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Analysis of an unstructured finite volume method

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Abstract: We give a discrete H^1 error estimate of a finite volume discretization of the H^2 regular Poisson problem on nonstructured meshes. The discrete finite volume solution is compared to a L^2 weighted projection of the exact solution to the finite volume space. As the projection is stable in L^2 (unlike in standard finite volume estimates which use Taylor's expansion) a weak approximation property is also satisfied. This makes the obtained results interesting in the framework of the multigrid convergence theory of Bramble, Ewing, Pasciak and Shen.

Key-words: finite volumes, error estimate, volume agglomeration, multigrid

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Analyse d'une méthode volumes finis nonstructurés

Résumé : On propose une estimation d'erreur dans une norme H^1 discrète pour la méthode des volumes finis nonstructurés appliquée au problème de Poisson dans H^2. La solution discrète volumes finis est comparée avec une projection L^2 ponderée de la solution exacte dans l'espace volumes finis. Comme cette projection est stable en L^2 (contrairement à des estimations usuelles qui utilisent un developpement de Taylor) une proprieté d'approximation faible est aussi verifiée, ce qui rend notre resultat interessant dans le cadre de la théorie de convergence de Bramble, Ewing, Pasciak et Shen.

Mots-clés : volumes finis, estimation d'erreur, agglomeration des volumes, multi-grille

1 Introduction

The finite volume methods have become a very popular way of discretizing the problems arizing from conservative laws of physics, namely in fluid dynamics, heat and mass transfer and petroleum engineering. It may be used to discretize computational domains with complex geometries and can give reasonable results even when applied to difficult non-linear problems, which are still beyond the scope of numerical analysis. Thanks to their property of local conservativity of fluxes (inherited from the basic idea of finite volumes – conservation of local balance of fluxes) these schemes are attractive for problems where flux should be preserved.

The mathematical analysis of various finite volume schemes is only recently being developed. Namely, the H^1-seminorm (energy-norm) error estimates of such schemes are quite recent (cf. [2, 3, 4, 5, 8]).

Ewing, Lazarov and Vassilevski [4] proposed for the discretization of the Poisson equation by various finite volume schemes on regular rectangular structured locally refined meshes a discrete H^1 error estimate in the form

$$|u_h - Q_h u|_{1,h} \leq Ch^{\alpha} \|u\|_{2,\Omega}, \tag{1}$$

where u_h is the discrete finite volume solution of the Poisson equation, $u \in H^2(\Omega)$ is the exact solution, $Q_h u = u(x_i)$ on the control volume C_i, $x_i \in C_i$ being some points, $|\cdot|_{1,h}$ is the discrete H^1 seminorm induced by the finite volume matrix and $\|\cdot\|_{2,\Omega}$ is a standard H^2 norm. The crucial point in their estimate (1) is the local application of Bramble-Hilbert lemma to get consistency of fluxes. The result (1) was generalized by Vassilevski, Petrova and Lazarov [8] to triangular cells and by Lazarov, Mishev and Vassilevski [7] to convection diffusion problems.

In Eymard, Gallouët and Herbin [5] we can find similar results in discrete H^1 seminorm for convection-diffusion, with the same Q_h as in [4] on a class of (admissible) nonstructured meshes. Their results were obtained by Taylor expansion locally on each cell interface. For such an expansion, it is necessary that Q_h is defined pointwise as in [4].

Problems arised for other choice of Q_h, such as eg. L^2 projection from H^2 to the piecewise constant finite volume solution space M_h, while still using the Taylor expansion of [5], are studied by Coudière, Vila and Villedieu [2]. It seems, that in this case, one needs a special regularity condition on the exact solution u, $u \in W^{2,p}$, $p > 2$.

Finally, Coudière and Villedieu [3] combined the original approach with the Bramble-Hilbert lemma as in [4] with Q_h being the L^2 projection to get the estimate (1) on structured rectangular locally refined grids.

In this presentation, we develop the idea of [3] for nonstructured grids, where instead of taking Q_h to be an L^2 projection, we rather chose it to be a certain weighted L^2 projection. Thus, we also obtain a result like (1), but this time for an operator Q_h which is stable in L^2 norm, ie. there exists a constant $C > 0$ independent of mesh size h, such that for all $u \in L^2$ there is

$$\|Q_h u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}.$$

For further application of the presented convergence theory this fact might be crucial. For example in multigrid applications like in [1] it is necessary that the same $Q_h : H^2 \to M_h$ appearing in (1) verifies also a weak approximation property, ie. there exists a constant $C > 0$ independent of mesh size h such that for all $u \in H^2$ there is

$$\|u - Q_h u\|_{L^2(\Omega)} \leq Ch\|u\|_{1,\Omega}.$$

This estimate, which is not known to hold in the case of pointwise Q_h, is verified when for example Q_h preserves a constant and is stable in L^2 norm. Hence the interest of this paper.

Still with multigrid in mind (cf. [6]), we add to this presentation a straightforward generalization of the proposed convergence proof for finite volume schemes where the right-hand side is integrated over control volumes generated by agglomeration of polygonal convex finest-level control volumes.

Although we give rigorous proof for simplicity only for Poisson equation, the same technique (cf. [3]) might be applied to get similar results for convection-diffusion problems.

2 Preliminaries

Consider the following Dirichlet problem: for a convex polygonal domain $\Omega \in \mathbb{R}^d$ and the right-hand side $f \in L^2(\Omega)$ find $u \in H^2(\Omega)$ such that

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Let us discretize the problem (2) by a finite-volume method on a system of polygonal control volumes (cf. Fig 1) with the following properties.

Definition 2.1. (admissible mesh) Let us have a set of polygonal finite volume cells C_i, $i = 1, \ldots, n$ with characteristic diameter h, such that for $i \neq j$ either

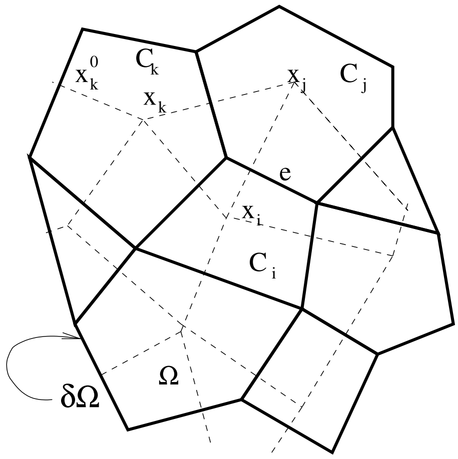

Figure 1: Admissible mesh

$\partial C_i \cap \partial C_j = \emptyset$ or $\partial C_i \cap \partial C_j$ is a part of a straight line ($d = 2$) or a part of a plane ($d = 3$). Let us denote by E_h the set of all cell interfaces

$$E_h = \{e, e = \partial C_i \cap \partial C_j \text{ or } e = \partial C_i \cap \partial\Omega\}.$$

Let $\mathbf{n}_e$ be a unit normal vector to $e \in E_h$ with the orientation chosen apriori. Based on the orientation of the normal $\mathbf{n}_e$, we define for every cell interface e of the cell C_i a sign function $s_{i,e}$, $s_{i,e} = 1$ if $\mathbf{n}_e$ points out of C_i, $s_{i,e} = -1$ otherwise.

Let us have a system of points $x_i \in C_i$ such that for two neighbouring cells C_i and C_j the line led through x_i and x_j is perpendicular to the common interface $e = \partial C_i \cap \partial C_j$. For the boundary cells C_i, $\bar{C}_i \cap \partial\Omega \neq \emptyset$, let us also define points $x_i^0 \in \partial\Omega \cap \partial C_i$ such that the line led through x_i and x_i^0 is perpendicular to $\partial\Omega \cap \partial C_i$ (cf. Fig 2).

For two neighbouring cells C_i and C_j, we will use the indices W (west) and E (east) to denote x_i by x^W or x^E and x_j by x^E or x^W such that the normal vector $\mathbf{n}_e$ of the common interface e points from west to east (cf. Fig 2).

For the boundary cell C_i and $e = \partial C_i \cap \partial\Omega$, we take $x^W = x_i$, $x^E = x_i^0$ if $\mathbf{n}_e$ points outside the cell, $x^W = x_i^0$, $x^E = x_i$ otherwise. We denote h_e the euclidean distance of x^W and x^E.

Let us denote by M_h the standard finite-volume control space of piecewise constant functions on cells C_i, $i = 1, \ldots, n$. In the same way as above, we will for every

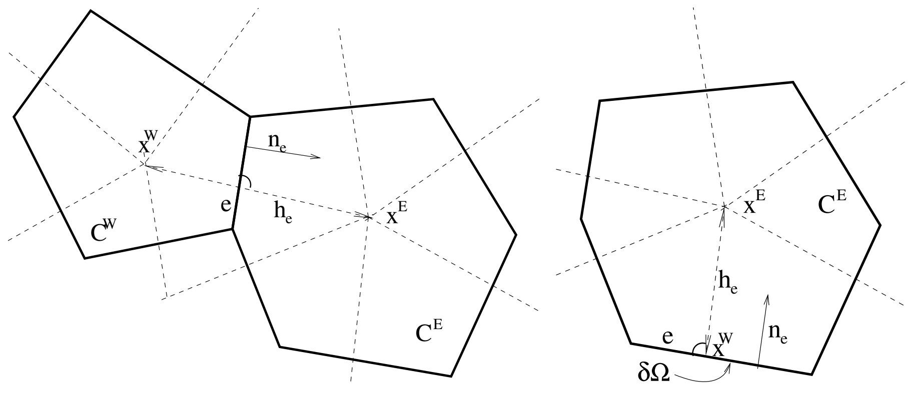

Figure 2: Typical interior and boundary cells

$e \in E_h$ denote u_h^W and u_h^E the values of $u_h \in M_h$ corresponding to two neighbouring cells with the common interface e, ie. $u_h^W = u_h(x^W)$, $u_h^E = u_h(x^E)$.

Let us formulate the discrete problem by the standard finite-volume approach using the volumes from Definition 2.1. By integrating the continuous problem (2) on C_i, using Gauss theorem and the fact that the cells are polygonals we get for the solution $u \in H^2(\Omega)$ that

$$-\sum_{e \in E_h \cap \partial C_i} s_{i,e} \int_e \nabla u \, \mathbf{n}_e d\Gamma = \int_{C_i} f d\Omega.$$

Following the spirit of a finite volume method, we should in the next step approximate the continuous solution $u \in H^2(\Omega)$ by a function $u_h \in M_h$, which is discontinuous on the integration path e. Therefore, we also approximate the gradient ∇u by a simple difference formula in u_h.

Let $u_h \in M_h$ be the solution of the following discrete problem.

$$-\sum_{e \in E_h \cap \partial C_i} s_{i,e} \Phi(u_h, \mathbf{n}_e) \mu(e) = \int_{C_i} f d\Omega, \quad (3)$$

where $\mu(e)$ is the measure (length) of the interface e and $\Phi(u_h, \mathbf{n}_e)$ is the numerical diffusive flux accross this cell interface, calculated by

$$\Phi(u_h, \mathbf{n}_e) = \frac{u_h^E - u_h^W}{h_e}.$$

The values of u_h^W and/or u_h^E corresponding to the boundary points x_i^0 are set to 0.

Now we are ready to define our continuous and discrete operators.

Definition 2.2. Let us define an averaged continuous-problem operator $\mathcal{L} : H^2(\Omega) \to M_h$ and the discrete-problem operator $\mathcal{L}_h : M_h \to M_h$, respectively, by

$$\mathcal{L}u = \left\{ -\frac{1}{\mu(C_i)} \sum_{e\in E_h\cap\partial C_i} s_{i,e} \int_e \nabla u \, \mathbf{n}_e d\Gamma \right\}_{C_i},$$
$$\mathcal{L}_h u_h = \left\{ -\frac{1}{\mu(C_i)} \sum_{e\in E_h\cap\partial C_i} s_{i,e} \Phi(u_h, \mathbf{n}_e)\mu(e) \right\}_{C_i},$$

for all $u \in H^2(\Omega)$ and $u_h \in M_h$. The corresponding bilinear forms $A(\cdot,\cdot) : H^2 \times L^2 \to \mathbb{R}$ and $A_h(\cdot,\cdot) : M_h \times M_h \to \mathbb{R}$ are defined as

$$A(u,v) = (\mathcal{L}u, v)_{L^2(\Omega)}, \qquad A_h(u_h, v_h) = (\mathcal{L}_h u_h, v_h)_{L^2(\Omega)},$$

for all $u \in H^2(\Omega)$, $v_h, u_h \in M_h$, $v \in L^2(\Omega)$.

Thus, for the discrete solution $u_h \in M_h$, we can write equivalently to (3) that

$$\mathcal{L}_h u_h = f_h, \tag{4}$$

where $f_h \in M_h$ is the L^2 projection of $f \in L^2(\Omega)$ to M_h.

The continuous solution $u \in H^2(\Omega)$ of (2) and the discrete solution $u_h \in M_h$ of (4) satisfy

$$A(u, v_h) = (f, v_h)_{L^2(\Omega)}, \ \forall v_h \in M_h,$$
$$A_h(u_h, v_h) = (f, v_h)_{L^2(\Omega)}, \ \forall v_h \in M_h.$$

Thus, by subtracting the above equations we get

$$A_h(u_h, v_h) = A(u, v_h), \ \forall v_h \in M_h. \tag{5}$$

The bilinear form $A_h(\cdot,\cdot)$ induces an energy norm $\|\cdot\|_{A_h} = A_h(\cdot,\cdot)^{\frac{1}{2}}$ which coincides with the H^1 discrete seminorm $|\cdot|_{1,h}$. Namely, defining

$$|v_h|_{1,h}^2 = \sum_{e\in E_h} \left| \frac{v_h^W - v_h^E}{h_e} \right|^2 h_e \mu(e) \qquad \forall v_h \in M_h,$$

and using the notation $(v_h)_i = v_h^W$ if $s_{i,e} = 1$, $(v_h)_i = v_h^E$ otherwise, we get by discrete integration by parts for all $v_h \in M_h$

$$
\begin{aligned}
A_h(v_h, v_h) &= \sum_{C_i} \sum_{e \in E_h \cap \partial C_i} s_{i,e} \frac{v_h^W - v_h^E}{h_e} \mu(e)(v_h)_i \\
&= \sum_{e \in E_h} \frac{v_h^W - v_h^E}{h_e} \mu(e)(v_h^W - v_h^E) \\
&= \sum_{e \in E_h} \left| \frac{v_h^W - v_h^E}{h_e} \right|^2 \mu(e) h_e = |v_h|_{1,h}^2.
\end{aligned} \tag{6}
$$

It is clear, that $|v_h|_{1,h} \geq 0$ for all $v_h \in M_h$ and $|v_h|_{1,h} = 0$ only if v_h is constant. Hence the name discrete H^1 seminorm.

3 Error estimate

The aim of this presentation is to give an error estimate in H^1 discrete seminorm for the finite volume method described above, ie. to estimate

$$A_h(u_h - Q_h u, u_h - Q_h u) \sim \mathcal{O}(h^2)$$

for some mapping $Q_h : H^2(\Omega) \to M_h$.

We will show, that the above estimate is verified, under some assumptions, for the following mapping Q_h.

Definition 3.1. (mapping Q_h) Let us subdivide each polygonal control volume C_i into N_i triangles ($d = 2$) or tetrahedra ($d = 3$) $\{C_{i,j}\}$ by connecting the vertices of the polygonal cell interface with the point $x_i \in C_i$ (cf. Fig 3). Let us denote by $N = \max_i\{N_i\}$.

Given positive weights $w_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, N_i$, we can define a mapping $Q_h : H^2(\Omega) \to M_h$ by

$$Q_h u = \sum_{j=1}^{N_i} \frac{w_{i,j}}{\mu(C_{i,j})} \int_{C_{i,j}} u d\Omega \quad \text{on } C_i.$$

Assumption 3.2. Let the system of finite-volume cells as described in Definition 2.1 have the following properties:

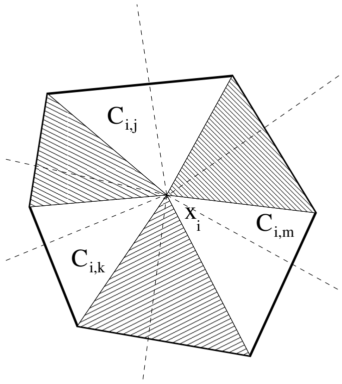

Figure 3: Subdivision of finite-volume cell C_i into triangles $C_{i,j}$

1. The set of cells $\{C_i\}$ is quasiuniform, ie. there exist constants $c_d, C_d > 0$ independent of h such that for all $i = 1, \ldots, n$ we have $c_d h \leq \operatorname{diam}(C_i) \leq C_d h$.

2. There exist $C_e, c_e > 0$ such that $\operatorname{diam}(e) \leq C_e h$ and $\operatorname{diam}(C_i)/\operatorname{diam}(e) \leq c_e$, for all $e \in E_h \cap \partial C_i$.

3. There exist $C_N > 0$ such that $N_i \mu(C_{i,j}) \leq C_N \mu(C_i)$ for all subdivisions $C_{i,j}$ of any finite-volume cell C_i.

4. There exist constants $c_h, C_h > 0$ such that for all $e \in E_h$ there is $c_h h \leq h_e \leq C_h h$.

5. The number of neighbouring cells N is bounded by a constant independent of h.

Lemma 3.3. Let us have the finite volume mesh as in Definition 2.1 satisfying Assumption 3.2. Then there exist weights $w_{i,j}$, for which $w_{i,j} \in [0, 1]$ such that for the mapping $Q_h : H^2(\Omega) \to M_h$ from Definition 3.1 we have:

1. If u is linear then $Q_h u = u(x_i)$ on each C_i,

2. If u is constant then $Q_h u = u$,

3. The mapping Q_h is stable in L^2 norm, ie. there exists a constant $C > 0$ such that for all $u \in H^2(\Omega)$

$$\|Q_h u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}.$$

Proof . Let us denote $x_{i,j}$ the barycenters of the triangles (resp. of the tetrahedra) $C_{i,j}$. We search for each C_i the positive weights $w_{i,j}$ such that

$$\sum_{j=1}^{N_i} w_{i,j} x_{i,j} = x_i$$
$$\sum_{j=1}^{N_i} w_{i,j} = 1.$$

This gives a system of 3 linear equations for N_i unknowns, which assures, that the statements 1. and 2. of the lemma are satisfied. From all possible solutions, we would be interested by the one for which $\max_j(w_{i,j})$ is minimal. It is easy to see from the fact that x_i lies within the convex envelope of the barycenters $x_{i,j}$ that there exists a set of $\{w_{i,j}\}$, for which $w_{i,j} \in [0,1]$, $j = 1, \ldots, N_i$.

Now, let us prove the L^2 stability of Q_h. From Definition 3.1, using twice Cauchy-Schwarz inequality, the facts that $\sum_j w_{i,j}^2 \leq \max_j(w_{i,j}) \sum_j w_{i,j}$, $w_{i,j} \leq 1$ and Assumption 3.2 we get

$$\begin{aligned}
\|Q_h u\|_{L2(\Omega)}^2 &= \sum_{i=1}^{n} \int_{C_i} \left(\sum_{j=1}^{N_i} \frac{w_{i,j}}{\mu(C_{i,j})} \int_{C_{i,j}} u d\Omega \right)^2 d\Omega \\
&\leq \sum_{i=1}^{n} \mu(C_i) \left(\sum_{j=1}^{N_i} w_{i,j}^2 \right) \left(\sum_{j=1}^{N_i} \left(\frac{1}{\mu(C_{i,j})} \int_{C_{i,j}} u d\Omega \right)^2 \right) \\
&\leq \sum_{i=1}^{n} \mu(C_i) \max_j(w_{i,j}) \left(\sum_{j=1}^{N_i} \|u\|_{L^2(C_{i,j})}^2 \mu(C_{i,j})^{-1} \right) \\
&\leq \sum_{i=1}^{n} \mu(C_i) \frac{\max_j(w_{i,j})}{\min_j(\mu(C_{i,j}))} \|u\|_{L^2(C_i)}^2 \\
&\leq C \|u\|_{L^2(\Omega)}^2.
\end{aligned}$$

■

Now, we are going to estimate the A_h-norm of the error between the discrete solution $u_h \in M_h$ of (4) and $Q_h u$, u is the solution of the continuous problem (2). From (5) we have by taking $v_h = u_h - Q_h u$,

$$A_h(u_h - Q_h u, u_h - Q_h u) = A(u, u_h - Q_h u) - A_h(Q_h u, u_h - Q_h u).$$

The convergence of $A_h(u_h - Q_h u, u_h - Q_h u) \to 0$ as $h \to 0$ will follow from the discrete coercivity of A_h and from the consistency.

Here we will show, that the consistency of A_h follows from the consistency of numerical fluxes in the finite-volume scheme.

Definition 3.4. For any $u \in H^2(\Omega)$ let $\bar{\Phi}(u, \mathbf{n}_e)$ be the averaged exact flux, ie:

$$\bar{\Phi}(u, \mathbf{n}_e) = \frac{1}{\mu(e)} \int_e \nabla u \, \mathbf{n}_e d\Gamma.$$

Let us define the consistency error $R_e(u)$ of fluxes by

$$R_e(u) = \bar{\Phi}(u, \mathbf{n}_e) - \Phi(Q_h u, \mathbf{n}_e).$$

Definition 3.5. (consistency) The operator $\mathcal{L}_h$ is said to be consistent in $H^2(\Omega)$ if there exists $c > 0$ such that for all $u \in H^2(\Omega)$, $u = 0$ on $\partial\Omega$ there is

$$\sum_{e \in E_h} |R_e(u)|^2 \mu(e) h_e \le c h^2 \|u\|_{2,\Omega}^2,$$

where $\|\cdot\|_{2,\Omega}$ is the standard $H^2(\Omega)$ norm.

While the Definition 3.5 defines consistency in terms of the fluxes, we need consistency of the bilinear form $A_h(\cdot,\cdot)$ which follows from it; indeed, for all $v_h \in M_h$ we have after discrete integration by parts (cf. (6))

$$\begin{aligned}
|A(u, v_h) - A_h(Q_h u, v_h)| &= \\
&= \left| -\sum_{C_i} \sum_{e \in E_h \cap C_i} s_{i,e} \left(\frac{1}{\mu(e)} \int_e \nabla u \, \mathbf{n}_e d\Gamma - \frac{(Q_h u)^E - (Q_h u)^W}{h_e} \right) \mu(e) (v_h)_i \right| \\
&= \left| \sum_{e \in E_h} \left(\frac{1}{\mu(e)} \int_e \nabla u \, \mathbf{n}_e d\Gamma - \frac{(Q_h u)^E - (Q_h u)^W}{h_e} \right) \mu(e) (v_h^W - v_h^E) \right| \\
&\le \sum_{e \in E_h} \left| R_e(u) \frac{v_h^W - v_h^E}{h_e} \right| \mu(e) h_e.
\end{aligned}$$

Using Cauchy-Schwarz inequality and using consistency of fluxes, we find that for all $v_h \in M_h$ there is

$$|A(u, v_h) - A_h(Q_h u, v_h)| \le c h \|u\|_{2,\Omega} \left(\sum_{e \in E_h} \left| \frac{v_h^W - v_h^E}{h_e} \right|^2 \mu(e) h_e \right)^{\frac{1}{2}} = c h \|u\|_{2,\Omega} |v_h|_{1,h},$$

where $|\cdot|_{1,h}$ denotes the H^1 discrete seminorm.

Also, in the proof of convergence, we will use the coercivity of $A_h(\cdot,\cdot)$.

Definition 3.6. (coercivity) The operator $\mathcal{L}_h$ is said to be coercive, if there exists $c^* > 0$ independent of h such that for all $v_h \in M_h$ there is

$$A_h(v_h, v_h) = (\mathcal{L}_h v_h, v_h)_{L^2(\Omega)} \geq c^* |v_h|_{1,h}^2.$$

For our case, this is obviously true with equal sign and $c^* = 1$, thanks to the fact that the line connecting of x^W and x^E is perpendicular to the corresponding interface e, which allows us to use the discrete integration by parts (6).

Thus, the convergence estimate follows from coercivity and consistency.

Theorem 3.7. If the discrete operator $\mathcal{L}_h$ is consistent (as in Definition 3.5) and coercive (as in Definition 3.6), then we have the following error estimate: for the solution $u \in H^2(\Omega)$ of the continuous problem (2), the solution $u_h \in M_h$ of (4) and the mapping $Q_h : H^2(\Omega) \to M_h$ from Definition 3.1, there exists $C > 0$ independent of h such that

$$A_h(u_h - Q_h u, u_h - Q_h u) \leq C h^2 \|u\|_{2,\Omega}^2.$$

Proof . Indeed, by we have

$$\begin{aligned} A_h(u_h - Q_h u, u_h - Q_h u) &\leq |A(u, u_h - Q_h u) - A_h(Q_h u, u_h - Q_h u)| \\ &\leq C h \|u\|_{2,\Omega} \, |u_h - Q_h u|_{1,h} \\ &\leq \frac{C}{c^*} h \|u\|_{2,\Omega} \, A_h(u_h - Q_h u, u_h - Q_h u)^{\frac{1}{2}}. \end{aligned}$$

Cancelling $A_h(u_h - Q_h u, u_h - Q_h u)^{\frac{1}{2}}$ on both sides of the inequality finishes the proof.

■

3.1 Consistency

It rests to verify that under Assumption 3.2 the discrete operator $\mathcal{L}_h$ is consistent. As we are going to use scaling technique together with Bramble-Hilbert lemma, we shall first define our local reference elements.

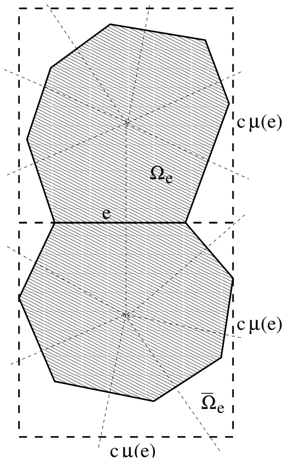

Figure 4: Neighbourhood envelope

Definition 3.8. For every interface $e \in E_h$ let us define local neighbourhood of e to be a domain Ω_e, such that

$$\Omega_e = \left\{ \bigcup_k C_k ; C_k \cap e \neq \emptyset \right\}.$$

Also, let us define for every interior interface $e \in E_h$, $e \cap \partial\Omega = \emptyset$ the neighbourhood envelope $\bar{\Omega}_e \supset \Omega_e$ composed of two squares ($d = 2$) or cubes ($d = 3$), as small as possible, such that their common intersection $\bar{e}$ contains the cell interface e (cf. Fig 4). In the same way, for all boundary interfaces, let the envelope be composed of one square or cube circumscribed to Ω_e whose side contains e (cf. Fig 4).

Lemma 3.9. Let us for all $e \in E_h$ have $R_e(u)$ as in Definition 3.4, the mapping Q_h as in Definition 3.1 and let $u \in H^2(\Omega)$. Under the Assumption 3.2 there exists a constant $C > 0$ independent of h such that for all $e \in E_h$ there is

$$\sum_{e \in E_h} |R_e(u)|^2 \leq C h^{2-d} \|u\|^2_{2,\Omega}.$$

Proof . Let us take one $e \in E_h$. First, we introduce an affine transformation of coordinates $\psi : \mathbb{R}^d \to \mathbb{R}^d$, $x = \psi(\hat{x})$, which scales the reference neighbourhood envelope $\hat{\bar{\Omega}}_e$, which consists of two unit squares ($d = 2$) or cubes ($d = 3$) if e is from

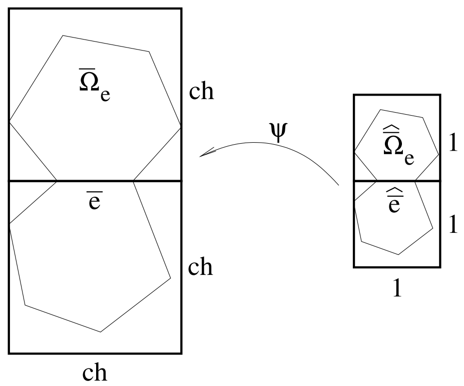

Figure 5: Mapping ψ of the reference envelope to the neighbourhood envelope

the interior, and of one unit square (or cube) if e is a boundary interface, to the neighbourhood envelope $\bar{\Omega}_e$ (cf. Fig 5).

In the following we will denote by hat the objects corresponding to the reference envelope $\hat{\bar{\Omega}}_e$. Namely, we denote $\hat{e} = \psi^{-1}(e)$, $\hat{\bar{e}} = \psi^{-1}(\bar{e})$, $\hat{C}_{i,j} = \psi^{-1}(C_{i,j})$, $\hat{h}_e = \mathrm{dist}(\psi^{-1}(x^W), \psi^{-1}(x^E))$.

We can define the functional $\hat{R}_e : H^2(\hat{\bar{\Omega}}_e) \to \mathbb{R}$ by

$$\hat{R}_e(\hat{u}) = \frac{1}{\mu(\hat{e})} \int_{\hat{e}} \nabla \hat{u}\, \mathbf{n}_{\hat{e}} d\Gamma - \hat{\Phi}_e(\hat{Q}_h \hat{u}),$$

where

$$\hat{\Phi}_e(\hat{u}_h) = \frac{\hat{u}_h^E - \hat{u}_h^W}{\hat{h}_e},$$

$$\hat{Q}_h \hat{u} = \sum_{j=1}^{N_i} \frac{w_{i,j}}{\mu(\hat{C}_{i,j})} \int_{\hat{C}_{i,j}} \hat{u} d\Omega \quad \text{on } \hat{C}_i.$$

Let us for further purposes denote by $\tilde{u} \in H^2(\mathbb{R}^d)$ the extension of $u \in H^2(\Omega)$, $u = 0$ on $\partial\Omega$, such that by Extension Theorem $\tilde{u}$ verifies

$$\|\tilde{u}\|_{2,\mathbb{R}^d} \leq C(\Omega)\|u\|_{2,\Omega}.$$

We can state the following properties of ψ:

1. for $\hat{u}(\hat{x}) = \tilde{u}(\psi(\hat{x}))$ there is

(a) $\hat{Q}_h \hat{u} = \widehat{Q_h u}$,

(b) $\hat{R}_e(\hat{u}) = \mathrm{diam}(\bar{e})^{2-d} \mu(\bar{e}) R_e(u)$,

2. if u is linear, $\hat{u}$ is also linear and $R_e(u) = 0$, $\hat{R}_e(\hat{u}) = 0$,

3. by scaling we have $|\hat{u}|_{2,\hat{\bar{\Omega}}_e} = \mathrm{diam}(\bar{e})^{2-\frac{d}{2}} |\tilde{u}|_{2,\bar{\Omega}_e}$.

Now, we will show that $\hat{R}_e(\hat{u})$ is continuous in $H^2(\hat{\bar{\Omega}}_e)$. Indeed, by Cauchy-Schwarz inequality and by Assumption 3.2 it follows that $\mu(\hat{e}) \geq c_1^{-1}$, $\mu(\hat{C}_i)\hat{h}_e \geq \bar{c}_2^{-1}$, $\mu(\hat{\Omega}_e) \leq 2$, and

$$
\begin{aligned}
|\hat{R}_e(\hat{u})| &\leq \frac{1}{\mu(\hat{e})} \int_{\hat{e}} |\nabla \hat{u}|\, |\mathbf{n}_{\hat{e}}| d\Gamma + \frac{1}{\hat{h}_e} \left(|(\hat{Q}_h \hat{u})^W| + |(\hat{Q}_h \hat{u})^E| \right) \\
&\leq \frac{1}{\mu(\hat{e})^{\frac{1}{2}}} \left(\int_{\hat{e}} |\nabla \hat{u}|^2 d\Gamma \right)^{\frac{1}{2}} + \frac{1}{\hat{h}_e} \left(\frac{1}{\mu(\hat{C}^W)} \int_{\hat{C}^W} |\hat{Q}_h \hat{u}| d\Omega + \frac{1}{\mu(\hat{C}^E)} \int_{\hat{C}^E} |\hat{Q}_h \hat{u}| d\Omega \right) \\
&\leq c_1 \left(\int_{\hat{e}} |\nabla \hat{u}|^2 d\Gamma \right)^{\frac{1}{2}} + \bar{c}_2 \int_{\hat{\Omega}_e} |\hat{Q}_h \hat{u}| d\Omega \\
&\leq c_1 \left(\int_{\hat{e}} |\nabla \hat{u}|^2 d\Gamma \right)^{\frac{1}{2}} + c_2 \left(\int_{\hat{\Omega}_e} (\hat{Q}_h \hat{u})^2 d\Omega \right)^{\frac{1}{2}} \\
&\leq c_1 \|\nabla \hat{u}\|_{[L^2(\hat{e})]^d} + c_2 \|\hat{Q}_h \hat{u}\|_{L^2(\hat{\Omega}_e)},
\end{aligned}
$$

where we denoted $\|\nabla \hat{u}\|^d_{[L^2(\hat{e})]^2} = \int_{\hat{e}} |\nabla \hat{u}|^2 d\Gamma$. Moreover, by Lemma 3.3, Q_h is stable in $L^2(\Omega)$, implying that also $\hat{Q}_h$ is stable in $L^2(\hat{\Omega})$. Thus, applying the Trace Theorem for the first term, we can continue by

$$
\begin{aligned}
|\hat{R}_e(\hat{u})| &\leq C \left(\|\nabla \hat{u}\|_{[H^1(\hat{\bar{\Omega}}_e)]^d} + \|\hat{u}\|_{L^2(\hat{\bar{\Omega}}_e)} \right) \\
&\leq C \|\hat{u}\|_{H^2(\hat{\bar{\Omega}}_e)}.
\end{aligned}
$$

Thanks to the properties of ψ mentioned above and as $|\hat{R}_e(\hat{u})|$ is continuous in H^2, we can now apply Bramble-Hilbert lemma to get

$$
|\hat{R}_e(\hat{u})| \leq C(\hat{\bar{\Omega}}_e) |\hat{u}|_{2,\hat{\bar{\Omega}}_e}.
$$

Scaling back to $\bar{\Omega}_e$ and denoting $\tilde{u}$ the H^2 extension of $u \in H^2(\Omega)$ to $H^2(\mathbb{R}^d)$ gives

$$
|R_e(u)| \leq \frac{\mathrm{diam}(\bar{e})^{\frac{d}{2}}}{\mu(\bar{e})} C(\hat{\bar{\Omega}}_e) |\tilde{u}|_{2,\bar{\Omega}_e}.
$$

The quasiuniformity in Assumption 3.2 implies that the number of overlaps of $\bar{\Omega}_e$ is bounded and that $\mu(\bar{e}) \leq ch^{d-1}$ and $\mathrm{diam}(\bar{e}) \leq ch$. Thus, summing the squares of the above through all cell interfaces $e \in E_h$ and applying the Extension Theorem we get

$$\sum_{e \in E_h} |R_e(u)|^2 \leq C(\hat{\hat{\Omega}}_e) h^{2-d} \sum_{e \in E_h} |\tilde{u}|^2_{2,\bar{\Omega}_e} \leq C(\hat{\hat{\Omega}}_e) h^{2-d} |\tilde{u}|^2_{2,\mathbb{R}^d} \leq C(\hat{\hat{\Omega}}_e, \Omega) h^{2-d} \|u\|^2_{2,\Omega}.$$

This is the statement of the lemma.

■

Theorem 3.10. For the finite-volume scheme based on control volumes from Definition 2.1 satisfying Assumption 3.2, with the numerical flux from Definition 3.4 and the discrete operator $\mathcal{L}_h$ from Definition 2.2, we have the following error estimate: for the solution $u \in H^2(\Omega)$ of the continuous problem (2), the solution $u_h \in M_h$ of (4) and the mapping $Q_h : H^2(\Omega) \to M_h$ from Definition 3.1, there exists $C > 0$ independent of h such that

$$A_h(u_h - Q_h u, u_h - Q_h u) \leq C h^2 \|u\|^2_{2,\Omega}.$$

Proof . The coercivity (cf. Definition 3.6) of the discrete operator $\mathcal{L}_h$ is trivially verified for our case. The statement of the theorem then follows from Lemma 3.9 applied to Theorem 3.7.

■

4 Generalized admissible meshes

In this section we are going to investigate the case in which the finite-volume cells are neither convex nor polygonal. The discretization of the operator $\mathcal{L}$ is, however, based on the regularized system of cells from Section 3, while the right-hand side is integrated over the new non-convex cells. The resulting scheme might be also interpreted as the scheme from Setion 3 with non-exact "numerical" integration of the right-hand side. The interest of this section is to show the H^1 discrete norm convergence estimate of the scheme with control cells of a generalized shape. This is important, for example, in the framework of multigrid algorithms with coarsening by agglomeration of control volumes (eg. [6]), where the shape of the coarse-grid control cells cannot be controled.

Definition 4.1. (generalized admissible mesh) Based on the system of regularized finite-volume cells $\{C_i\}$ as in Definition 2.1, let us define a new set of finite volumes $\{\tilde{C}_i\}$ in the following way (cf. Fig 6): let us have a continuous (but generally non-smooth) one-to-one mapping $\theta : \Omega \to \Omega$ such that

$$\begin{aligned} &\theta(x_i) = x_i, \\ &\partial C_i \cap \partial C_j = \emptyset \Leftrightarrow \theta(\partial C_i) \cap \theta(\partial C_j) = \emptyset \\ &\theta(z) = z \quad \text{for all } z \in \partial\Omega. \end{aligned}$$

Let us denote the deformed cell interfaces and the deformed finite volumes, respectively, by $\tilde{e}$ and $\tilde{C}_i$, where

$$\tilde{e} = \theta(e) \qquad \tilde{C}_i = \theta(C_i).$$

Moreover, let for all $i, j = 1, \ldots, n$, $i \neq j$ either $\partial\tilde{C}_i \cap \partial\tilde{C}_j = \emptyset$, or there exists $\tilde{e} = \theta(e)$, $e \in E_h$ such that $\partial\tilde{C}_i \cap \partial\tilde{C}_j = \tilde{e}$. Also, let us denote the set of all deformed cell interfaces by $\tilde{E}_h$,

$$\tilde{E}_h = \{\tilde{e} = \theta(e), \forall e \in E_h\}.$$

In the sequel, we also need the extended deformed cell interface $\tilde{\varepsilon} = \varepsilon(\tilde{e})$ comprising the deformed cell interface $\tilde{e}$ and lines ($d = 2$) or surfaces ($d = 3$) connecting the border-points ξ of e to the corresponding points $\theta(\xi)$ of $\tilde{e}$. Let us also define an extension $\mathbf{n}_{\tilde{\varepsilon}}$ of the normal $\mathbf{n}_{\tilde{e}}$ onto $\tilde{\varepsilon}$. The set of all extended deformed interfaces will be denoted by $\tilde{\mathcal{E}}_h$,

$$\tilde{\mathcal{E}}_h = \{\tilde{\varepsilon} = \varepsilon(\tilde{e}), \forall \tilde{e} \in \tilde{E}_h\}.$$

Further, let us denote by $\tilde{\Omega}_e$ the domain bounded by the interfaces e and $\tilde{\varepsilon}$.

Accordingly, let us have a finite volume solution space $\tilde{M}_h$ of piecewise constant functions on $\tilde{C}_i$.

Assumption 4.2. Let the system of finite volume cells in Definition 4.1 based on the regular cells from Definition 2.1 satisfy the following:

1. The regular cells $\{C_i\}$ satisfy Assumtpion 3.2
2. There exist a constat $C > 0$ independent of h such that for all $i = 1, \ldots, n$ there is $\mu(\tilde{C}_i) \leq C\mu(C_i)$.
3. For each $e \in E_h$ there is a constant $c > 0$ independent of h such that $\operatorname{diam}(\tilde{\Omega}_e)/\operatorname{diam}(e) \leq c$.

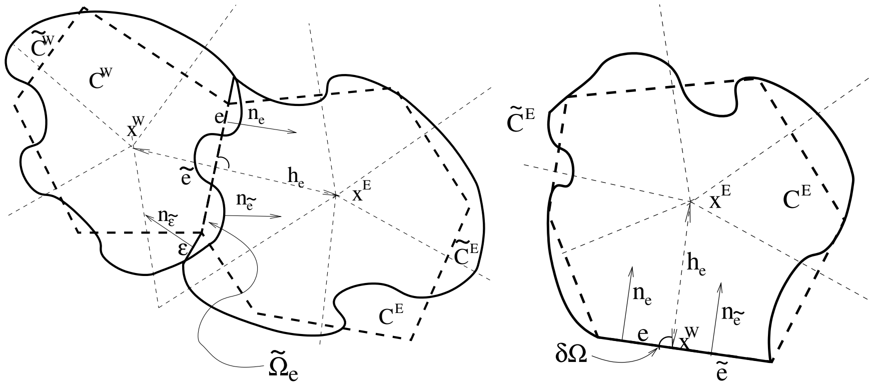

Figure 6: Generalized mesh: typical interior and boundary cells

Definition 4.3. Let us define as in Definition 2.2 the continuous-problem operator $\tilde{\mathcal{L}} : H^2(\Omega) \to \tilde{M}_h$ and the discrete-problem operator $\tilde{\mathcal{L}}_h : \tilde{M}_h \to \tilde{M}_h$, respectively, by

$$\tilde{\mathcal{L}}u = \Big\{ -\frac{1}{\mu(\tilde{C}_i)} \sum_{\tilde{e}\in\tilde{E}_h\cap\partial\tilde{C}_i} s_{i,e} \int_{\tilde{\varepsilon}=\varepsilon(\tilde{e})} \nabla u\,\mathbf{n}_{\tilde{\varepsilon}} d\Gamma \Big\}_{\tilde{C}_i},$$

$$\tilde{\mathcal{L}}_h\tilde{u}_h = \Big\{ -\frac{1}{\mu(\tilde{C}_i)} \sum_{e\in E_h\cap\partial C_i} s_{i,e}\tilde{\Phi}(\tilde{u}_h, \mathbf{n}_{\tilde{e}})\mu(e) \Big\}_{\tilde{C}_i=\theta(C_i)},$$

where $u \in H^2(\Omega)$, $\tilde{u}_h \in \tilde{M}_h$ and the fluxes $\tilde{\Phi}(\tilde{u}_h, \mathbf{n}_{\tilde{e}})$ are given by

$$\tilde{\Phi}(\tilde{u}_h, \mathbf{n}_{\tilde{e}}) = \frac{\tilde{u}_h^E - \tilde{u}_h^W}{h_e}.$$

The corresponding bilinear forms $\tilde{A}(\cdot,\cdot) : H^2 \times L^2 \to \mathbb{R}$ and $\tilde{A}_h(\cdot,\cdot) : \tilde{M}_h \times \tilde{M}_h \to \mathbb{R}$ are defined by

$$\tilde{A}(u,v) = (\tilde{\mathcal{L}}u, v)_{L^2(\Omega)}, \qquad \tilde{A}_h(\tilde{u}_h, \tilde{v}_h) = (\tilde{\mathcal{L}}_h\tilde{u}_h, \tilde{v}_h)_{L^2(\Omega)},$$

for all $u \in H^2(\Omega)$, $\tilde{v}_h, \tilde{u}_h \in \tilde{M}_h$, $v \in L^2(\Omega)$.

Thus, we can pose the discrete problem: find $\tilde{u}_h \in \tilde{M}_h$ such that

$$\tilde{\mathcal{L}}_h\tilde{u}_h = \tilde{f}_h, \qquad (7)$$

where $\tilde{f}_h$ is an L^2 projection of the continuous right-hand side $f \in L^2(\Omega)$ to $\tilde{M}_h$.

Definition 4.4. (mapping $\tilde{Q}_h$) Given a set of positive weights $w_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, N_i$, we can define a mapping $\tilde{Q}_h : H^2(\Omega) \to \tilde{M}_h$ by

$$\tilde{Q}_h u = \sum_{j=1}^{N_i} \frac{w_{i,j}}{\mu(C_{i,j})} \int_{C_{i,j}} u d\Omega \quad \text{on } \tilde{C}_i.$$

Lemma 4.5. (L^2 stability of $\tilde{Q}_h$) Let us have the mapping $\tilde{Q}_h$ as in Definition 4.4. Then $\tilde{Q}_h$ is stable in L^2, ie. there exists $C > 0$ independent of h such that for all $u \in H^2(\Omega)$ there is

$$\|\tilde{Q}_h u\|_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega)}.$$

Proof . By the definition of $\tilde{Q}_h$, using Cauchy-Schwarz inequality and existence of weights $w_{i,j} \in [0, 1]$ together with the Assumption 4.2, it follows that

$$\begin{aligned}
\|\tilde{Q}_h u\|^2_{L^2(\Omega)} &= \sum_i \int_{\tilde{C}_i} \left(\sum_{j=1}^{N_i} \frac{w_{i,j}}{\mu(C_{i,j})} \int_{C_{i,j}} u d\Omega \right)^2 d\Omega \\
&\le \sum_i \left[\mu(\tilde{C}_i) \cdot \sum_{j=1}^{N_i} (w_{i,j})^2 \cdot \sum_{j=1}^{N_i} \left(\frac{1}{\mu(C_{i,j})} \int_{C_{i,j}} u d\Omega \right)^2 \right] \\
&\le \sum_i \left[\mu(\tilde{C}_i) \cdot \max_j (w_{i,j}) \cdot \sum_{j=1}^{N_i} \frac{1}{\mu(C_{i,j})} \|u\|^2_{L^2(C_{i,j})} \right] \\
&\le \sum_i \left[\frac{\mu(\tilde{C}_i) \cdot \max_j (w_{i,j})}{\min(\mu(C_{i,j}))} \|u\|^2_{L^2(C_i)} \right] \\
&\le C\|u\|^2_{L^2(\Omega)},
\end{aligned}$$

which finishes the proof.

■

Definition 4.6. For any $u \in H^2(\Omega)$ let $\bar{\bar{\Phi}}(u, \mathbf{n}_{\tilde{e}})$ be the averaged exact flux, ie:

$$\bar{\bar{\Phi}}(u, \mathbf{n}_{\tilde{e}}) = -\frac{1}{\mu(e)} \int_{\tilde{\varepsilon} = \varepsilon(\tilde{e})} \nabla u \, \mathbf{n}_{\tilde{\varepsilon}} d\Gamma.$$

Let us define the consistency error $\tilde{R}_e(u)$ of fluxes by

$$\tilde{R}_e(u) = \bar{\bar{\Phi}}(u, \mathbf{n}_{\tilde{e}}) - \tilde{\Phi}(\tilde{Q}_h u, \mathbf{n}_{\tilde{e}}).$$

Lemma 4.7. Let $\tilde{R}_e(u)$ be defined as in Definition 4.6. Then the flux is consistent, ie. there exists a constant $C > 0$ independent of h such that

$$\sum_{e \in E_h} |\tilde{R}_e(u)|^2 \leq C h^{2-d} \|u\|^2_{2,\Omega}.$$

Proof . Let us take one $e \in E_h$. We have

$$\begin{aligned}
|\tilde{R}_e(u)| &= \left| \frac{1}{\mu(e)} \int_{\tilde{\varepsilon}} \nabla u\, \mathbf{n}_{\tilde{\varepsilon}} d\Gamma - \frac{1}{h_e} \left((\tilde{Q}_h u)^E - (\tilde{Q}_h u)^W \right) \right| \\
&\leq \left| \frac{1}{\mu(e)} \left(\int_{\tilde{\varepsilon}} \nabla u\, \mathbf{n}_{\tilde{\varepsilon}} d\Gamma - \int_e \nabla u\, \mathbf{n}_e d\Gamma \right) \right| \\
&\quad + \left| \frac{1}{\mu(e)} \int_e \nabla u\, \mathbf{n}_e d\Gamma - \frac{1}{h_e} \left((\tilde{Q}_h u)^E - (\tilde{Q}_h u)^W \right) \right| \\
&= |\tilde{R}_{1,e}(u)| + |\tilde{R}_{2,e}(u)|,
\end{aligned}$$

where we have simplified the formula by denoting the first and the second terms by $\tilde{R}_{1,e}(u)$ and $\tilde{R}_{2,e}(u)$, respectively.

Using the fact that $(\tilde{Q}_h u)^W = (Q_h u)^W$ and $(\tilde{Q}_h u)^E = (Q_h u)^E$ we have that the second term equals the flux error from Definition 3.4, ie.

$$|\tilde{R}_{2,e}(u)| = \left| \frac{1}{\mu(e)} \int_e \nabla u\, \mathbf{n}_e d\Gamma - \frac{1}{h_e} \left((\tilde{Q}_h u)^E - (\tilde{Q}_h u)^W \right) \right| = |R_e(u)|.$$

For the first term, we can sum the two integration domains to get an integral over the closed surface $\partial\tilde{\Omega}_e$. By Gauss Theorem over the domain $\tilde{\Omega}_e$ (cf. Fig) we find that

$$|\tilde{R}_{1,e}(u)| = \left| \frac{1}{\mu(e)} \oint_{\partial\tilde{\Omega}_e} \nabla u\, \mathbf{n} d\Gamma \right| = \left| \frac{1}{\mu(e)} \int_{\tilde{\Omega}_e} \Delta u d\Omega \right|.$$

Let us now, like in the proof of Lemma 3.9, introduce an affine transformation of coordinates $\psi : \mathbb{R}^d \to \mathbb{R}^d$, $x = \psi(\hat{x})$, such that for the reference interface $\hat{e}$, $e = \psi(\hat{e})$ there is $\mathrm{diam}(\hat{e}) = 1$.

Let us further define a functional $\hat{\tilde{R}}_{1,e}(\hat{u}) : H^2(\hat{\tilde{\Omega}}_e) \to \mathbb{R}$ by

$$\hat{\tilde{R}}_{1,e}(\hat{u}) = \left| \int_{\hat{\tilde{\Omega}}_e} \Delta \hat{u} d\Omega \right|.$$

It is clear that

1. $\hat{\tilde{R}}_{1,e}(\hat{u}) = \mathrm{diam}(\bar{e})^{2-d} \mu(e) \tilde{R}_{1,e}(u)$,

2. $|\hat{u}|_{2,\hat{\tilde{\Omega}}_e} = \mathrm{diam}(e)^{2-\frac{d}{2}}|u|_{2,\tilde{\Omega}_e}$.

Also, by Cauchy-Schwarz inequality and by Assumption 4.2 by which $\mu(\hat{\tilde{\Omega}}_e) \leq C$ we get

$$\begin{aligned}\hat{\tilde{R}}_{1,e}(\hat{u}) &= \left|\int_{\hat{\tilde{\Omega}}_e} \Delta\hat{u} d\Omega\right| \\ &\leq \mu(\hat{\tilde{\Omega}}_e)^{\frac{1}{2}} \cdot \left(\int_{\hat{\tilde{\Omega}}_e} (\Delta\hat{u})^2 d\Omega\right)^{\frac{1}{2}} \\ &\leq C|\hat{u}|_{2,\hat{\tilde{\Omega}}_e}.\end{aligned}$$

By scaling we thus obtain

$$|\tilde{R}_{1,e}(u)| \leq C\frac{\mathrm{diam}(e)^{\frac{d}{2}}}{\mu(e)}|u|_{2,\tilde{\Omega}_e}.$$

Summing the squares of $|\tilde{R}_e(u)|$ over all interfaces e and applying the results of Lemma 3.9 gives

$$\begin{aligned}\sum_{e\in E_h} |\tilde{R}_e(u)|^2 &\leq \sum_{e\in E_h} \left(|\tilde{R}_{1,e}(u)| + |\tilde{R}_{2,e}(u)|\right)^2 \\ &\leq 2\left[\sum_{e\in E_h} |\tilde{R}_{1,e}(u)|^2 + \sum_{e\in E_h} |\tilde{R}_{2,e}(u)|^2\right] \\ &\leq C_1 h^{2-d}\|u\|^2_{2,\Omega} + C_2 h^{2-d} \sum_{e\in E_h} \|u\|^2_{2,\tilde{\Omega}_e}.\end{aligned}$$

Now, by the fact that $\tilde{\Omega}_e$, $e \in E_h$ do not overlap we get

$$\sum_{e\in E_h} |\tilde{R}_e(u)|^2 \leq (C_1 + C_2)h^{2-d}\|u\|^2_{2,\Omega},$$

which is the statement of the lemma.

■

Lemma 4.8. The discrete operator $\tilde{\mathcal{L}}_h$ from Definition 4.3 is coercive, ie. there exists a constant $c^* > 0$ such that for all $\tilde{v}_h \in \tilde{M}_h$ we have

$$\tilde{A}_h(\tilde{v}_h, \tilde{v}_h) \leq c^*|\tilde{v}_h|_{1,h},$$

where $|\tilde{v}_h|_{1,h}$ is the discrete H^1 seminorm defined by

$$|\tilde{v}_h|^2_{1,h} = \sum_{e\in E_h} \left| \frac{\tilde{v}_h^W - \tilde{v}_h^E}{h_e} \right|^2 \mu(e) h_e. \tag{8}$$

Proof . In fact, for our case $c^* = 1$ and coercivity holds with equality. Indeed, by discrete integration by parts we have for all $\tilde{v}_h \in \tilde{M}_h$

$$\begin{aligned}\tilde{A}_h(\tilde{v}_h, \tilde{v}_h) &= \sum_{C_i} \sum_{e\in E_h\cap\partial C_i} s_{i,e} \frac{\tilde{v}_h^W - \tilde{v}_h^E}{h_e} (\tilde{v}_h)_i \mu(e) \\ &= \sum_{e\in E_h} \frac{\tilde{v}_h^W - \tilde{v}_h^E}{h_e} (\tilde{v}_h^W - \tilde{v}_h^E) \mu(e) \\ &= \sum_{e\in E_h} \left| \frac{\tilde{v}_h^W - \tilde{v}_h^E}{h_e} \right|^2 \mu(e) h_e,\end{aligned}$$

which finishes the proof.

■

Theorem 4.9. For the finite volume scheme on a generalized admissible mesh (cf. Definition 4.1) with the numerical flux $\tilde{\Phi}(\tilde{u}_h, \mathbf{n}_{\tilde{e}})$ from Definition 4.3 and the mapping $\tilde{Q}_h$ from Definition 4.4 we have for the solution $u \in H^2(\Omega)$ of the continuous problem (2) and $\tilde{u}_h \in \tilde{M}_h$ the solution of the discrete problem (7) the following error estimate:

$$\tilde{A}_h(\tilde{u}_h - \tilde{Q}_h u, \tilde{u}_h - \tilde{Q}_h u) \le C h^2 \|u\|^2_{2,\Omega},$$

where $C > 0$ is independent of h.

Proof . The solutions $u \in H^2(\Omega)$ and $\tilde{u}_h \in \tilde{M}_h$ of (2) and (7), respectively, satisfy

$$\tilde{A}(u, \tilde{v}_h) = (f, \tilde{v}_h)_{L^2(\Omega)} \quad \text{and} \quad \tilde{A}_h(\tilde{u}_h, \tilde{v}_h) = (f, \tilde{v}_h)_{L^2(\Omega)} \quad \forall \tilde{v}_h \in \tilde{M}_h.$$

Thus, we have

$$\tilde{A}(u, \tilde{v}_h) = \tilde{A}_h(\tilde{u}_h, \tilde{v}_h) \quad \forall \tilde{v}_h \in \tilde{M}_h.$$

Choosing $\tilde{v}_h = \tilde{u}_h - \tilde{Q}_h u$ and using triangle inequality and discrete integration by parts we get

$$\tilde{A}_h(\tilde{u}_h - \tilde{Q}_h u, \tilde{u}_h - \tilde{Q}_h u) = \tilde{A}(u, \tilde{u}_h - \tilde{Q}_h u) - \tilde{A}_h(\tilde{Q}_h u, \tilde{u}_h - \tilde{Q}_h u)$$

$$= \tilde{A}(u, \tilde{v}_h) - \tilde{A}_h(\tilde{Q}_h u, \tilde{v}_h)$$
$$= \left| -\sum_{C_i} \sum_{e \in E_h \cap C_i} s_{i,e} \left(\frac{1}{\mu(e)} \int_{\tilde{e}=\theta(e)} \nabla u \, \mathbf{n}_{\tilde{e}} d\Gamma - \frac{(\tilde{Q}_h u)^E - (\tilde{Q}_h u)^W}{h_e} \right) \mu(e)(\tilde{v}_h)_i \right|$$
$$\leq \sum_{e \in E_h} \left| \frac{1}{\mu(e)} \int_{\tilde{e}=\theta(e)} \nabla u \, \mathbf{n}_{\tilde{e}} d\Gamma - \frac{(\tilde{Q}_h u)^E - (\tilde{Q}_h u)^W}{h_e} \right| \cdot \left| \frac{\tilde{v}_h^W - \tilde{v}_h^E}{h_e} \right| \mu(e) h_e.$$

Using Definition 4.6 of $\tilde{R}_e(u)$, definition (8), Cauchy-Schwarz inequality and the fact that $\mu(e) \leq ch$ and $h_e \leq ch$, it follows from Lemmas 4.7 and 4.8 that

$$\begin{aligned} \tilde{A}_h(\tilde{u}_h - \tilde{Q}_h u, \tilde{u}_h - \tilde{Q}_h u) &\leq \left(\sum_{e \in E_h} |\tilde{R}_e(u)|^2 \right)^{\frac{1}{2}} \|\tilde{u}_h - \tilde{Q}_h u\|_{1,h} \\ &\leq Ch \|u\|_{2,\Omega} \, \|\tilde{u}_h - \tilde{Q}_h u\|_{1,h} \\ &\leq Ch \|u\|_{2,\Omega} \, \tilde{A}_h(\tilde{u}_h - \tilde{Q}_h u, \tilde{u}_h - \tilde{Q}_h u)^{\frac{1}{2}}. \end{aligned}$$

Cancelling $\tilde{A}_h(\tilde{u}_h - \tilde{Q}_h u, \tilde{u}_h - \tilde{Q}_h u)^{\frac{1}{2}}$ completes the proof.

■

5 Final remarks

5.1 Weak approximation property

In the previous sections we have given the analysis of a finite-volume scheme for Poisson problem. The interest of this paper lies in the choice of the mappings $Q_h, \tilde{Q}_h : H^2 \to M_h$ which are stable in L^2 and preserve the constant. As a result, we can also state the important weak approximation property of Q_h or $\tilde{Q}_h$ on the spaces M_h or $\tilde{M}_h$, respectively.

Lemma 5.1. (weak approximation property) Let M_h be the finite-volume space as in Definition 2.1 and $Q_h : H^2 \to M_h$ be defined as in Definition 3.1. Further, we consider that Assumptions 3.2 and 4.2 are verified. Then there exists a constant $C > 0$ independent of h such that for all $u \in H^1(\Omega)$ there is

$$\|(I - Q_h)u\|_{0,\Omega} \leq Ch|u|_{1,\Omega},$$

where $\| \cdot \|_{0,\Omega}$ is the standard norm on $L^2(\Omega)$ and $| \cdot |_{1,\Omega}$ is the H^1-seminorm. The same holds if we replace Q_h by $\tilde{Q}_h$ and M_h by $\tilde{M}_h$.

Proof . From Lemma 3.3 and Definition 3.1 the Q_h preserves constant functions, ie. for all constant functions c over Ω there is

$$(I - Q_h)u = (I - Q_h)(u - c).$$

Hence choosing $c = \mu(\Omega)^{-1} \int_\Omega u d\Omega$ and using L^2 stability of Q_h and scaled Poincaré inequality we get

$$\begin{aligned}\|(I - Q_h)u\|_{0,\Omega} = \|(I - Q_h)(u - c)\|_{0,\Omega} &\leq C\|u - c\|_{0,\Omega} \\ &\leq Ch|u|_{1,\Omega},\end{aligned}$$

which is the statement of the lemma.

■

5.2 Weaker consistency of fluxes

It is clear, that the crucial point in the proof of Lemma 3.9 is the application of Bramble-Hilbert lemma locally for each cell interface combined with the fact that the numerical fluxes $\Phi(u, \mathbf{n}_e)$ are calculated exactly for linear functions u, which implies that $R_e(u) = 0$ for those functions. This property depends on the differenciation scheme used to compute the approximation of ∇u in the numerical fluxes $\Phi(u, \mathbf{n}_e)$, more precisely on the fact that for each cell interface $e \in E_h$ the line connecting the flux-evaluation points x^W and x^E is perpendicular to the cell interface e. This property can, however, be violated on a certain number of cell interfaces. As soon as we know for these interfaces that $R_e(u) = 0$ for at least constant u's, we can employ the technique mentioned in [7, 8] using the so called Il'in inequality to get convergence of the finite-volume scheme of somewhat lower order in h than in Theorem 3.10.

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