partial B_{Rh}} \psi_h \bar{q}_h \cdot n \, ds - (\bar{\nu} \alpha + 1) \int_{\Lambda} (1 - \bar{\nu}) \bar{H}_0 \cdot \bar{q}_h \, dx + \alpha \int_{\Lambda} (1 - \bar{\nu}) \bar{H}_0 \cdot \nabla \varphi_h \, dx
\]

\[
\psi_h(\vec{x}_i) = \frac{R^2 - r^2}{4\pi r} \int_{\partial B_{sh}} \frac{\psi_h(\vec{y})}{||\vec{y} - \vec{x}_i||^3} \, ds(\vec{y}),
\]

pointwise for all vertices \( \vec{x}_i \) in the boundary mesh \( \tau_h(\partial B_{Rh}) \). Due to continuity of \( \psi_h \), we do not need the regularization operator \( R \) here, it is replaced by identity.

The function \( \bar{\nu} \) is the piecewise-constant approximation of \( \bar{\nu} \), defined here through the relative reluctivity \( \bar{\nu} \) (see 29). Recall that \( \bar{\nu} \) is a function of \( ||\bar{B}|| \), while \( \bar{\mu} \) is a function of \( ||\bar{H}|| \). We set

\[
\bar{\nu} = \begin{cases} \nu_r \left( ||Q_h(\bar{B}_0 - \mu_0 \bar{p}_h)|| \right) & \text{in } \Lambda, \\ 1 & \text{otherwise.} \end{cases}
\]

Here, as before, \( Q_h : L^2(B_{Rh})^3 \to P_{0h}(B_{Rh})^3 \) is the \( L^2 \)-orthogonal projection.

As before, all integrals are done exactly except for the one on \( \partial B_{Rh} \).

Again, the problem (73)-(74) is non-linear and the coupling of (73) and (74) is non-local. That is why, we employ an iterative fixed-point scheme to solve it, cf. Algorithm 3.
Algorithm 3 (Domain decomposition for mixed $\mathcal{P}_1\mathcal{P}_1$ formulation, fixed point).
Let us set $\psi_h^0 \in \mathcal{P}_1(B_{Rh})$, $\psi_h^0 = 0$, and calculate $\tilde{p}_h^0 \in \mathcal{P}_1(B_{Rh})^3$, $\tilde{p}_h^0 = \left(1 - \mu_c(\|Q_h \bar{H}_0\|)\right) Q_h \bar{H}_0$.
For $k = 1, \ldots, J$ do

1.) Define $\psi_h^{k-\frac{1}{2}} \in \mathcal{P}_1(\partial B_{Rh})$ such that
\[
\psi_h^{k-\frac{1}{2}}(\bar{x}_i) = \frac{R^2 - r^2}{4\pi r} \int_{\partial B_{Rh}} \psi_h^{k-1}(\bar{y}) \frac{ds(\bar{y})}{\|\bar{y} - \bar{x}_i\|^3},
\]
pointwise on each vertex $\bar{x}_i$ of the mesh $\tau_h(\partial B_{Rh})$.

2.) Evaluate $\tilde{\mu}^k \in \mathcal{P}_0(\partial B_{Rh})$ by
\[
\tilde{\mu}^k = \begin{cases} 
\nu_r \left(\|Q_h (\bar{B}_0 - \mu_0 \bar{p}_h^{k-1})\|\right), & \text{in } \Lambda, \\
1 & \text{otherwise}.
\end{cases}
\]

3.) Find $(\tilde{p}_h^k, \psi_h^k) \in \mathcal{P}_1(B_{Rh})^3 \times \mathcal{P}_1(B_{Rh})$ such that
\[
\int_{B_{Rh}} \tilde{\mu}^k \tilde{p}_h \cdot \tilde{q}_h \, dx + \int_{B_{Rh}} (\psi_h^k \text{ div } \tilde{q}_h + \varphi_h \text{ div } \tilde{p}_h) \, dx + \alpha \int_{B_{Rh}} \left(\tilde{\nu} \tilde{p}_h^k - \nabla \psi_h^k\right) \left(\tilde{\nu} \tilde{q}_h - \nabla \varphi_h\right) \, dx \\
= \int_{\partial B_{Rh}} \psi_h^{k-\frac{1}{2}} \tilde{q}_h \cdot \tilde{\nu} \, ds - (\tilde{\nu} \alpha + 1) \left(1 - \tilde{\nu}\right) \bar{H}_0 \cdot \tilde{q}_h \, dx + \alpha \int_{\Lambda} \left(1 - \tilde{\nu}\right) \bar{H}_0 \cdot \nabla \varphi_h \, dx,
\]
(76)
for all $(\tilde{q}_h, \varphi_h) \in \mathcal{P}_1(B_{Rh})^3 \times \mathcal{P}_1(B_{Rh})$
until estimated convergence.

4.1.4 Mixed $\mathcal{RT}_0$ method with Poisson boundary conditions
Let us consider again the problem (56),(57),(59). We are approximating now the functional spaces $L^2(B_R)$, resp. $H(\text{div}, B_R)$ by the space $\mathcal{P}_0(B_{Rh})$ of piecewise-constant functions, resp. by the space $\mathcal{RT}_0(B_{Rh})$, the lowest-order Raviart-Thomas space. Note that the dimension of $\mathcal{RT}_0$ is the number of all faces $F \in \mathcal{F}_h(B_{Rh})$.

The discrete formulation of the problem (56),(57),(59) reads in this case: find $(\tilde{p}_h, \psi_h) \in \mathcal{RT}_0(B_{Rh}) \times \mathcal{P}_0(B_{Rh})$ such that for all $(\tilde{q}_h, \varphi_h) \in \mathcal{RT}_0(B_{Rh}) \times \mathcal{P}_0(B_{Rh})$ there is
\[
\int_{B_{Rh}} \tilde{\nu} \tilde{p}_h \cdot \tilde{q}_h \, dx + \int_{B_{Rh}} \psi_h \text{ div } \tilde{q}_h \, dx = \int_{\partial B_{Rh}} \psi_h \tilde{q}_h \cdot \tilde{\nu} \, ds - (\tilde{\nu} \alpha + 1) \left(1 - \tilde{\nu}\right) \bar{H}_0 \cdot \tilde{q}_h \, dx,
\]
(77)
\[
\int_{B_{Rh}} \varphi_h \text{ div } \tilde{p}_h \, dx = 0,
\]
(78)
\[
\psi_h(\bar{x}_i) = \frac{R^2 - r^2}{4\pi r} \int_{\partial B_{Rh}} \frac{\bar{R}_h \psi_h(\bar{y})}{\|\bar{y} - \bar{x}_i\|^3} \frac{ds(\bar{y})}{\|\bar{y} - \bar{x}_i\|^3},
\]
(79)
pointwise for all barycenters $\bar{x}_i$ of boundary faces $F \in \tau_h(\partial B_{Rh})$. The regularizing operator $\bar{R}_h$ needs be only given for functions with trace on $\partial B_{Rh}$. Let us define $\bar{R}_h : \mathcal{P}_0(B_{Rh}) \to \mathcal{P}_0(\partial B_{Rh})$ in the following way: for each face $F \in \partial B_{Rh}$ and for each function $\eta_h \in \mathcal{P}_0(B_{Rh})$, the value of $\bar{R}_h \eta_h$ on this face is the arithmetic mean of $\eta_h$ on the two tetrahedra in $\tau_h(B_{Rh})$ sharing the face $F$.  

Again, the function \( \tilde{\nu} \in P_0(B_{Rh}) \) is the piecewise-constant approximation of \( \tilde{\nu} \) which satisfies
\[
\tilde{\nu} = \begin{cases} 
\nu_r \left( \|Q_h(\tilde{B}_0 - \mu_0\tilde{p}_h)\| \right) & \text{in } \Lambda, \\
1 & \text{otherwise.}
\end{cases}
\]
where \( Q_h : L^2(B_{Rh}) \rightarrow P_0(B_{Rh}) \) is the \( L^2 \) orthogonal projection.

As before, all integrals are done exactly except for the one on \( \partial B_{rh} \).

As before, the problem (77)-(80) is non-linear, and the coupling of (77),(78) with (79) is non-local. Therefore, we solve the non-linear problem by using a fixed point iteration. For one fixed \( \tilde{\nu} \), we solve the interior-exterior subproblem (77)-(79) by a multiplicative Dirichlet-Dirichlet domain decomposition.

Algorithm 4 (Domain decomposition for mixed \( RT_0h \) formulation).
Let us set \( \psi_h^0 \in P_0(B_{Rh}) \), \( \psi_h^0 = 0 \), and let \( \tilde{p}_h^0 \in RT_0h(B_{Rh}) \) be an approximation of \( \left( 1 - \tilde{\mu}(\cdot,\|\tilde{H}_0\|) \right) \tilde{H}_0 \).
For \( k = 1, \ldots, J \) do

1.) Evaluate \( \tilde{\psi}^k \in P_0(B_{Rh}) \) by
\[
\tilde{\psi}^k = \begin{cases} 
\nu_r \left( \|Q_h(\tilde{B}_0 - \mu_0\tilde{p}_h^{k-1})\| \right) & \text{in } \Lambda, \\
1 & \text{otherwise.}
\end{cases}
\]

2.) Define \( \psi_h^{k-\frac{1}{2}} \in P_0(\partial B_{Rh}) \) such that
\[
\psi_h^{k-\frac{1}{2}}(\bar{x}_i) = \frac{R^2 - r^2}{4\pi r} \int_{\partial B_{rh}} \frac{\tilde{R}_h \psi_h^{k-1}(\bar{y})}{||\bar{y} - \bar{x}_i||^2} \, ds(\bar{y}),
\]
for all barycenters \( \bar{x}_i \) for faces \( F \in F_h(\partial B_{Rh}) \).

3.) Find \( (\tilde{p}_h^k, \psi_h^k) \in RT_0h(B_{Rh}) \times P_0h(B_{Rh}) \) such that
\[
\int_{B_{Rh}} \tilde{p}_h^k \cdot \tilde{q}_h \, dx + \int_{B_{Rh}} \psi_h^k \text{div} \tilde{q}_h \, dx = \int_{\partial B_{Rh}} \psi_h^{k-\frac{1}{2}} \tilde{q}_h \cdot \tilde{n} \, ds - \int_{\Lambda} (1 - \tilde{\nu}^k) \tilde{H}_0 \cdot \tilde{q}_h \, dx,
\]
\[
\int_{B_{Rh}} \varphi_h \text{div} \tilde{p}_h^k \, dx = 0,
\]
for all \( (\tilde{q}_h, \varphi_h) \in RT_0h(B_{Rh}) \times P_0h(B_{Rh}) \).

until estimated convergence.

4.2 Discretization of vector potential formulation
Let us discretize the problem (62)-(63). We approximate the space \( H(\text{curl},B_{Rh}) \cap H(\text{div},B_{Rh}) \) by the space \( P_{1h}(B_{Rh})^3 \). The discrete formulation then reads: find \( \tilde{A}_h \in P_{1h}(B_{Rh})^3 \) such that for all \( \tilde{w}_h \in P_{1h0}(B_{Rh})^3 \) there is
\[
\int_{B_{Rh}} \left[ \tilde{\nu} \text{curl} \tilde{A}_h \cdot \text{curl} \tilde{w}_h + \tilde{\nu} \text{div} \tilde{A}_h \text{div} \tilde{w}_h \right] \, dx = \int_{\Lambda} (1 - \tilde{\nu}) \tilde{B}_0 \cdot \text{curl} \tilde{w}_h \, dx,
\]
\[
\tilde{A}_h(\bar{x}_i) = \frac{R^2 - r^2}{4\pi r} \int_{\partial B_{Rh}} \frac{\tilde{A}_h(\bar{y})}{||\bar{y} - \bar{x}_i||^2} \, ds(\bar{y}),
\]
for all nodes $\vec{x}_i$ of the boundary mesh $\tau_h(\partial B_{Rh})$. The function $\bar{\nu} \in \mathcal{P}_0(\mathcal{B}_{Rh})$ is the piecewise-constant approximation of $\bar{\nu}$ defined here through the relative reluctivity $\bar{\nu} = \frac{1}{\bar{\mu}}$

$$\bar{\nu} = \begin{cases} \nu_r \left( ||Q_h \vec{B}_0 + \text{curl} \vec{A}_h|| \right), & \text{in } \Lambda, \\ 1, & \text{otherwise}, \end{cases}$$

where $Q_h : L^2(\Lambda)^3 \to \mathcal{P}_0(\Lambda)^3$ is the $L^2$-orthogonal projection.

As before, all integrals are done exactly except for the one on $\partial B_{Rh}$.

The discrete problem (81)-(82) is non-linear and the coupling of (81) with (82) is non-local. This is why we solve this problem by a fixed-point iteration combined with a multiplicative Dirichlet-Dirichlet interior-exterior domain decomposition, cf. Algorithm 5.

Algorithm 5 (Domain decomposition for gauged vector-potential, fixed point).

Let us set $\vec{A}_h^0 \in \mathcal{P}_1(\mathcal{B}_{Rh})^3$, $\vec{A}_h^0 = 0$.

For $k = 1, \ldots, J$

1.) Define $\vec{A}_{k-\frac{1}{2}}^h \in \mathcal{P}_1(\partial B_{Rh})^3$ such that

$$\vec{A}_{k-\frac{1}{2}}^h(\vec{x}_i) = \frac{R^2 - r^2}{4\pi r} \int_{\partial B_h} \vec{A}_{k-1}^h(\vec{y}) \frac{1}{||\vec{y} - \vec{x}_i||^3} \, ds(\vec{y}),$$

pointwise for each vertex $\vec{x}_i$ of the boundary mesh $\tau_h(\partial B_{Rh})$.

2.) Evaluate $\bar{\nu}^k \in \mathcal{P}_0(\mathcal{B}_{Rh})$ by

$$\bar{\nu}^k = \begin{cases} \nu_r \left( ||Q_h \vec{B}_0 + \text{curl} \vec{A}_{k-1}^h|| \right), & \text{in } \Lambda, \\ 1, & \text{otherwise}. \end{cases}$$

3.) Find $\vec{A}_h^k \in \mathcal{P}_1(\mathcal{B}_{Rh})^3$, $\vec{A}_h^k = \vec{A}_{k-\frac{1}{2}}^h$ on $\partial B_{Rh}$, such that

$$\int_{B_{Rh}} \left[ \bar{\nu}^k \text{curl} \vec{A}_h^k \cdot \text{curl} \vec{w}_h + \bar{\nu}^k \text{div} \vec{A}_h^k \cdot \text{div} \vec{w}_h \right] \, dx = \int_{B_{Rh}} (1 - \bar{\nu}^k) \vec{B}_0 \cdot \text{curl} \vec{w}_h \, dx,$$

for all $\vec{w}_h \in \mathcal{P}_{1,h}(\mathcal{B}_{Rh})^3$. until estimated convergence.

5 Numerical experiments

Let us proceed with numerical experiments, in order to compare the results and the computational complexity of the above-mentioned algorithms. As it was said at the beginning, special focus is brought to ferromagnetic plates and their screening effects.

5.1 Definition of the test-case.

We consider a ferromagnetic rectangular plate, 5 meters wide, 4 meters high and 2 centimeters thick which is placed in front of an idealized infinitely long wire with zero section (see Figure 1, where the plate is represented in dark red). A given constant electric current runs in the wire. Omitting the plate, this current would produce an induction field $\vec{B}_0$ (suggested on Figure 1) given by Biot-Savart law. In the presence of the plate, the induction field $\vec{B}$ will be modified, due to ferromagnetic response of the steel plate.
We want to simulate:

1°) The screen effect of the ferromagnetic plate, i.e., the attenuation of the induction field "behind" the plate; we will compare $\overrightarrow{B}$ and $\overrightarrow{B}_0$.

2°) The magnetization in the plate, i.e., the physical phenomenon which is responsible for the screen effect; we will compare the relative permeability $\mu_r(\|\overrightarrow{H}\|)$ in the plate.

Our aim is here to compare the results obtained using each of the above described formulations for $\overrightarrow{B}$ inside and outside the plate as well as $\mu_r(\|\overrightarrow{H}\|)$ in the plate. In particular, we will focus on the approximation of $\overrightarrow{B}$ inside the plate which for some methods present strong "oscillations". Moreover, we want to check the $h$-convergence for each formulation; to this aim, we will use four different meshes from coarse to fine.

Let us now introduce the following system of coordinates: the origin $O$ is placed at the center of the plate; the $Ox$ axis is in the direction of the thickness of the plate; the $Oy$ axis is in the direction of the length of the plate; the $Oz$ axis is in the direction of the height of the plate.

With this coordinate system, the electric current of 143'000A, which is parallel to $Oy$, goes from $y = -20$ meters to $y = +20$ meters (which simulates correctly an infinite wire) and passes through the point $(1.01, 0, 0)$ (see Figure 1 where the current in the wire is suggested by the generated field $\overrightarrow{B}_0$ in a plane perpendicular to the wire).

The nonlinear material behaviour is characterized by the $B - H$ diagram, or by the relative magnetic permeability or relucitivity functions $\mu_r(\|\overrightarrow{H}\|)$, resp. $\nu_r(\|\overrightarrow{B}\|)$ given in Fig. 2.

Fig. 3 gives a sketch of some lines we have chosen to compare the results obtained by each of the methods. The lines 1, 2 and 3 were chosen to observe the screening effect when we go away perpendicularly from the plate, while the line 4 is meant to check the behaviour of the field $\overrightarrow{B}$ very near (10 centimeters) the plate. Because it goes through a corner of the plate, line 2 will also be used to check the singular behaviour of the field $\overrightarrow{B}$ when approaching this corner. Lines 5 and 6, which are located in the middle of the plate thickness, will be used to measure the magnetization state in the plate.

Our aim is now to apply each of the above presented algorithms. They will be referred to now as given in Table 1.
Figure 2: Material properties of the ferro-material: the $H-B$ diagram without hysteresis (left), relative permeability $\mu_r(\|H\|)$ as a function of $\|H\|$ (middle) and relative reluctivity $\nu_r(\|B\|)$ as a function of $\|B\|$ (right).

Figure 3: Geometry of the test-case with observation lines 1-3, 4 and 5-6, a 3D view).

<table>
<thead>
<tr>
<th>label</th>
<th>short description</th>
<th>defined in</th>
</tr>
</thead>
<tbody>
<tr>
<td>bem-fem</td>
<td>scalar potential formulation with boundary-elements coupled to finite elements</td>
<td>Algorithm 1</td>
</tr>
<tr>
<td>scal</td>
<td>non-mixed scalar potential formulation on $P^1_h$ nodal finite element spaces</td>
<td>Algorithm 2</td>
</tr>
<tr>
<td>mix</td>
<td>mixed scalar potential formulation on $P^1_h$-$P^1_h$ nodal finite element spaces with GLS stabilisation</td>
<td>Algorithm 3</td>
</tr>
<tr>
<td>rt0</td>
<td>scalar potential mixed formulation with $RT0_h$ finite elements</td>
<td>Algorithm 4</td>
</tr>
<tr>
<td>vec</td>
<td>vector potential formulation with Coulomb’s gauge on $P^1_h$ nodal finite element spaces</td>
<td>Algorithm 5</td>
</tr>
</tbody>
</table>

Table 1: Labels for different algorithms to be used for the presentations of results
5.2 Mesh for the plate and balls.

Let us now define the balls and meshes used in our algorithms. The small ball $B_r$ is the ball of radius $r = 3.5$ meters centered at the origin while $B_R$ has radius $r = 4.4$ meters (see Fig. 3).

**Finite element mesh of $B_R$:** To compare also $h$-convergence of the finite element discretizations of the test-case, we are using four levels of refinement of the same coarse mesh. The coarsest quasi-uniform isotropic mesh, called meshH0.25, with a representative mesh-size $h = 1/4$ has been three times refined to produce meshes meshH0.12, meshH0.06 and meshH0.03. This has been done by successive halving the edges, dividing each triangular face between two tetrahedra into four smaller similar triangles and dividing each coarse tetrahedron to eight finer tetrahedra. By this division, the quality of tetrahedra might be deteriorated only in the first refinement, all subsequent refinements do not deteriorate the mesh quality, cf. [1]. Hence the resulting finer finite element nodal spaces are nested to the coarse ones, except near the small sphere boundary $\partial B_r$, where finer meshes have been adapted to capture better the curved surface. Table 2 gives the number of nodes, edges, faces and tetrahedrons for each mesh while Fig. 4 gives a simple representation of these meshes.

<table>
<thead>
<tr>
<th>mesh name</th>
<th>no. nodes</th>
<th>no. edges</th>
<th>no. faces</th>
<th>no. tetras</th>
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<td>194.657</td>
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<td>7.267.548</td>
<td>12.449.648</td>
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<td>meshH0.03</td>
<td>8.308.873</td>
<td>58.107.464</td>
<td>99.585.984</td>
<td>49.787.392</td>
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</tbody>
</table>

Table 2: Statistics of the used finite element mesh with four levels of refinement for $h = \{1/4, 1/8, 1/16, 1/32\}$.

![Figure 4: Mesh of the plate and of the balls $BB_R$ and $BB_r$.](image)

**Algorithmic complexity of all algorithms:** One of the important criteria for measuring the efficiency of the method might be the size of the underlying linear system of equations which is to be solved in each iteration of the non-linear (domain-decomposition) algorithm.

Table 3 gives the order of the systems of equations as well as the number of non-zero terms to store for each algorithm and for each mesh.
Table 3: Size of the assembled linear system for each algorithm and mesh; column dofs gives the number of unknowns and nonzeros gives the number of non zeros terms in the resulting matrix.

<table>
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<th>algo</th>
<th>dofs</th>
<th>nonzeros</th>
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<td>16.967</td>
<td>131.349</td>
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<td></td>
<td>vec</td>
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<td>1.182.141</td>
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<td></td>
<td>mix</td>
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<td>9.511.092</td>
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<td>2.139.477</td>
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<td>909.976</td>
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<td>vec</td>
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<td>8.189.784</td>
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<td></td>
<td>mix</td>
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<td>151.061.943</td>
</tr>
<tr>
<td></td>
<td>rt0</td>
<td>2.334.464</td>
<td>17.115.116</td>
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<td>meshH0.06</td>
<td>scal</td>
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<td></td>
<td>rt0</td>
<td>18.673.072</td>
<td>136.918.128</td>
</tr>
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</table>

5.3 Magnetic induction $\vec{B}$ outside the plate

Let us compare all presented algorithms on the described test case on the coarsest and the finest mesh. As a criterion of quality, we observe the behaviour of the magnetic induction $\vec{B}$ on each of the lines 1-4, defined above.

All plots given in Fig. 5-9 are presented in the same way:

- Column 1 gives plots of components $B_x$, $B_y$ and $B_z$ of induction on the coarsest mesh ($h = \frac{1}{4}$);
- Column 2 gives plots of components $B_x$, $B_y$ and $B_z$ of induction on the finest mesh ($h = \frac{1}{16}$); note that some algorithms do not appear for the plots in this column, since the mesh is too fine;
- The dotted line refers to induction $\vec{B}_0$ computed with no ferromagnetic plate using Biot-Savart’s formula.
- On the horizontal axis we give the distance to the plate, the plate being at the right of the plot.

We can now discuss the results obtained on observation lines 1-4.

Screen effect along line 1-3 (see Fig. 5-7): All algorithms produce roughly the same results on lines 1-3. Even for coarse mesh, the approximation is already acceptable. The biggest differences can be observed, as expected with finite element approximation, on small components of $\vec{B}$. Algorithm mix seems to have more difficulties near the plate. We assume it is due to the rather non natural approximation for the space $H(\text{div}, \Lambda)$.

Behaviour near the plate - line 4 - (see Fig. 8): Even quite near the plate, approximation seems correct for all methods. Once more, algorithm mix presents some difficulties.

Behaviour near singularity - line 2 - (see Fig. 9): It is well known that in the vicinity of edges and corners of the plate, solution $\vec{B}$ will present a singularity. In order to measure the singular behaviour on numerically, we superimpose on log-log plots reference lines with prescribed slopes. Let us note that here the mesh of the plate, and as a consequence also the mesh of $\vec{B}$, near the plate has been intentionally refined. Please note that here, for logistic reasons, the plate is placed to the right of the plot.
Figure 5: Magnetic induction $\vec{B} = (B_x, B_y, B_z)$ on the observation line 1, on the screened side of the plate. Left column: computations on coarse mesh, right column: computations on fine mesh.
Figure 6: Magnetic induction \( \vec{B} = (B_x, B_y, B_z) \) on the observation line 2, on the screened side of the plate. Left column: computations on coarse mesh, right column: computations on fine mesh.
Figure 7: Magnetic induction $\vec{B} = (B_x, B_y, B_z)$ on the observation line 2, on the screened side of the plate. Left column: computations on coarse mesh, right column: computations on fine mesh.
Figure 8: Magnetic induction $\vec{B} = (B_x, B_y, B_z)$ on the observation line 4, just near the plate. Left column: computations on coarse mesh, right column: computations on fine mesh.
Figure 9: Observation line 2 with log-log scaled axes. One can observe the asymptotic behaviour of the singularity in the field $\vec{B}$ when approaching the corner of the plate from the screened side of the plate. Left column: computations on coarse mesh, right column: computations on fine mesh.
5.4 Magnetic induction inside the ferro-plate $\Lambda$

Oscillation of induction vector inside the plate. One of the reasons for comparing different solution methods for ferromagnetism was the reported oscillations problems when using non-mixed scalar potential formulation of the problem. Indeed, as we show here, numerical spurious oscillations of $\vec{M}$, $\vec{B}$ or $\vec{H}$ are present inside the ferromagnetic plate $\Lambda$. Even if such oscillations do not appear outside the plate, it is worth studying this problem because, $\vec{H}$ inside the plate is used for computing the relative permeability.

![Figure 10: meshH0.12: oscillation of the field $\vec{B}$ inside the plate $\Lambda$ for methods scal and vec.](image)

It can be shown, using an asymptotic analysis of the solution for infinitely thin plates and high relative permeabilities from [3], that the magnetic induction vector $\vec{B}$ should be a vector parallel to the plane of the plate. Even though our test case plate is not infinitely thin, let us verify to which extent the ratio $|B_x|/\|\vec{B}\|$ is small in the plate (i.e., the field $\vec{B}$ is almost tangential to the plane of the plate).

Fig. 10 shows the results obtained using scal and vec algorithms. The first line of plots gives results obtained using the scal algorithm while second line plots shows results obtained using the vec algorithm. Every line of plots is arranged (from left to right) as follows:

- Plots 2, 3, 4: show the elements of the plate for which $|B_x|/\|\vec{B}\|$ is bigger than 8%, 4%, resp. 2%.
- Plot 1: gives an histogram of the percent of tetrahedra in $\Lambda$ (vertical axis) having a given ratio $|B_x|/\|\vec{B}\|$ (horizontal axis).

We clearly see, that the method vec presents much less oscillations than the scal method. Method rt0 gives results quite similar to the results of vec method while all others give results comparable to the scal method. For simplicity we give only results for rt0. As a conclusion here we can say that only vec and rt0 methods give satisfactory results inside the plate.
Relative permeability plots. In Fig. 11 and 12 we see the plots of $\mu_r$ on the observation lines 5 and 6 inside the plate. The first row of plots corresponds to the observation line 5 (horizontal line), the second one corresponds to the observation line 6 (vertical line). Results for all methods are superposed in one plot, for the coarsest mesh meshH0.25 (left), up to the finest mesh meshH0.03 (right).

Figure 11: Relative permeability $\mu_r$ on observation line 5 (horizontal line), for the coarsest mesh (left), up to the finest one (right).

While almost all methods give the same result on the observation line 5, unphysical peaks of $\mu_r$ appear for the method mix on the edges of the plate. This is probably caused by the nonphysical regularity imposed by the $P_{1h}$ continuous finite element space for $\vec{H}$. For the observation line 6 (plots below), we see that all methods tend to converge roughly to the same profile of $\mu_r$, but for the methods scal and bem-fem we see spurious oscillations appear, especially for finer meshes (bottom right). This behaviour is caused by spurious oscillations of $\vec{H}$ inside the plate.

5.5 Artificial boundary conditions vs. the Poisson formula

In the domain decomposition method of Algorithm 2–4, resp. Algorithm 5, we take the initial guess $\psi^0 = 0$, resp. $\vec{A}^0 = 0$, in particular, the trace of $\psi^0$, resp. of $\vec{A}^0$ on $\partial B_R$ vanish. The domain decomposition algorithm then iteratively updates the boundary values of $\psi$ resp. $\vec{A}$ on $\partial B_R$ by using the Poisson formula.

Considering that the potentials $\psi$ or $\vec{A}$ are vanishing at infinity with the rate $O(1/||r||)$, one may expect that already the initial guess $\psi = 0$, resp. $\vec{A} = 0$ on $\partial B_R$ is quite reasonable, so that we
can avoid computing the Poisson formula. It is an interesting question to see to which extent the update of the boundary condition through the domain decomposition algorithm is really necessary.

To this end, we consider formulation (50), (51) for the scalar potential $\psi$. If we replace equation (51) by $\psi(\vec{x}) = 0$, on $\partial B_R$ we get a new formulation we could call "scalar potential problem with homogeneous Dirichlet boundary conditions". We insist on the fact that this formulation does not correspond anymore to our original ferromagnetic problem, it is only an approximation of that problem. We then call Algorithm 2' the algorithm obtained from Algorithm 2 by just replacing in step 1.) equation (71) by $\psi_h^{k-\frac{1}{2}} = 0$, one $\partial B_{Rh}$. Algorithm 2' will converge to an approximated solution for our homogeneous Dirichlet boundary conditions formulation. In Algorithm 2', we do not use the Poisson formula anymore (and then we do not really need the small ball $B_{Rh}$). It is also clear that iterations are now only used to solve the non-linear problem.

Clearly, the answer to the question at the beginning of this section will depend mainly on the ratio $R/r$. So we present here two simulations of our test-case using Algorithm 2 and Algorithm 2': one with $R/r = 1.25$ using a new mesh of $B_R$, one with $R/r = 1.5$ using a new mesh of $B_R$. In order to also compare convergence of the two algorithms, we define the non-linear residue as follows: if $A_k \psi_k = b_k$ is the system of equations resulting from (72) after standard finite element calculations, we define the non-linear residue at iteration $k$ to be the vector $\mathbf{r}_k = b_k - A_k \psi_k - 1$. So, when Algorithm 2 (or 2') converges, the non-linear residue $\mathbf{r}_k$ should tend to zero when $k$ increases.

**Homogeneous Dirichlet vs. Poisson formula:** We call $\vec{B}_1$ the induction field computed using Algorithm 2' and $\vec{B}_2$ the induction field computed using Algorithm 2. We evaluate the relative difference $\|\vec{B}_2 - \vec{B}_1\|/\|\vec{B}_2\|$ at every point of $B_R$. Here, $\| \cdot \|$ denotes the euclidean norm in $\mathbb{R}^3$.

![Figure 13: meshH0.12 comparison of boundary condition by Poisson formula ($\vec{B}_2$) vs. the homogeneous Dirichlet condition ($\vec{B}_1$) for $R/r = 1.5$ (top line) and $R/r = 1.25$ (bottom line)](image)

Fig. 13 compares $\vec{B}_1$ and $\vec{B}_2$ for both $R/r = 1.25$ and $R/r = 1.5$. The right, resp. middle plot
shows all tetrahedra on which the relative difference $\|\vec{B}_2 - \vec{B}_1\|/\|\vec{B}_2\|$ exceeds 7% resp. 5%. The left plot shows a histogram of how many tetrahedra (vertical axis) there are with a given relative difference $\|\vec{B}_2 - \vec{B}_1\|/\|\vec{B}_2\|$ (horizontal axis). More precisely, we cluster all tetrahedra according to their corresponding relative error $\|\vec{B}_2 - \vec{B}_1\|/\|\vec{B}_2\|$ into 0.1%-large bins (on the horizontal axis) and we plot the number of tetrahedra in each bin on the vertical axis (in dark blue). In light blue, we plot the corresponding volume of the computational domain $\mathcal{B}_R$ occupied by the tetrahedra in each bin.

We clearly see, that the quality of the resulting $\vec{B}_1$ for the homogeneous Dirichlet boundary condition depends on $R/r$. For $R/r = 1.5$ we commit at most 6% of relative error, while for $R/r = 1.25$ it is already 9%.

It seems from Figs. 13 that we may approximate the decay condition at infinity by a homogeneous Dirichlet condition on $\partial \mathcal{B}_R$, provided that $\mathcal{B}_R$ is large enough, typically $R/r > 2$. However, we must consider that the number of unknowns of the underlying discretization is proportional to $(R/r)^3$, once $\mathcal{B}_r$ has been fixed.

On the other hand, we can use smaller $\mathcal{B}_R$ within the domain decomposition framework with Poisson formula. The solved discrete problem has in this case considerably less unknowns than in the previous case. Moreover, Fig. 14, which shows the convergence of the non-linear residue as a function of iteration number $k$ for both meshes and both algorithms, seems to indicate, that the domain decomposition present just a minor computational overhead, provided that $R/r$ is large enough, typically $1.5 \geq R/r > 1.2$.

6 Conclusions

We have presented both a scalar and a vector potential model for a static Maxwell problem including ferromagnetic effects with different formulations leading to finite element approximations: algorithms $\text{scal}$, $\text{bemfem}$, $\text{rt0}$, $\text{mix}$ and $\text{vec}$ were developed. We have applied all these algorithms to a simple but representative test-case of a ferromagnetic plate in front of an electric conductor and compared the results for induction inside and outside the plate. It must be noticed here that we had no exact solution for our test-case, we just compared the results obtained from the different
algorithms.

It comes out from our observations that the scal algorithm is by far the most efficient in terms of CPU time (and memory use) for obtaining quite a reasonable precision in its results.

If we aim the best approximation of induction near the plate or even in the plate, algorithms like rt0 or vec give better results at the price of much more CPU time (and memory use).

The bemfem algorithm, which gives comparable results as the scal algorithm, could be made efficient by using multipole techniques which we have not tried here. It is important to note that this algorithm is the only one presented here which does not need a mesh outside the ferromagnetic parts.

References


