University of West Bohemia in Plzeň
Faculty of Applied Sciences

Overlapping Schwarz Domain Decomposition With Coarse Space Appropriate to Linear Elasticity

Master thesis

Rychlý paralelní řešič pro úlohy lineární elasticity

Diplomová práce

Aleš Janka

Declaration

I, Aleš Janka, declare that the contents on this publication is entirely my own work, except when I have clearly indicated otherwise.

I have only used the literature inclued in the Bibliography when completing research for this work.

I wish to express my sincerest gratitude to my advisor Dr. Ing. Petr Vaněk from the University of Colorado in Denver, whose encouragement, patience and expert guidance have made this diploma work possible.

I would also like to honour the memory of RNDr. Jitka Křížková, CSc. who introduced me to the secrets of numerical mathematics.

Předkládám k posouzení a obhajobě diplomovou práci vypracovanou na závěr studia na Fakultě aplikovaných věd Západočeské univerzity v Plzni. Prohlašuji, že jsem předkládanou diplomovou práci vypracoval samostatně s použitím odborné literatury a pramenů, jejichž úplný seznam je její součástí.

Chtěl bych poděkovat mému školiteli, Dr. Ing. Petru Vaňkovi, za podporu, odborné vedení a péči, kterou mi věnoval. Chtěl bych také vzpomenout na paní doc. RNDr. Jitku Křížkovou,CSc., která mne zasvětila do tajů numerické matematiky.

Plzeň, 20. 5. 1998

Contents

1 Introduction

Many problems in both linear and nonlinear mechanics reduce to the numerical solution of large, sparse, unstructured, poorly conditioned linear systems. The problem-size and lack of structure prevent to solve these systems by direct solvers. Also, their large condition number makes it hard to solve them by purely iterative solvers. In this framework, domain decomposition techniques are attractive as they combine direct solvers used at the subdomain level and iterative solvers used at the interface level. Moreover, these techniques are ideally suited to modern parallel computers, because of the built-in parallelism of the algorithm and the good localisation of the associated data.

The primary objective of domain decomposition methods is the efficient solution of partial differential equations set on complex geometries and approximated on very fine grids. This objective is achieved by splitting the original domain of computation in smaller, simpler subdomains, computing local simplified solutions, and using efficient algebraic iterative solvers to interface these local solutions.

Here, we will design a black-box overlapping Schwarz method of domain decomposition with an efficient coarse space. The overlapping subdomains will be created purely by algebraic means using a smoothed aggregation technique proposed by Mandel and Vaněk in [1]. We will start from a system of nonoverlapping subdomains (aggregates) and we will obtain the overlapping subdomains by a special smoothing procedure based on the stiffness matrix A.

The coarse space of possibly small dimension is designed by smoothing the tentative coarse space basis functions (tentative prolongator) by a polynomial prolongator smoother. This prolongator smoother will transform the above mentioned aggregates into a system of overlapping computational subdomains, and at the same time it will ensure that the resulting smoothed coarse space basis functions have sufficiently small energy.

Keeping in mind, that we shall design a black-box solver, we will avoid any assumption on geometry of the whole domain, on the measure and quality of Dirichlet's boundary conditions, or on the fine or coarse space resolutions. However, we can make some geometrical assumptions on subdomains and aggregates, because we can influence their shape by the way they are created in the solver - we propose a simple algorithm generating subdomains which satisfy the assumed properties.

Uniform convergence is proved for second order elliptic problems of linear elasticity discretized on quasiuniform P1 and Q1 finite element meshes. Except for the ellipticity and quasiuniformity, our theory does not use any other assumption.

The paper is organized as follows: in the Section 2 we describe the problem and we will mention the cornerstones for stating the theorem on uniform convergence. In Section 3 we will verify assumptions of the convergence theorem with the help of the technical tools listed in Section 4. Section 5 describes the construction of the coarse space based on smoothed aggregation of nodes. Here, we describe a smoothing procedure allowing us to satisfy the assumptions intruduced in Section 3. Verifying the assumptions on the smoother from the Section 5 is the subject of Section 6. Section 7 gives an altervative to Section 3 - it contains verification of assumptions in the convergence theorem solely by algebraic means, giving even better results than Section 3. Section 8 provides an overview of the used algorithms followed by computational complexity estimates for our implementation of the method. Finally, Section 9 demonstrates the applicability of the method on several numerical experiments.

2 Preliminaries

In this section, fundamentals of the convergence theory are presented. Following Vaněk in [2] and Pasciak, Bramble, Wang and Xu in [4], [5], and [6], we are going to state the Theorem 2.11 about the h and H independent rate of convergence of the method. All proofs are provided for completeness.

Definition 2.1 *(problem setting) Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a Lipschitz domain and $\mathcal{T}_h$ be a quaziuniform finite-element mesh on Ω of a characteristic meshsize h. Let V be a P1 or Q1 finite-element space associated with the mesh $\mathcal{T}_h$ with homogeneous Dirichlet boundary conditions imposed on $\Gamma_D \subset \partial\Omega$. We consider the finite element discretization*

$$Ax = b$$

of the following elliptic problem: Find $u \in V$ such that

$$(u, v)_\varepsilon = (f, v)_{[L^2(\Omega)]^d} \quad \forall v \in V,$$

where

$$(u, v)_\varepsilon = \int \sum_{i,j=1}^{3} \varepsilon_{ij}(u)\varepsilon_{ij}(v) d\Omega, \qquad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Definition 2.2 *Let J be the number of subspaces involved in our analysis. Let us define the index set I, $I = \{1, 2, \ldots, J\}$.*

Let V be a Hilbert space equipped with an energy dot product $a(\cdot,\cdot) = (\cdot,\cdot)_\varepsilon$ and a corresponding norm $|\cdot|_\varepsilon$ and let V_i be subspaces of V,

$$V : V = \sum_{i \in I} V_i$$

Let us further have operators $T_i : V \to V_i$, T_i is an ε-orthogonal projection onto V_i.

Further, set

$$E_i = \prod_{j=1}^{i} (I - T_j) = (I - T_i) \cdot (I - T_{i-1}) \cdots (I - T_2) \cdot (I - T_1), \qquad i \in I$$

and put $E_0 = I$, $E_J = E$.

Lemma 2.3 *(Lions) Let V, V_i, I, $(\cdot,\cdot)_\varepsilon$ and T_i be as in Definition 2.2. Let us further have an operator T,*

$$T : V \to V, \qquad T \equiv \sum_{i \in I} T_i \quad ,$$

where T_i are ε-orthogonal projectors onto V_i.

Then if we find a constant $C_L > 0$ such that

$$\forall v \in V \quad \exists \{v_i \in V_i\}_{i \in I} : \qquad v = \sum_{i \in I} v_i \qquad \text{and}$$

$$\sum_{i \in I} (v_i, v_i)_\varepsilon \le C_L (v, v)_\varepsilon \tag{1}$$

then

$$(Tv, v)_\varepsilon \ge \frac{1}{C_L} (v, v)_\varepsilon \tag{2}$$

Proof

$$
\begin{aligned}
(v,v)_\varepsilon &= \Big(\sum_{i\in I} v_i, v\Big)_\varepsilon = \Big(\sum_{i\in I} T_i v_i, v\Big)_\varepsilon \\
&= \sum_{i\in I} (T_i v_i, v)_\varepsilon = \sum_{i\in I} (v_i, T_i v)_\varepsilon \\
&\le \text{ using twice Schwarz inequality for } (\cdot,\cdot)_\varepsilon \text{ and for } \textstyle\sum(\cdot,\cdot)^{\frac{1}{2}}(\cdot,\cdot)^{\frac{1}{2}} : \\
&\le \sum_{i\in I} (v_i, v_i)_\varepsilon^{\frac{1}{2}} \cdot (T_i v, v)_\varepsilon^{\frac{1}{2}} \le \Big(\sum_{i\in I} (v_i, v_i)_\varepsilon\Big)^{\frac{1}{2}} \cdot \Big(\sum_{i\in I} (T_i v, v)_\varepsilon\Big)^{\frac{1}{2}} \\
&\le \text{ using the assumption (1):} \\
&\le C_L^{\frac{1}{2}} (v,v)_\varepsilon^{\frac{1}{2}} \cdot (Tv, v)_\varepsilon^{\frac{1}{2}}
\end{aligned}
$$

Now, we cancel $(v,v)_\varepsilon^{\frac{1}{2}}$ from both sides of the above inequality and get

$$(v,v)_\varepsilon \le C_L \,(Tv,v)_\varepsilon$$

And thus, (2) is proved.

■

Definition 2.4 *(interaction property) Let us have operators T_i, spaces V_i and the index set I as in Definition 2.2. Let us define new, possibly empty index sets I_1, I_2 such that $I_1 = \{1, 2, \ldots, N\}$ and $I_2 = \{N+1, \ldots, J\}$ for some $N \in I$.*

Let $\gamma = |I_2|$ and let us define a symmetric matrix $\boldsymbol{\epsilon} = \{\varepsilon_{ij}\}_{i,j\in I_1}$ by

$$\varepsilon_{ij} = \begin{cases} 0, & \text{if } (V_i, V_j)_\varepsilon = 0, \qquad i,j \in I_1 \\ 1, & \text{otherwise} \end{cases}$$

Also, let $\rho(\boldsymbol{\epsilon})$ be the spectral radius of the matrix $\boldsymbol{\epsilon}$.

Remark 2.5 Note that we have excluded some subspaces from the definition of the interaction matrix $\boldsymbol{\epsilon}$, namely those V_i's for which $i \in I_2$. This was done with the intention to keep $(\rho(\boldsymbol{\epsilon}) + \gamma)$ small, because our estimates of $\|E\|_\varepsilon$ (as it is to be seen below) will depend on $(\rho(\boldsymbol{\epsilon}) + \gamma)$.

Lemma 2.6 *Let T_i and E_i be as in Definition 2.2. Then*

$$\sum_{i=1}^{J} (T_i E_{i-1} v, E_{i-1} v)_\varepsilon \le \|v\|_\varepsilon^2 - \|Ev\|_\varepsilon^2 \tag{3}$$

Proof It follows from the Definition 2.2 that:

$$E_{i-1} - E_i = T_i E_{i-1}, \qquad i = 1, 2, \ldots, J \tag{4}$$

and by summing over i we get:

$$I - E_i = \sum_{j=1}^{i} T_j E_{j-1} \tag{5}$$

From (4) it follows that

$$\|E_{i-1}v\|_\varepsilon^2 - \|E_i v\|_\varepsilon^2 = \|T_i E_{i-1} v\|_\varepsilon^2 + 2\left(T_i E_{i-1} v, E_i v\right)_\varepsilon \tag{6}$$

and from (4) we get (using the self-adjointness of T_i):

$$\begin{aligned}(T_i E_{i-1} v, E_i v)_\varepsilon &= (T_i E_{i-1} v, (I - T_i) E_{i-1} v)_\varepsilon \\ &= (T_i (I - T_i) E_{i-1} v, E_{i-1} v)_\varepsilon\end{aligned} \tag{7}$$

Hence the right-hand side of (6) can be bounded (using the self-adjointness of T_i) by:

$$\begin{aligned}\|T_i E_{i-1} v\|_\varepsilon^2 + 2\left(T_i E_{i-1} v, E_i v\right)_\varepsilon &= ((2I - T_i) T_i E_{i-1} v, E_{i-1} v)_\varepsilon \\ &\geq \left(2 - \|T_i\|_\varepsilon\right)\left(T_i E_{i-1} v, E_{i-1} v\right)_\varepsilon\end{aligned} \tag{8}$$

and thus combining (8) and (6) with the fact that $\|T_i\|_\varepsilon \leq 1$ and summing over $i = 1 \dots J$ completes the proof.

■

Proposition 2.7 *It follows from the definition of the interaction matrix (Definition 2.4) that for T_i, T_j, $i, j \in I_1$ a strengthened Schwarz inequality holds:*

$$\left(T_i u_i, T_j v_j\right)_\varepsilon^2 \leq \varepsilon_{ij}\left(T_i u_i, u_i\right)_\varepsilon \cdot \left(T_j v_j, v_j\right)_\varepsilon, \qquad \forall i, j \in I_1 \tag{9}$$

Lemma 2.8 *Let us first define index sets S, S_{11}, S_{22} and S_{21}: $S = I \times I$, $S_{11} = I_1 \times I_1$, $S_{22} = I_2 \times I_2$, $S_{21} = I_2 \times I_1$. Then the following estimate holds:*

$$\sum_{\substack{(i,j)\in S \\ i>j}} (T_i u_i, T_j v_j)_\varepsilon \leq \Big(\rho(\boldsymbol{\epsilon}) + \gamma - 1\Big) \cdot \Big(\sum_{i\in I} (T_i u_i, u_i)_\varepsilon\Big)^{\frac{1}{2}} \Big(\sum_{j\in I} (T_j v_j, v_j)_\varepsilon\Big)^{\frac{1}{2}} \tag{10}$$

Proof First, let us prove the following:

$$\Big(\sum_{\substack{(i,j)\in S_{11} \\ i>j}} (T_i u_i, T_j v_j)_\varepsilon\Big)^2 \leq \Big(\rho(\boldsymbol{\epsilon}) - 1\Big)^2 \cdot \sum_{i\in I_1} (T_i u_i, u_i)_\varepsilon \cdot \sum_{j\in I_1} (T_j v_j, v_j)_\varepsilon \tag{11}$$

$$\Big(\sum_{\substack{(i,j)\in S_{22} \\ i>j}} (T_i u_i, T_j v_j)_\varepsilon\Big)^2 \leq \gamma(\gamma - 1) \cdot \sum_{i\in I_2} (T_i u_i, u_i)_\varepsilon \cdot \sum_{j\in I_2} (T_j v_j, v_j)_\varepsilon \tag{12}$$

$$\Big(\sum_{\substack{(i,j)\in S_{21} \\ i>j}} (T_i u_i, T_j v_j)_\varepsilon\Big)^2 \leq \gamma\Big(2\rho(\boldsymbol{\epsilon}) - 1\Big) \cdot \sum_{i\in I_2} (T_i u_i, u_i)_\varepsilon \cdot \sum_{j\in I_1} (T_j v_j, v_j)_\varepsilon \tag{13}$$

Inequality (11) follows from (9) in the following way:

$$\begin{aligned}\Big(\sum_{\substack{(i,j)\in S_{11} \\ i>j}} (T_i u_i, T_j v_j)_\varepsilon\Big)^2 &\leq \Big(\sum_{\substack{(i,j)\in S_{11} \\ i>j}} \varepsilon_{ij} (T_i u_i, u_i)_\varepsilon^{\frac{1}{2}} (T_j v_j, v_j)_\varepsilon^{\frac{1}{2}}\Big)^2 \\ &\leq \Big(\sum_{\substack{(i,j)\in S_{11} \\ i\neq j}} \varepsilon_{ij} (T_i u_i, u_i)_\varepsilon^{\frac{1}{2}} (T_j v_j, v_j)_\varepsilon^{\frac{1}{2}}\Big)^2 \\ &\leq \Big(\sum_{(i,j)\in S_{11}} \varepsilon_{ij} (T_i u_i, u_i)_\varepsilon^{\frac{1}{2}} (T_j v_j, v_j)_\varepsilon^{\frac{1}{2}} - \sum_{\substack{(i,j)\in S_{11} \\ i=j}} (T_i u_i, u_i)_\varepsilon^{\frac{1}{2}} (T_j v_j, v_j)_\varepsilon^{\frac{1}{2}}\Big)^2\end{aligned}$$

$$\leq \left(\Big(\rho(\boldsymbol{\epsilon})-1\Big)\sum_{\substack{i\in I_1\\ i=j}}(T_iu_i,u_i)_\varepsilon^{\frac12}\,(T_jv_j,v_j)_\varepsilon^{\frac12}\right)^2$$

$$\leq \text{ using Schwarz inequality}$$

$$\leq \Big(\rho(\boldsymbol{\epsilon})-1\Big)^2\cdot\sum_{i\in I_1}(T_iu_i,u_i)_\varepsilon\cdot\sum_{j\in I_1}(T_jv_j,v_j)_\varepsilon$$

which proves (11).

Now, the inequality (12):

$$\left(\sum_{\substack{(i,j)\in S_{22}\\ i>j}}(T_iu_i,T_jv_j)_\varepsilon\right)^2 \leq \left(\sum_{i\in I_2}(T_iu_i,u_i)_\varepsilon^{\frac12}\left[\sum_{\substack{j\in I_2\\ j<i}}(T_jv_j,v_j)_\varepsilon^{\frac12}\right]\right)^2$$

$$\leq \text{ using Schwarz inequality for } \textstyle\sum(\cdot,\cdot)^{\frac12}(\cdot,\cdot)^{\frac12}$$

$$\leq \sum_{i\in I_2}(T_iu_i,u_i)_\varepsilon\sum_{i\in I_2}\left(\sum_{\substack{j\in I_2\\ j<i}}(T_jv_j,v_j)_\varepsilon^{\frac12}\right)^2$$

$$\leq \text{ using Schwarz inequality for } \textstyle\sum 1\cdot(\cdot,\cdot)^{\frac12}$$

$$\leq \sum_{i\in I_2}(T_iu_i,u_i)_\varepsilon\sum_{i\in I_2}\left(\sum_{\substack{j\in I_2\\ j<i}}1\cdot\sum_{\substack{j\in I_2\\ j<i}}(T_jv_j,v_j)_\varepsilon\right)$$

$$\leq (\gamma-1)\sum_{i\in I_2}(T_iu_i,u_i)_\varepsilon\sum_{i\in I_2}\sum_{j\in I_2}(T_jv_j,v_j)_\varepsilon$$

$$\leq \gamma(\gamma-1)\sum_{i\in I_2}(T_iu_i,u_i)_\varepsilon\sum_{j\in I_2}(T_jv_j,v_j)_\varepsilon$$

which proves (12).

Finally, using that T_i is a projector and applying Schwarz inequality to (13) we get:

$$\left(\sum_{\substack{(i,j)\in S_{21}\\ i>j}}(T_iu_i,T_jv_j)_\varepsilon\right)^2 \leq \left(\sum_{i\in I_2}\Big(T_iu_i,\sum_{\substack{j\in I_1\\ j<i}}T_jv_j\Big)_\varepsilon\right)^2$$

$$\leq \left(\sum_{i\in I_2}(T_iu_i,u_i)_\varepsilon^{\frac12}\left(\sum_{\substack{j\in I_1\\ j<i}}T_jv_j,\sum_{\substack{k\in I_1\\ k<i}}T_kv_k\right)_\varepsilon^{\frac12}\right)^2$$

$$\leq \text{ using Schwarz inequality for } \textstyle\sum(\cdot,\cdot)^{\frac12}(\cdot,\cdot)^{\frac12}$$

$$\leq \sum_{i\in I_2}(T_iu_i,u_i)_\varepsilon\sum_{i\in I_2}\left(\sum_{\substack{j\in I_1\\ j<i}}T_jv_j,\sum_{\substack{k\in I_1\\ k<i}}T_kv_k\right)_\varepsilon$$

$$= \gamma\sum_{i\in I_2}(T_iu_i,u_i)_\varepsilon\left(\sum_{\substack{j\in I_1\\ j<i}}T_jv_j,\sum_{\substack{k\in I_1\\ k<i}}T_kv_k\right)_\varepsilon$$

Now, let us investigate the ε-product of sums on the right-hand side:

$$\left(\sum_{\substack{j\in I_1\\ j<i}}T_jv_j,\sum_{\substack{k\in I_1\\ k<i}}T_kv_k\right)_\varepsilon = \sum_{\substack{j\in I_1\\ j<i}}\sum_{\substack{k\in I_1\\ k<i}}(T_jv_j,T_kv_k)_\varepsilon$$

$$= 2 \sum_{\substack{j\in I_1\\ j<i}} \sum_{\substack{k\in I_1\\ j>k}} (T_j v_j, T_k v_k)_\varepsilon + \sum_{\substack{j\in I_1\\ j<i}} (T_j v_j, T_j v_j)_\varepsilon$$

$$\leq \text{ using (11) for the first term}$$

$$\leq 2\Big(\rho(\boldsymbol{\epsilon}) - 1\Big) \sum_{j\in I_1} (T_j v_j, v_j)_\varepsilon + \sum_{j\in I_1} (T_j v_j, v_j)_\varepsilon$$

$$\leq \Big(2\rho(\boldsymbol{\epsilon}) - 1\Big) \sum_{j\in I_1} (T_j v_j, v_j)_\varepsilon$$

Combining the two latter inequalities gives the proof of (13).

The statement (10) of the lemma follows easilly from (11), (12), (13) when realizing that

$$\sum_{\substack{(i,j)\in S\\ i>j}} (T_i u_i, T_j v_j)_\varepsilon = \sum_{\substack{(i,j)\in S_{11}\\ i>j}} (T_i u_i, T_j v_j)_\varepsilon + \sum_{\substack{(i,j)\in S_{22}\\ i>j}} (T_i u_i, T_j v_j)_\varepsilon + \sum_{\substack{(i,j)\in S_{21}\\ i>j}} (T_i u_i, T_j v_j)_\varepsilon$$

$$\leq \Big(\rho(\boldsymbol{\epsilon}) - 1\Big)\Big(\sum_{i\in I_1} (T_i u_i, u_i)_\varepsilon \cdot \sum_{j\in I_1} (T_j v_j, v_j)_\varepsilon\Big)^{\frac{1}{2}}$$

$$+\Big(\gamma(\gamma - 1)\Big)^{\frac{1}{2}} \Big(\sum_{i\in I_2} (T_i u_i, u_i)_\varepsilon \sum_{j\in I_2} (T_j v_j, v_j)_\varepsilon\Big)^{\frac{1}{2}}$$

$$+\Big(\gamma\Big(2\rho(\boldsymbol{\epsilon}) - 1\Big)\Big)^{\frac{1}{2}} \Big(\sum_{i\in I_2} (T_i u_i, u_i)_\varepsilon \cdot \sum_{j\in I_1} (T_j v_j, v_j)_\varepsilon\Big)^{\frac{1}{2}}$$

$$\leq \text{ using Schwarz inequality for } \mathbb{R}^3$$

$$\leq \Big[(\rho(\boldsymbol{\epsilon}) - 1)^2 + \gamma(\gamma - 1) + \gamma\,[2\rho(\boldsymbol{\epsilon}) - 1]\Big]^{\frac{1}{2}} \Big(\sum_{i\in I} (T_i u_i, u_i)_\varepsilon \cdot \sum_{j\in I} (T_j v_j, v_j)_\varepsilon\Big)^{\frac{1}{2}}$$

$$= \Big(\rho(\boldsymbol{\epsilon}) + \gamma - 1\Big) \cdot \Big(\sum_{i\in I} (T_i u_i, u_i)_\varepsilon\Big)^{\frac{1}{2}} \Big(\sum_{j\in I} (T_j v_j, v_j)_\varepsilon\Big)^{\frac{1}{2}}$$

Which concludes the proof.

■

Lemma 2.9 *(Uniform bound for E) Let $\rho(\boldsymbol{\epsilon})$ and γ be defined as in Definition 2.4 and let E be defined as in Definition 2.2. Then if we have $C_L > 0$, C_L being the constant from the Lions' lemma (Lemma 2.3), and if we have the strengthened Schwarz inequality (9) then*

$$\|E\|_\varepsilon^2 \leq 1 - \frac{1}{C_L\Big[\gamma + \rho(\boldsymbol{\epsilon})\Big]^2} \tag{14}$$

Proof It is easy to see, that it suffices to prove

$$\|v\|_\varepsilon^2 \leq C_L\Big[\gamma + \rho(\boldsymbol{\epsilon})\Big]^2\Big(\|v\|_\varepsilon^2 - \|Ev\|_\varepsilon^2\Big)$$

Using Lions' lemma (Lemma 2.3) and Lemma 2.6 we can say, that the lemma would be proved, if we show that

$$\sum_{i\in I} (T_i v, v)_\varepsilon \leq \Big[\gamma + \rho(\boldsymbol{\epsilon})\Big]^2 \sum_{i\in I} (T_i E_{i-1} v, E_{i-1} v)$$

By (5) we have:

$$\begin{aligned}\sum_{i\in I}(T_i v, v)_\varepsilon &= \sum_{i=1}^{J}(T_i v, E_{i-1}v)_\varepsilon + \sum_{i=1}^{J}\sum_{j=1}^{i-1}(T_i v, T_j E_{j-1}v)_\varepsilon \\ &\leq \text{using Schwarz inequality and Lemma 2.8} \\ &\leq \Big(\sum_{i=1}^{J}(T_i v, v)_\varepsilon\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{J}(T_i E_{i-1}v, E_{i-1}v)_\varepsilon\Big)^{\frac{1}{2}} \\ &\quad + \Big(\rho(\boldsymbol{\epsilon}) + \gamma - 1\Big)\Big(\sum_{i=1}^{J}(T_i v, v)_\varepsilon\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{J}(T_i E_{i-1}v, E_{i-1}v)_\varepsilon\Big)^{\frac{1}{2}}\end{aligned}$$

Summing and cancelling $\Big(\sum_{i=1}^{J}(T_i v, v)_\varepsilon\Big)^{\frac{1}{2}}$ finishes the proof.

■

Remark 2.10 So far, we have devised an abstract convergence theory for broad scope of methods. Now, we are going to confine ourselves to Schwarz overlapping domain decomposition method with one coarse space. Thus, $N = J - 1$ (from Definition 2.4) is now the number of fine-level subdomains, $J = N + 1$ is the total number of subspaces including the coarse space. The index set I_1 includes indices of all fine-level subdomains, and $I_2 = \{J\}$, ie. it contains just the coarse-space, which means that $\gamma = 1$. The following convergence theorem is just a re-statement of Lemma 2.9 for our purposes.

Theorem 2.11 *(Uniform bound for E) Let $\rho(\boldsymbol{\epsilon})$ be defined as in Definition 2.4 and let E be the error-propagation operator for multiplicative (alternating) Schwarz domain decomposition with one coarse-space. Then if we have $C_L > 0$, C_L being the constant from the Lions' lemma (Lemma 2.3), and if we have the strengthened Schwarz inequality (9) for fine-level subdomains then*

$$\|E\|_\varepsilon^2 \leq 1 - \frac{1}{C_L\Big[1 + \rho(\boldsymbol{\epsilon})\Big]^2} \tag{15}$$

Remark 2.12 Theorem 2.11 provides us with a uniform bound for E, provided that neither C_L nor $\rho(\boldsymbol{\epsilon})$ depend on h or H (fine and coarse-level resolutions). We can control $\rho(\boldsymbol{\epsilon})$ (by Gershgorin's theorem) with a maximum number of subdomains having non-empty intersection with one arbitrary subdomain, which does not depend on h or H. Proving the h and H independence of C_L for one particular choice of the coarse-space is the main objective of this work.

Remark 2.13 Now, it suffices to prove the assumptions of the Lions' lemma (Lemma 2.3) for our particular case, ie. the fine-level subdomains and the coarse-space appropriate to linear elasticity and our work would be thus finished.

3 Estimates for Overlapping Schwarz with Coarse Space

The purpose of this section is to specify requirements on the coarse space and overlapping fine-level subdomains that will allow us to prove uniform convergence. Requirements on the coarse space and the smoother will be formulated in terms of coarse-space basis functions.

Keeping in mind that our goal is to design a black-box solver and following Vaněk and Brezina in [2], we avoid any assumption on the L^∞ boundedness of the gradient of basis functions.

As we have already stated in the previous sections, it is sufficient to verify the assumption of the Lions' lemma, namely that there exists such a constant C_L independent of h and H such that

$$\sum_{i\in I}(v_i,v_i)_\varepsilon \le C_L\,(v,v)_\varepsilon$$

ie. our task is to find some decomposition $\{v_i\}$ of v such that the above holds. This is the same as finding operators $Q_i : V \to V_i$, $Q_i v = v_i$, $i \in I$ (ie. both for the fine-level subdomains and the coarse-space).

3.1 Fine-level subdomains

Let us suppose for the moment that we have already constructed the projector Q_0 onto the coarse space. The following should state what properties we need when verifying assumptions of Lions' lemma for fine-level subdomains. But first, let us state one general assumption on subdomain geometry (cf. [2]):

Assumption 3.1 (subdomain geometry) Let Ω be a union of simply connected clusters Ω_k of finite elements such that

1. $\operatorname{diam}(\Omega_k) \le CH, \quad k \in I_1$
2. $\forall x \in \Omega \quad \exists \Omega_k : x \in \Omega_k \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega_k \setminus \partial\Omega) \ge cH, \quad k \in I_1$
3. For each number $C_R > 0$ there is a number K such that for every $x \in \Omega$ the ball

   $$B(x, C_R H) = \{y \in \Omega : \operatorname{dist}(y,x) \le C_R H\},$$

   intersects at most K subdomains Ω_k (ie. an object of a diameter $O(H)$ intersects at most $O(1)$ subdomains).
4. $\operatorname{meas}(\Omega_k) \ge CH^d, \quad k \in I_1$, where $d \in \{2,3\}$ is the dimension of $\Omega \subset \mathbb{R}^d$.

Assumption 3.2 (on Q_0 needed for Lions on fine level) Let $Q_0 : V \to V_0$ be the projector of v onto the coarse space. Then for proving the assumptions of Lions' lemma for finite level we need to assume that

1. (approximation property) There exists a constant C independent of h or H such that for $\forall v \in V$ there is

   $$\|Q_0 v\|_{[L^2]^d} \le CH|v|_\varepsilon$$

2. (ε-stability) There exists a constant C independent of h or H such that for $\forall v \in V$ there is

   $$|Q_0 v|_\varepsilon \le C|v|_\varepsilon$$

Lemma 3.3 *Let Assumption 3.1 be satisfied. Then, there exists a set of functions* $\{\psi_k\}_{k\in I_1}$, $\psi_k \in W^{1,\infty}(\Omega)$ *such that*

1. $|\psi_k|_{W^{1,\infty}(\Omega)} \le CH^{-1}$,

2. $\sum_{k \in I_1} \psi_k = 1$ *on* Ω,

3. $\operatorname{supp}(\psi_k) \subset \Omega_k$, $k \in I_1$.

(cf. [2]).

Proof For each Ω_k we define

$$\tilde{\psi}_k(x) = \begin{cases} H^{-1} \operatorname{dist}(x, \partial\Omega_k \setminus \partial\Omega) & \text{for } x \in \Omega_k, \\ 0 & \text{for } x \in \Omega \setminus \Omega_k. \end{cases}$$

Due to Assumption 3.1 1) and 3),

$$\sum_{k \in I_1} \tilde{\psi}_k \leq C \tag{16}$$

and from Assumption 3.1 2) we also have

$$\sum_{k \in I_1} \tilde{\psi}_k \geq c. \tag{17}$$

We will show that

$$|\tilde{\psi}_k|_{W^{1,\infty}} \leq CH^{-1} \tag{18}$$

Consider two nodes $u, v \in \Omega_k$. Without the loss of generality, we assume that $\tilde{\psi}_k(u) \leq \tilde{\psi}_k(v)$. Then we have

$$\tilde{\psi}_k(u) = H^{-1} \operatorname{dist}(u, \partial\Omega_k \setminus \partial\Omega) = H^{-1} \operatorname{dist}(u, P)$$

for some node $P \in \partial\Omega_k \setminus \partial\Omega$. Further,

$$\begin{aligned} \tilde{\psi}_k(v) &= H^{-1} \operatorname{dist}(v, \partial\Omega_k \setminus \partial\Omega) \\ &\leq H^{-1} \operatorname{dist}(v, P) \\ &\leq H^{-1} \Big(\operatorname{dist}(v, u) + \operatorname{dist}(u, P) \Big) \\ &\leq H^{-1} \operatorname{dist}(v, u) + \tilde{\psi}_k(u) \end{aligned}$$

Therefore,

$$|\tilde{\psi}_k|_{Lip(\Omega)} := \sup\{ \frac{|\tilde{\psi}_k(x) - \tilde{\psi}_k(y)|}{\operatorname{dist}(x, y)} : x, y \in \Omega \ ; \ x \neq y \} \leq H^{-1}$$

Now, (18) follows from the well-known equivalence $|\cdot|_{Lip(\Omega)} \approx |\cdot|_{W^{1,\infty}}$.

Let us define

$$w(x) = \frac{1}{\sum_{k \in I_1} \tilde{\psi}_k(x)}, \quad x \in \bar{\Omega}$$

Due to (16) and (17),

$$\|w\|_{L^\infty(\bar{\Omega})} \leq C. \tag{19}$$

Further from (18), denoting the Euclidean norm in $\mathbb{R}^d$ by $\|\cdot\|$,

$$\begin{aligned} \|\nabla w(x)\| &\leq \|\nabla\Big(\sum_{k \in I_1} \tilde{\psi}_k\Big)(x)\| \left(\min_{y \in \bar{\Omega}} \sum_{k \in I_1} \tilde{\psi}_k(y) \right)^{-2} \\ &\leq \sum_{k : x \in \Omega_k} \|\nabla \tilde{\psi}_k(x)\| \left(\min_{y \in \bar{\Omega}} \sum_{k \in I_1} \tilde{\psi}_k(y) \right)^{-2} \leq CH^{-1} \end{aligned} \tag{20}$$

almost everywhere on Ω. Finally, we set

$$\psi_k = w\tilde{\psi}_k, \quad k \in I_1.$$

Statements 2) and 3) of the lemma are trivially satisfied by functions ψ_k. Further, (18), (19), and (20) imply

$$\begin{aligned}\|\nabla\psi_k(x)\| &= \|w(x)\Big(\nabla\tilde{\psi}_k\Big)(x) + \tilde{\psi}_k(x)\Big(\nabla w\Big)(x)\| \\ &\le \|\nabla\tilde{\psi}_k(x)\| \cdot |w(x)| + |\tilde{\psi}_k(x)| \cdot \|\nabla w(x)\| \\ &\le CH^{-1}\end{aligned}$$

almost everywhere, which proves the statement 1) of the lemma.

■

Definition 3.4 *(fine-level decomposition) Let us define an operator* $I_h : [C^0(\bar{\Omega})]^d \to V$ *as an* ε*-orthogonal projection to fine finite-element space.*

Assuming we have the set $\{\psi_k\}_{k\in I_1}$ *as in Lemma 3.3, we can define the decomposition of* $v \in V$ *onto the fine-level subdomains as:*

$$v_k = Q_k v = I_h\Big(\psi_k(I - Q_0)v\Big), \quad \forall k \in I_1$$

where Q_0 *is an interpolator onto the coarse space.*

Remark 3.5 It is clear from the construction of the Q_i's and from the properties of I_h and $\{\psi_k\}$ (Lemma 3.3 2)), that $\sum\limits_{i\in I} Q_i v = \sum\limits_{i\in I} v_i = v$, which is required by Lions' lemma.

Lemma 3.6 *(verification of Lions' assumptions on fine level) Let* $Q_k : V \to V_k$ *be as in Definition 3.4,* $\{\psi_k\}_{k\in I_1}$ *be as in Lemma 3.3, and let* Q_0 *satisfy the Assumption 3.2, then the following hols for all fine-level subdomains and for all* $v \in V$ *(cf.[2]):*

$$|v_k|_{\varepsilon(\Omega)} = |Q_k v|_{\varepsilon(\Omega_k)} \le C|v|_{\varepsilon(\Omega_k)}, \quad \forall k \in I_1 \ , \tag{21}$$

where C *is independent of* h *and* H*.*

Proof We have for the k-th subdomain, using ε-stability of I_h:

$$\begin{aligned}|v_k|^2_{\varepsilon(\Omega)} &= |v_k|^2_{\varepsilon(\Omega_k)} = |Q_k v|^2_{\varepsilon(\Omega_k)} = |I_h\Big(\psi_k(I - Q_0)v\Big)|^2_{\varepsilon(\Omega_k)} \\ &= \text{ putting } (I - Q_0)v = w \in V \\ &= C|\psi_k w|^2_{\varepsilon(\Omega_k)} = C\int\limits_{\Omega_k}\sum_{i,j=1}^{3}\varepsilon^2_{ij}(\psi_k w)d\Omega\end{aligned} \tag{22}$$

And by investigating only the integrand we have:

$$\begin{aligned}\varepsilon_{ij}(\psi_k w) &= \frac{1}{2}\Big(\frac{\partial}{\partial x_j}(\psi_k w_i) + \frac{\partial}{\partial x_i}(\psi_k w_j)\Big) \\ &= \frac{1}{2}\Big(\frac{\partial\psi_k}{\partial x_j}w_i + \frac{\partial\psi_k}{\partial x_i}w_j + \frac{\partial w_i}{\partial x_j}\psi_k + \frac{\partial w_j}{\partial x_i}\psi_k\Big) \\ &\le \text{using Lemma 3.3} \\ &\le \frac{1}{H}\frac{C}{2}(w_i + w_j) + \frac{1}{2}\Big(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\Big)\end{aligned}$$

Now continuing with (22) we get:

$$\begin{aligned}
\int\limits_{\Omega_k} \sum_{i,j=1}^{3} \varepsilon_{ij}^2(\psi_k w) d\Omega &\leq \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \left(\frac{1}{H} \frac{C}{2}(w_i + w_j) + \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) \right)^2 d\Omega \\
&\leq \frac{C}{H^2} \int\limits_{\Omega_k} \left(\sum_{i=1}^{3} w_i^2 \right) d\Omega + C \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \varepsilon_{ij}^2(w) d\Omega \\
&\leq C \left(\frac{1}{H^2} \|w\|_{[L^2]^d}^2 + |w|_\varepsilon^2 \right) \\
&= C \left(\frac{1}{H^2} \|(I - Q_0)v\|_{[L^2]^d}^2 + |w|_\varepsilon^2 \right) \\
&\leq \text{using Assumption 3.2 1)} \\
&\leq C \Big(|v|_{\varepsilon(\Omega_k)}^2 + |w|_{\varepsilon(\Omega_k)}^2 \Big) \\
&= C \Big(|v|_{\varepsilon(\Omega_k)}^2 + |(I - Q_0)v|_{\varepsilon(\Omega_k)}^2 \Big) \\
&\leq C \Big(|v|_{\varepsilon(\Omega_k)}^2 + |Q_0 v|_{\varepsilon(\Omega)}^2 \Big) \\
&\leq \text{using Assumption 3.2 2)} \\
&\leq C |v|_{\varepsilon(\Omega_k)}^2
\end{aligned}$$

Which together with (22) proves (21).

■

3.2 Coarse space

We were able to verify the assumptions of the Lions' lemma for fine-level subdomains just thanks to the assumed ε-stability and approximation property of the coarse space, or in other words of the interpolator onto the coarse space $Q_0 : V \to V_0$ (Assumption 3.2).

In this section we shall combine the geometrical properties of fine-level subdomains with an assumption on coarse-space basis functions (which follows in Assumption 3.9) to be able to verify the ε-stability and approximation property of Q_0.

It is clear, that the property of ε-stability assumed when dealing with fine-level subdomains coincides with the assumption in Lions' lemma for coarse space V_0.

Definition 3.7 *Let us define $\mathcal{N}_i = \{j : \Omega_j \cap \Omega_i \neq \emptyset\}$. For $i \in I_1$ we define $B_i' = \bigcup_{j \in \mathcal{N}_i} \Omega_j$ and B_i to be a ball circumscribed about B_i'. Also, let us set $\tilde{B}_i = B_i \cap \Omega$. From Assumption 3.1 it immediately follows that*

$$\operatorname{diam}(B_i) \leq CH.$$

Definition 3.8 *Let us denote by Γ_i, $i = 1 \dots G$ the various parts of $\partial\Omega$, with different Dirichlet constraints imposed on them, with the Dirichlet boundary conditions of the form: $g_i^T \cdot v = 0$ on Γ_i, $g_i \in \mathbb{R}^d$, $v \in [H^1(\Omega)]^d$ and let $V_{\Gamma_i}(\Omega)$ be a Hilbert space such that*

$$V_{\Gamma_i}(\Omega) = \{v \in [H^1(\Omega)]^d : g_i^T \cdot v = 0 \ \textit{on}\ \Gamma_i\}$$

Assumption 3.9 (coarse-space basis functions) Let us assume the following properties of the coarse-space basis functions $\Phi^i_j \in V$, i denoting the fine-level subdomain corresponding to this function and $j = \{1 \dots 6\}$ being the number of corresponding local RBM's (rigid body modes):

1. (a) There is a domain Ω^{int} for which for all $r \in \text{RBM}(\Omega^{\text{int}}) \cap V$ there exist $\{\alpha^i_j\}$ so that:
$$\sum_{i \in I_1} \sum_{j=1}^{6} \alpha^i_j \Phi^i_j = r \quad \text{restricted on } \Omega^{\text{int}}$$
and, furthermore, it holds that
$$\text{dist}(x, \Gamma_D) \le CH \tag{23}$$
for all $x \in \Omega \setminus \Omega^{\text{int}}$.
   (b) There exists a set of patches $\{G_i\}$, $G_i \subset \Omega$, $i = 1 \dots G$ and index sets $\mathcal{G}_i$, such that $\mathcal{G}_i = \{j : \Omega_i \cap G_j \neq \emptyset\}$, and for which for $\forall r \in \text{RBM}(\Omega_i) \cap \left(\bigcap_{j \in \mathcal{G}_i} V_{\Gamma_j}\right)$ there exist $\{\alpha^i_j\}$ so that
$$\sum_{k \in \mathcal{N}_i} \sum_{j=1}^{6} \alpha^k_j \Phi^k_j = r \quad \text{restricted on } \Omega_i.$$
and, furthermore, it holds that
$$\text{dist}(x, \Gamma_i) \le CH, \tag{24}$$
for all $x \in G_i$, $i = 1 \dots G$.

   Consequently, we can put $\Omega^{\text{int}} = \Omega \setminus \left(\bigcup_{\forall j} G_j\right)$ to see that 1a) follows from 1b).

   Also, if $\mathcal{G}_i \neq \emptyset$ then for $\forall k \in \mathcal{G}_i$ $\exists j \in \mathcal{N}_i$ such that $\partial\Omega_j \cap \Gamma_k \neq \emptyset$. This holds for all subdomains Ω_i, $i \in I_1$.

2. There exists a correspondence between Φ^i_j and $r^i_j \in \text{RBM}(\Omega_i)$, $i \in I_1$ in the sense that
$$r = \sum_{k \in \mathcal{N}_i} \sum_{j=1}^{6} \alpha^k_j(r) \Phi^k_j \quad \text{restricted on } \Omega_i,$$
ie.
$$r = \sum_{j=1}^{6} \alpha^i_j(r) \Phi^i_j + \sum_{\substack{k \in \mathcal{N}_i \\ k \neq i}} \sum_{j=1}^{6} \alpha^k_j(r) \Phi^k_j \quad \text{restricted on } \Omega_i,$$
where
$$r = \sum_{j=1}^{6} \alpha^i_j(r) r^i_j$$
for all $r \in \text{RBM}(\Omega_i) \cap V$ if $\Omega_i \subset \Omega^{\text{int}}$, or for all $r \in \text{RBM}(\Omega_i) \cap \left(\bigcap_{j \in \mathcal{G}_i} V_{\Gamma_j}\right)$ if $\Omega_i \subset \Omega \setminus \Omega^{\text{int}}$.

3. It holds for all γ^i_j that

   (a) $$\|\sum_{j=1}^{6} \gamma^i_j \Phi^i_j\|_{[L^2(\Omega_i)]^d} \le C \|\sum_{j=1}^{6} \gamma^i_j r^i_j\|_{[L^2(\Omega_i)]^d}$$

(b) $$|\sum_{j=1}^{6}\gamma_j^i\Phi_j^i|_{\varepsilon(\Omega_i)} \leq CH^{-1}\|\sum_{j=1}^{6}\gamma_j^i r_j^i\|_{[L^2(\Omega_i)]^d},$$

where the constants C are independent of H and h.

4. For all $i \in I_1$, there is: $\operatorname{supp}\Phi_j^i \subset \Omega_i \quad j = \{1\dots 6\}$.

Remark 3.10 Assumptions 3.9 1a), and 1b) are to be understood in this sense: the coarse-space basis functions are able to generate all RBM's locally on Ω^{int}. On $\Omega \setminus \Omega^{\text{int}}$, ie. on the CH neighbourhood of Dirichlet's constraints (patches G_i around each type of Dirichlet boundary condition), they can generate only a subspace of local RBM's, because the capability of generating some local RBM's on the patches G_i is suppressed by local Dirichlet boundary condition Γ_i. The patches G_i might overlap each other.

The second part of Assumption 3.9 1) says that if the coarse-space basis cannot generate some RBM's locally on some Ω_i due to Dirichlet constraints, then these constraints are included in the boundaries of the subdomain clusters B_i' and $B_i \cap \Omega$ (see Definition 3.7).

Assumption 3.9 2) relates the coarse-space basis functions $\{\Phi_j^i\}$ with local rigid body modes $\{r_j^i\}$. Assumption 3.9 3) states requirements on the smoother and Assumption 3.9 4) gives hints of how to construct fine-level subdomains.

Definition 3.11 *For convenience of proving, let us consider the extension u_E of the function $u \in V \subset [H^1(\Omega)]^d$ satisfying*

$$|u_E|_{[H^1(\mathbb{R}^d)]^d} \leq C(\Omega)|u|_{[H^1(\Omega)]^d}\ , \quad u_E = u \text{ on } \Omega,$$

which by using the continuity of ε-norm in H^1 and Korn's inequality gives

$$|u_E|_{\varepsilon(\mathbb{R}^d)} \leq C(\Omega)|u|_{\varepsilon(\Omega)}$$

where $C = C(\Omega)$ depends just on shape of Ω and not its size.

Definition 3.12 *(coarse space) Let us define the coarse space V_0, $V_0 \subset V$ by*

$$V_0 = \operatorname{span}\{\Phi_j^i\}_{j=1\dots6}^{i\in I_1}$$

Definition 3.13 *(interpolator Q_0) Let us define the linear interpolator onto the coarse space $Q_0 : V \to V_0$ by*

$$Q_0 v = \sum_{i\in I_1}\sum_{j=1}^{6}\alpha_j^i\Phi_j^i,$$

where $\{\alpha_j^i(v)\}$'s are such that $\sum_{j=1}^{6}\alpha_j^i(v)r_j^i$, $r_j^i \in \mathrm{RBM}(\Omega_i)$ is a L^2-orthogonal projection of v onto $\mathrm{RBM}(\Omega_i)$ for all $i \in I_1$.

Further, let us denote by Q_0^i the restriction of Q_0 onto Ω_i, ie. $Q_0^i v = Q_0 v|_{\Omega_i}$.

Now, having Assumption 3.9 we want to prove the approximation property and the ε-stability of Q_0, which we needed when verifying assumptions of Lions' lemma on fine level (Assumption 3.2). For technical reasons, we will divide this task into two parts - proving of the above properties of Q_0 separately on Ω^{int} and on $\Omega \setminus \Omega^{\text{int}}$.

3.2.1 Coarse space on Ω^{int}

Lemma 3.14 *(approximation property of Q_0 on Ω^{int}) Let us have interpolators Q_0^i as in Definition 3.13, and a coarse space V_0 as in Definition 3.12 satisfying Assumption 3.9, then Q_0^i satisfies the approximation property of Assumption 3.2 1) on Ω_i, ie. for all $\Omega_i \subset \Omega^{\text{int}}$ there is:*

$$\|(I - Q_0^i)v\|_{[L^2(\Omega_i)]^d} \leq CH|v|_{\varepsilon(B_i)}, \tag{25}$$

where the constant C is independent of h and H.

Proof Let us for every $j \in I_1 \ : \ \Omega_j \subset \Omega^{\text{int}}$ define $\bar{v}_j$ on B_j by

$$\bar{v}_j = v_E - k, \quad k \in \text{RBM}(B_j) \ : \ \bar{v}_j \text{ is } L^2\text{-orthogonal to } \text{RBM}(B_j)$$

Due to Assumption 3.9 2), for every $x \in \Omega_j \subset \Omega^{\text{int}}$ and for every $k \in \text{RBM}(\Omega_j)$ it holds that

$$\begin{aligned}(Q_0^j v)(x) = (Q_0^j v_E)(x) &= \Big(Q_0^j(\bar{v}_j + k)\Big)(x) = (Q_0^j \bar{v}_j)(x) + (Q_0^j k)(x) \\ &= (Q_0^j \bar{v}_j)(x) + k(x) = Q_0^j \bar{v}_j + k\end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}\|(I - Q_0^j)v\|^2_{[L^2(\Omega_j)]^d} &= \|(I - Q_0^j)(\bar{v}_j + k)\|^2_{[L^2(\Omega_j)]^d} \\ &= \|(I - Q_0^j)\bar{v}_j\|^2_{[L^2(\Omega_j)]^d} \\ &\leq 2\Big(\|\bar{v}_j\|^2_{[L^2(\Omega_j)]^d} + \|Q_0^j \bar{v}_j\|^2_{[L^2(\Omega_j)]^d}\Big) \\ &\leq 2\Big(\|\bar{v}_j\|^2_{[L^2(B_j)]^d} + \|Q_0^j \bar{v}_j\|^2_{[L^2(\Omega_j)]^d}\Big).\end{aligned} \tag{27}$$

Further, using Assumption 3.9 2) together with bounded overlaps,

$$\begin{aligned}\|Q_0^j \bar{v}_j\|^2_{[L^2(\Omega_j)]^d} &= \|\sum_{k\in\mathcal{N}_j}\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k\|^2_{[L^2(\Omega_j)]^d} \leq C\sum_{k\in\mathcal{N}_j}\|\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k\|^2_{[L^2(\Omega_j)]^d} \\ &\leq \text{extending the domain over which the norm is taken to } \Omega_k \\ &\leq C\sum_{k\in\mathcal{N}_j}\|\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k\|^2_{[L^2(\Omega_k)]^d} \\ &\leq \text{using Assumption 3.9 3a) and property of } L^2\text{-projection} \\ &\leq C\sum_{k\in\mathcal{N}_j}\|\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)r_l^k\|^2_{[L^2(\Omega_k)]^d} \\ &\leq C\sum_{k\in\mathcal{N}_j}\|\bar{v}_j\|^2_{[L^2(\Omega_k)]^d} \leq C\|\bar{v}_j\|^2_{[L^2(B_j)]^d}\end{aligned}$$

Now, using this result in (27) we get:

$$\begin{aligned}\|(I - Q_0^j)v\|^2_{[L^2(\Omega_j)]^d} &\leq C\|\bar{v}_j\|^2_{[L^2(B_j)]^d} = C\|v_E - k\|^2_{[L^2(B_j)]^d} \\ &\leq \text{scaled Poincaré-Korn with Assumption 3.9 1)} \\ &\leq CH^2|v_E|^2_{\varepsilon(B_j)} \leq CH^2|v|^2_{\varepsilon(B_j)}\end{aligned}$$

which proves the statement of the lemma.

■

Lemma 3.15 *(ε-stability of Q_0 on Ω^{int}) Let us have interpolators Q_0^i as in Definition 3.13, and a coarse space V_0 as in Definition 3.12 satisfying Assumption 3.9, then Q_0^i satisfies ε-stability of Assumption 3.2 2) on Ω_i, ie. for all $\Omega_i \subset \Omega^{\text{int}}$ there is:*

$$|Q_0^i v|_{\varepsilon(\Omega_i)} \leq C|v|_{\varepsilon(B_i)}, \tag{28}$$

where the constant C is independent of h and H.

Proof Let us use the same decomposition of v_E as in Lemma 3.14 - for $\forall j \in I_1 \ : \ \Omega_j \subset \Omega^{\text{int}}$ we define $\bar{v}_j$ on B_j by

$$\bar{v}_j = v_E - k, \quad k \in \text{RBM}(B_j) \ : \ \bar{v}_j \text{ is } L^2\text{-orthogonal to RBM}(B_j)$$

Then, using bounded overlaps,

$$\begin{aligned}
|Q_0^j v|^2_{\varepsilon(\Omega_j)} &= |Q_0^j v_E|^2_{\varepsilon(\Omega_j)} = |Q_0^j(\bar{v}_j + k)|^2_{\varepsilon(\Omega_j)} = |Q_0^j(\bar{v}_j)|^2_{\varepsilon(\Omega_j)} \\
&= |\sum_{k\in\mathcal{N}_j}\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k|^2_{\varepsilon(\Omega_j)} \leq C \sum_{k\in\mathcal{N}_j} |\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k|^2_{\varepsilon(\Omega_j)} \\
&\leq \text{extending norm to } \Omega_k \text{ and using Assumption 3.9 3b)} \\
&\leq C \sum_{k\in\mathcal{N}_j} |\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k|^2_{\varepsilon(\Omega_k)} \leq CH^{-2} \sum_{k\in\mathcal{N}_j} \|\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j) r_l^k\|^2_{[L^2(\Omega_k)]^d} \\
&\leq \text{using property of } L^2\text{-projection and bounded overlaps} \\
&\leq CH^{-2} \sum_{k\in\mathcal{N}_j} \|\bar{v}_j\|^2_{[L^2(\Omega_k)]^d} \leq CH^{-2}\|\bar{v}_j\|^2_{[L^2(B_j)]^d} \\
&\leq \text{Poincare's and Korn's inequalities with Assumption 3.9 1)} \\
&\leq CH^{-2}\|v_E - k\|^2_{[L^2(B_j)]^d} \leq C|v_E|^2_{\varepsilon(B_j)} \\
&\leq \text{property of extension in Definition 3.11} \\
&\leq C|v|^2_{\varepsilon(B_j)}
\end{aligned}$$

which proves the statement of the lemma.

■

Remark 3.16 It is clear, that constants in the proofs of both approximation property and stability of Q_0^i depend through the extension v_E on the shape of Ω (see Definition 3.11), but they are wholly independent of the shape of subdomains.

However, omitting the extension v_E and writing just v in its place would provide constants independent of the shape of Ω, but dependent on the shape of the subdomains, through the dependence of the Korn's inequality on the chunkiness parameter (see Remark 4.7).

Combining the two above mentioned approaches implies, that there exist constants in Lemma 3.14 and Lemma 3.15 which are independent of both the shape of subdomains and of the shape of Ω, except for the influence on the shape of the subdomains adjacent to $\partial\Omega$ and on the overlaps.

3.2.2 Coarse space on $\Omega \setminus \Omega^{\text{int}}$

In this part, we will handle the verification of the two required properties of Q_0 on $\Omega \setminus \Omega^{\text{int}}$. We are going to combine both approaches mentioned in the Remark 3.16, ie. we will end up with constants dependent on the shape of the subdomains adjacent to boundary with Dirichlet conditions (or, in other words on their chunkiness parameter $\gamma(\Omega_i)$); which, in fact, reflects dependency on the boundary of Ω. By using the regularized clusters of subdomains $B_i \cap \Omega$, we diminish the influence of the shape of subdomains itself, the theory will then only slightly depend on the boundary of Ω.

This little drawback, however, does not still hamper our efforts to design a black-box solver, because we are able to detect the subdomains adjacent to the boundary of Ω, ie. in $\Omega \setminus \Omega^{\text{int}}$, (via the incapacity of the coarse space to generate the whole space of RBM's) and we will pay extra attention to them in the solver.

The technique used for proving is very similar to that used with Ω^{int}, the few adjustments needed here concern, in particular, the RBM's supported by those coarse-space basis functions which are influenced by the Dirichlet constraints.

Lemma 3.17 *(approximation property of Q_0 on $\Omega \setminus \Omega^{\text{int}}$) Let us have interpolators Q_0^i as in Definition 3.13, and a coarse space V_0 as in Definition 3.12 satisfying Assumption 3.9, then Q_0^i satisfies the approximation property of Assumption 3.2 1) on Ω_i, ie. for all $\Omega_i \cap (\Omega \setminus \Omega^{\text{int}}) \neq \emptyset$ there is:*

$$\|(I - Q_0^i)v\|_{[L^2(\Omega_i)]^d} \leq CH|v|_{\varepsilon(\tilde{B}_i)}, \tag{29}$$

where the constant C is independent of h and H.

Proof Let us for every $j \in I_1 \ : \ \Omega_j \cap (\Omega \setminus \Omega^{\text{int}}) \neq \emptyset$ define $\bar{v}_j$ on $\tilde{B}_j$ by

$$\bar{v}_j = v - k, \quad k \in \text{RBM}(\tilde{B}_j) \cap \Big(\bigcap_{l \in \mathcal{G}_j} V_{\Gamma_l} \Big)$$

such that $\bar{v}_j$ is L^2-orthogonal to $\text{RBM}(\tilde{B}_j) \cap \Big(\bigcap_{l \in \mathcal{G}_j} V_{\Gamma_l} \Big)$.

Due to Assumption 3.9 2), for every $x \in \Omega_j$ and for every $k \in \text{RBM}(\tilde{B}_j) \cap \Big(\bigcap_{l \in \mathcal{G}_j} V_{\Gamma_l} \Big)$ it holds that

$$\begin{aligned} (Q_0^j v)(x) &= \Big(Q_0^j(\bar{v}_j + k) \Big)(x) = (Q_0^j \bar{v}_j)(x) + (Q_0^j k)(x) \\ &= (Q_0^j \bar{v}_j)(x) + k(x) = Q_0^j \bar{v}_j + k \end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned} \|(I - Q_0^j)v\|^2_{[L^2(\Omega_j)]^d} &= \|(I - Q_0^j)(\bar{v}_j + k)\|^2_{[L^2(\Omega_j)]^d} = \|(I - Q_0^j)\bar{v}_j\|^2_{[L^2(\Omega_j)]^d} \\ &\leq 2 \Big(\|\bar{v}_j\|^2_{[L^2(\Omega_j)]^d} + \|Q_0^j \bar{v}_j\|^2_{[L^2(\Omega_j)]^d} \Big) \\ &\leq 2 \Big(\|\bar{v}_j\|^2_{[L^2(\tilde{B}_j)]^d} + \|Q_0^j \bar{v}_j\|^2_{[L^2(\Omega_j)]^d} \Big). \end{aligned} \tag{31}$$

Further, using Assumption 3.9 2) together with bounded overlaps,

$$\|Q_0^j \bar{v}_j\|^2_{[L^2(\Omega_j)]^d} = \| \sum_{k \in \mathcal{N}_j} \sum_{l=1}^{6} \alpha_l^k(\bar{v}_j) \Phi_l^k \|^2_{[L^2(\Omega_j)]^d} \leq C \sum_{k \in \mathcal{N}_j} \| \sum_{l=1}^{6} \alpha_l^k(\bar{v}_j) \Phi_l^k \|^2_{[L^2(\Omega_j)]^d}$$

$$\le \text{extending the domain over which the norm is taken to } \Omega_k$$
$$\le C \sum_{k\in\mathcal{N}_j} \|\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k\|^2_{[L^2(\Omega_k)]^d}$$
$$\le \text{using Assumption 3.9 3a) and property of } L^2\text{-projection}$$
$$\le C \sum_{k\in\mathcal{N}_j} \|\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)r_l^k\|^2_{[L^2(\Omega_k)]^d}$$
$$\le C \sum_{k\in\mathcal{N}_j} \|\bar{v}_j\|^2_{[L^2(\Omega_k)]^d} \le C\|\bar{v}_j\|^2_{[L^2(\tilde{B}_j)]^d}$$

Now, using this result in (31) we get:

$$\begin{aligned}\|(I - Q_0^j)v\|^2_{[L^2(\Omega_j)]^d} &\le C\|\bar{v}_j\|^2_{[L^2(\tilde{B}_j)]^d} = C\|v_E - k\|^2_{[L^2(\tilde{B}_j)]^d}\\ &\le \text{scaled Poincaré-Friedrichs' and Korn's inequalities}\\ &\le CH^2|v|^2_{\varepsilon(\tilde{B}_j)}\end{aligned}$$

which proves the statement of the lemma.

■

Lemma 3.18 *(ε-stability of Q_0 on $\Omega \setminus \Omega^{\text{int}}$) Let us have interpolators Q_0^i as in Definition 3.13, and a coarse space V_0 as in Definition 3.12 satisfying Assumption 3.9, then Q_0^i satisfies ε-stability of Assumption 3.2 2) on Ω_i, ie. for all $\Omega_i \cap (\Omega \setminus \Omega^{\text{int}})$ there is:*

$$|Q_0^i v|_{\varepsilon(\Omega_i)} \le C|v|_{\varepsilon(\tilde{B}_i)}, \tag{32}$$

where the constant C is independent of H and h.

Proof Let us introduce the same decomposition of v as in Lemma 3.14 - for every $j \in I_1$: $\Omega_j \cap (\Omega \setminus \Omega^{\text{int}}) \neq \emptyset$ we define $\bar{v}_j$ on $\tilde{B}_j$ by

$$\bar{v}_j = v - k, \quad k \in \text{RBM}(\tilde{B}_j) \cap \Big(\bigcap_{l\in\mathcal{G}_j} V_{\Gamma_l}\Big)$$

such that $\bar{v}_j$ is L^2-orthogonal to $\text{RBM}(\tilde{B}_j) \cap \Big(\bigcap_{l\in\mathcal{G}_j} V_{\Gamma_l}\Big)$.

Then, using bounded overlaps,

$$\begin{aligned}|Q_0^j v|^2_{\varepsilon(\Omega_j)} &= |Q_0^j(\bar{v}_j + k)|^2_{\varepsilon(\Omega_j)} = |Q_0^j(\bar{v}_j)|^2_{\varepsilon(\Omega_j)}\\ &= |\sum_{k\in\mathcal{N}_j}\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k|^2_{\varepsilon(\Omega_j)} \le C \sum_{k\in\mathcal{N}_j} |\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k|^2_{\varepsilon(\Omega_j)}\\ &\le \text{extending the norm to } \Omega_k \text{ and using Assumption 3.9 3b)}\\ &\le C \sum_{k\in\mathcal{N}_j} |\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)\Phi_l^k|^2_{\varepsilon(\Omega_k)} \le CH^{-2} \sum_{k\in\mathcal{N}_j} \|\sum_{l=1}^{6} \alpha_l^k(\bar{v}_j)r_l^k\|^2_{[L^2(\Omega_k)]^d}\\ &\le \text{using property of } L^2\text{-projection and bounded overlaps}\\ &\le CH^{-2} \sum_{k\in\mathcal{N}_j} \|\bar{v}_j\|^2_{[L^2(\Omega_k)]^d} \le CH^{-2}\|\bar{v}_j\|^2_{[L^2(\tilde{B}_j)]^d}\\ &\le \text{ using Poincare's, Friedrichs' and Korn's inequalities}\\ &\le CH^{-2}\|v - k\|^2_{[L^2(\tilde{B}_j)]^d} \le C|v|^2_{\varepsilon(\tilde{B}_j)}\end{aligned}$$

which proves the statement of the lemma.

■

Remark 3.19 It is clear, that the constants in both stability and approximation property on $\Omega \setminus \Omega^{\text{int}}$ depend on the Dirichlet constraints on the boundary, namely on $\text{meas}(\Gamma_j)$. It may happen that for some $\tilde{B}_i$ the $\text{meas}(\partial \tilde{B}_i \cap \Gamma_j)$ is small, but $\text{meas}(\Gamma_j)$ alone is sufficient. In such case, we can enlarge the cluster $\tilde{B}_i$ so that it contains more of Γ_j.

Also, having clusters $\{\tilde{B}_i\}$ for some i's, we can join them into one when doing the estimates, provided that the same 'kind' of local RBM's cannot be generated on all those clusters.

We have mentioned the dependence of constants on $\text{meas}(\Gamma_j)$ at the beginning of this remark: it might occur that $\text{meas}(\Gamma_j)$ alone is very small (eg. $\text{meas}(\Gamma_j)$ is only $O(h)$, instead of $O(H)$), and as a result the constants deteriorate.

However, there is another way of verifying the approximation property and stability of Q_0, which is $\text{meas}(\Gamma_j)$-independent (refer to Section 7).

4 Technical Tools

Proposition 4.1 *(Poincaré) Let $u \in H^1(\Omega)$, then there exists a constant $C = C(\Omega, \partial\Omega)$ such, that for all $u \in H^1(\Omega)$, $\int_\Omega u d\Omega = 0$ there is*

$$\|u\|_{L^2} \leq C|u|_{H^1}.$$

(cf. [7] and [9])

Proposition 4.2 *(Friedrichs) Let us have $u \in H^1_{0,\Gamma_1}(\Omega)$, where $\Gamma_1 \subset \partial\Omega$, $\text{meas}(\Gamma_1) > 0$. There exists a constant $C = C(\Omega, \partial\Omega, \Gamma_1)$ such, that for all $u \in H^1_{0,\Gamma_1}(\Omega)$ there is*

$$\|u\|_{L^2} \leq C|u|_{H^1}.$$

(cf. [7] and [9])

Lemma 4.3 *(Scaled Poincaré) Let $u \in H^1(\Omega)$ and let H be characteristic size of Ω, then there exists a constant C independent of H such, that for all $u \in H^1(\Omega)$, $\int_\Omega u d\Omega = 0$ there is*

$$\|u\|_{L^2} \leq CH|u|_{H^1}. \tag{33}$$

Proof Let us have a mapping $F : \Omega \to \hat{\Omega}$ shrinking the domain Ω to the domain $\hat{\Omega}$ of the characteristic size 1 (see Fig 1), ie. $\hat{\Omega} = F(\Omega)$ and let us denote $\hat{u}(x) = u(F(x))$. Let $u \in H^1(\Omega)$ satisfy the assumption of the lemma, ie.

$$\int_\Omega u d\Omega = 0.$$

It is clear that transforming the coordinate system by the mapping F (ie. transforming u to $\hat{u}$) will not spoil this property:

$$\int_{\hat{\Omega}} \hat{u} d\hat{\Omega} = 0.$$

Now we can apply the normal Poincaré for the domain $\hat{\Omega}$:

$$\|\hat{u}\|_{L^2(\hat{\Omega})} \leq C|\hat{u}|_{H^1(\hat{\Omega})}.$$

where C is independent of H. As F is a mere shrinking, we have:

$$\|u\|^2_{L^2(\Omega)} = \int_{\Omega} u^2 d\Omega = \int_{\hat{\Omega}} \hat{u}^2 \det(F) d\hat{\Omega} = H^2 \|\hat{u}\|^2_{L^2(\hat{\Omega})}$$

$$|u|^2_{H^1(\Omega)} = \int_{\Omega} (\nabla u)^2 d\Omega = \int_{\hat{\Omega}} H^{-2} (\nabla \hat{u})^2 \det(F) d\hat{\Omega} = |\hat{u}|^2_{H^1(\hat{\Omega})}$$

Using these equalities in the Poincaré's inequality for $\hat{\Omega}$ completes the proof.

■

Lemma 4.4 *(Scaled Friedrichs) Let us have $u \in H^1_{0,\Gamma_1}(\Omega)$, where $\Gamma_1 \subset \partial\Omega$,* meas$(\Gamma_1) > 0$, *and let the following hold for distances of any $x \in \Omega$ from Γ_1:*

$$\exists B_1, B_2, \quad B_i = \{x, \text{dist}(x, \Gamma_1) \leq c_i H\}, \quad i = 1, 2$$

such that $B_1 \subset \Omega \subset B_2$, ie. Ω is a belt-shaped domain with the boundary condition along one side of the belt, H being the characteristic thickness of the belt. Then there exists a constant C independent of H such, that for all $u \in H^1_{0,\Gamma_1}(\Omega)$ there is

$$\|u\|_{L^2} \leq CH|u|_{H^1}. \tag{34}$$

Proof Similarly as in the proof of Lemma 4.3, let us have a mapping $F : \Omega \to \hat{\Omega}$ shrinking the belt-shaped domain Ω to the domain $\hat{\Omega}$ of the characteristic thickness 1 (see Fig 2), ie. $\hat{\Omega} = F(\Omega)$ and let us denote $\hat{u}(x) = u(F(x))$. Let $u \in H^1(\Omega)$ satisfy the assumption of the lemma, ie. $u = 0$ on Γ_1.

It is clear that transforming the coordinate system by the mapping F (ie. transforming u to $\hat{u}$) will not spoil this property, ie $\hat{u} = 0$ on $\hat{\Gamma}_1$. Let us now divide $\hat{\Omega}$ into $\hat{\Omega}_i$ such that the characteristic size of each $\hat{\Omega}_i$ is 1 and $\hat{\Omega} = \sum_i \hat{\Omega}_i$. Analogously, we obtain $\hat{\Gamma}_{1i}$. Now we can apply the normal Friedrichs for $\hat{\Omega}_i$:

$$\|\hat{u}\|_{L^2(\hat{\Omega}_i)} \leq C|\hat{u}|_{H^1(\hat{\Omega}_i)} \qquad \text{for all } i\text{'s}$$

where C's are independent of H. Summing up squares of the latter Friedrichs' inequalities we get:

$$\|\hat{u}\|_{L^2(\hat{\Omega})} \leq C|\hat{u}|_{H^1(\hat{\Omega})} \tag{35}$$

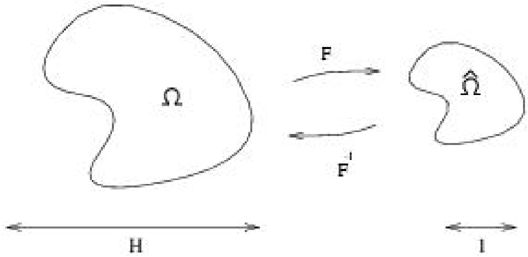

Figure 1: Mapping $F : \Omega \to \hat{\Omega}$ in Poincaré's inequality

where C is independent of H. As F is a mere shrinking, we have:

$$\|u\|^2_{L^2(\Omega)} = \int\limits_{\Omega} u^2 d\Omega = \int\limits_{\hat{\Omega}} \hat{u}^2 \det(F) d\hat{\Omega} = H^2 \|\hat{u}\|^2_{L^2(\hat{\Omega})}$$

$$|u|^2_{H^1(\Omega)} = \int\limits_{\Omega} (\nabla u)^2 d\Omega = \int\limits_{\hat{\Omega}} H^{-2} (\nabla \hat{u})^2 \det(F) d\hat{\Omega} = |\hat{u}|^2_{H^1(\hat{\Omega})}$$

These equalities and (35) complete the proof.

■

Proposition 4.5 *(Korn) Let V be a Hilbert space such that $\left[H^1_0(\Omega)\right]^d \subset V \subset \left[H^1(\Omega)\right]^d$. And let us denote*

$$\mathrm{RBM}(\Omega) = Ker\left(\int\limits_{\Omega} \sum_{i,j=1}^{3} \varepsilon_{ij}(u)^2 d\Omega\right) = Ker\left(|u|_\varepsilon\right), \qquad \forall u \in [H^1(\Omega)]^d.$$

Then there exists a constant C_K dependent on Ω such that for all $u \in V$ there is

$$\inf \|u - k\|_{[H^1]^d} \leq C_K |u|_\varepsilon \quad , \quad \forall k \in \mathrm{RBM}(\Omega) \cap V. \tag{36}$$

(cf. [8]).

Definition 4.6 *(cf. [10]) Suppose a domain Ω has diameter D and is star-shaped with respect to a ball B, ie. for $\forall x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω. Let*

$$\varrho_{max} = \sup\{\varrho : \Omega \text{ is star-shaped with respect to a ball of radius } \varrho\}.$$

Then the chunkiness parameter of Ω is defined by

$$\gamma(\Omega) = \frac{D}{\varrho_{max}}.$$

Remark 4.7 It is well known [11] for a star-shaped domain Ω that the Korn's constant $C_K(\Omega)$ on the factor space modulo rigid body modes (RBM's) (see Proposition 4.5) can be controlled in terms of the chunkiness parameter $\gamma(\Omega)$. Furthermore, for two Lipschitz domains Ω_1 and Ω_2, Korn's constant C_K on $\Omega_1 \cup \Omega_2$ can be estimated by [12] as

$$C_K(\Omega_1 \cup \Omega_2) \geq C_K(\Omega_1) + C_K(\Omega_2) + \frac{\min(\mathrm{meas}(\Omega_1), \mathrm{meas}(\Omega_2))}{\mathrm{meas}(\Omega_1 \cap \Omega_2)} \left(\sqrt{C_K(\Omega_1)} + \sqrt{C_K(\Omega_2)}\right)^2$$

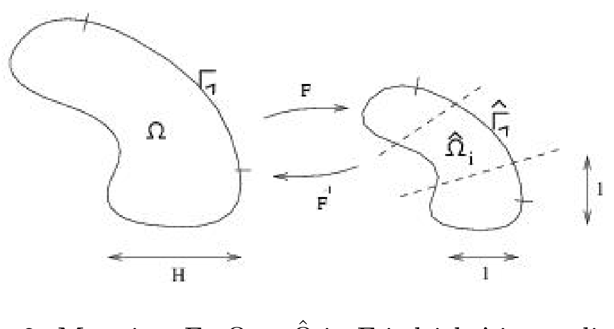

Figure 2: Mapping $F : \Omega \to \hat{\Omega}$ in Friedrichs' inequality

Lemma 4.8 *(Poincaré-Friedrichs in* $\mathbb{R}^d$*) Let us have a Hilbert space* V*,*

$$[H_0^1(\Omega)]^d \subseteq V \subseteq [H^1(\Omega)]^d$$

and let us have a vector function $u = (u_1, u_2, u_3)^T \in V$*, or* $u = (u_1, u_2)^T \in V$ u_i *being the components of* u *in some* $3D$ *or* $2D$ *coordinate system, respectively. Let us further denote* H *the characteristic size of* Ω*. Then there exist a constant function* $q \in V \cap \mathrm{RBM}(\Omega)$ *and a constant* $C > 0$ *independent of* H *such that for any* $u \in V$ *we have*

$$\|u - q\|_{[L^2]^d} \leq CH|u - q|_{[H^1]^d}$$

Proof Suppose we have $n \leq d$ independent Dirichlet's boundary constraints on $\Omega \subset \mathbb{R}^d$. The Dirichlet's constaints are, in general, of the form $g_j^T \cdot u = 0$, $j = 1 \ldots n$, $g_j, u \in \mathbb{R}^d$, Let $\{\tilde{g}_j\}_{j=1}^n$ be the set $\{g_j\}_{j=1}^n$ after ortonormalization and let us add (d-n) vectors to this set so that it forms an orthogonal basis $\langle \tilde{g}_i \rangle_{i=1}^d$ of $\mathbb{R}^d$.

By choosing $\langle \tilde{g}_i \rangle_{i=1}^d$ to be our new basis we have managed to decompose V into subspaces V_i such that $V = V_1 \times V_2 \times V_3$, or $V = V_1 \times V_2$,

$$H_0^1(\Omega) \subseteq V_i \subseteq H^1(\Omega) \quad , \quad i = 1 \ldots d,$$

for $d = 3$ or $d = 2$ respectively and where for n V_i's we have $V_i \subset H^1(\Omega)$ (constrained directions) and $V_i = H^1(\Omega)$ for the rest (unconstrained directions).

Now it suffices to choose $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)^T$ or $\tilde{q} = (\tilde{q}_1, \tilde{q}_2)^T$ in the new coordinate system as follows:

$$\tilde{q}_i = \begin{cases} \int_\Omega u_i d\Omega, & u_i \in V_i \text{ for unconstrained directions} \\ 0, & \text{for constrained directions} \end{cases}$$

And we can apply Poincaré's and Friedrichs' inequalities (Lemma 4.3 and Lemma 4.4) for unconstrained and constrained directions respectively to get the desired

$$\|\tilde{u} - \tilde{q}\|_{[L^2]^d} \leq CH|\tilde{u} - \tilde{q}|_{[H^1]^d}$$

Transforming $\tilde{q}$ and $\tilde{u}$ back to the original coordinate system (ie. to q and u) completes the proof.

■

Corollary 4.9 (Poincaré-Friedrichs-Korn) Let V be as in Lemma 4.8 then there exists $C > 0$ for all $u \in V$ such that

$$\inf_{k \in \mathrm{RBM}(\Omega) \cap V} \|u - k\|_{[L^2]^d} \leq CH|u|_\varepsilon$$

and the constant C is independent of H and h.

Proof Let $h \in \mathrm{RBM}(\Omega) \cap V$ be a H^1-orthogonal projection of u onto RBM's, then using Lemma 4.8 there exists a constant function $q \in \mathrm{RBM} \cap V$ such that

$$\begin{aligned} \inf_{k \in \mathrm{RBM} \cap V} \|u - k\|_{[L^2]^d} &\leq \|u - h - q\|_{[L^2]^d} \leq CH|u - h - q|_{[H^1]^d} \\ &\leq \text{using Korn's inequality (Proposition 4.5)} \\ &\leq CH|u - h|_{[H^1]^d} \leq CH|u|_\varepsilon \end{aligned}$$

which completes the proof.

■

5 Coarse Space by Smoothed Aggregation

In this section, we define a coarse-space based on the concept of smoothed aggregation introduced in [1]. Overlapping subdomains will be defined using the nonzero structure of the prolongator. The method described here allows black-box implementation; its only input is a system of linear algebraic equations $Ax = b$, a system of aggregates of nodes and a discrete representation of the local kernel of the bilinear form $a(\cdot,\cdot)$. By local kernel we mean the kernel in absence of essential boundary conditions, ie. the kernel of the unconstrained problem. Finite element packages usually provide this information.

Definition 5.1 *Let us assume we have the matrix A and the vector of the right-hand side b writen in block representation as $A = \{A^B_{ij}\}^n_{i,j=1}$ and $b = \{b^B_i\}^n_{i=1}$, where A^B_{ij} is a $(d \times d)$ matrix corresponding to the entries in the global stiffness matrix relating the dofs on the i-th and j-th nodes, and $b^B_i \in \mathbb{R}^d$ are the values of the right-hand side corresponding to the dofs on the i-th node. Let us further denote 0^B the $(d \times d)$ zero matrix.*

Let $\{\mathcal{A}_i\}^m_{i=1}$ be the given system of aggregates of nodes forming a disjoint covering of the set of all unconstrained nodes, ie.

$$\bigcup_{i=1}^{m} \mathcal{A}_i = \{1, \dots, n\} \quad \mathcal{A}_i \cap \mathcal{A}_j = \emptyset \quad \text{for } i \neq j,$$

where n is the number of all unconstrained nodes, each such node having d degrees of freedom (dofs) attached to it ($d = 2, 3$ is the dimension of the problem).

Let us further define the set $\mathcal{D}_i$ of all unconstrained degrees of freedom associated with nodes in $\mathcal{A}_i$, and let us denote $\{\hat{r}^j\}^{n_k}_{j=1}$ the set of vectors - discrete counterparts of the set of functions $\{r^j\}^{n_k}_{j=1}$, $r^j \in \mathrm{RBM}(\Omega)$ spanning the local kernel of $a(\cdot,\cdot)$ with the dimension n_k.

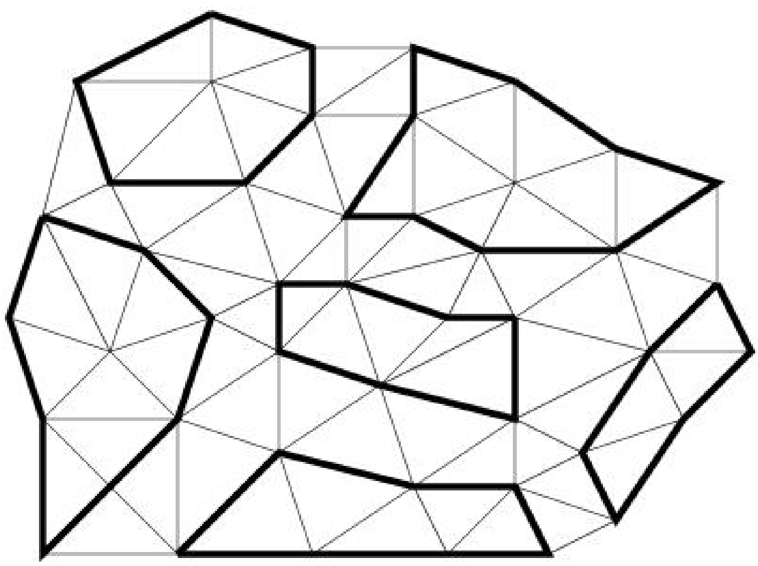

Figure 3: Typical aggregates in 2D

5.1 Prolongator and prolongator smoother

Our prolongators are constructed in two steps: in the first step, we generate a tentative prolongator $\tilde{P} : \mathbb{R}^{n_k \cdot m} \to \mathbb{R}^{d \cdot n}$ using the unknowns-aggregation technique. In the second step, we apply polynomial Chebyshev-like smoother $\mathcal{S}$ to suppress high energy components in $\mathrm{Rng}(\tilde{P})$ and thus we get our prolongator $P = \mathcal{S}\tilde{P}$.

Algorithm 5.2 (tentative prolongator, cf. [2]) For all aggregates $\mathcal{A}_i$ and for $j = 1 \dots n_k$ do

1. For $\hat{r}^j$, compute the vector $\hat{\bar{r}}^i_j \in \mathbb{R}^{dn}$ with components

$$(\hat{\bar{r}}^i_j)_k = \begin{cases} (\hat{r}^j)_k & \text{if } k \in \mathcal{D}_i, \\ 0 & \text{otherwise.} \end{cases}$$

2. Interpret the vector $\hat{\bar{r}}^i_j$ as the $[n_k(i-1)+j]$-th column of the tentative prolongator $\tilde{P}$.

Remark 5.3 In order to improve conditioning of the coarse problem, it is advisable to perform a discrete l^2-orthogonalization of vectors $\hat{\bar{r}}^i_j$ on each aggregate $\mathcal{A}_i$, as suggested in [1] and [2]. This is not required by the theory, but practical applications can benifit from such stabilization (see Fig. 4).

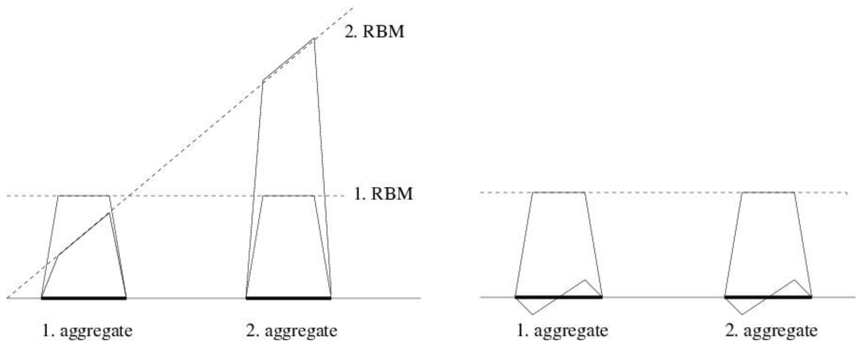

Figure 4: Orthogonalizing tentative basis

Note that the tentative coarse-space basis functions $\bar{r}^i_j = \Pi_h \hat{\bar{r}}^i_j$ already satisfy the assumption on decomposition of local kernel on Ω^{int}, however, they do not satisfy the assumption concerning their energy (Assumption 3.9 3b)). That is why, we will now construct and apply a symmetric polynomial smoother $\mathcal{S}$ of degree at most d_s (for given d_s) which reduces $|\bar{r}^i_j|_\varepsilon$, in other words we will seek $\mathcal{S}$ so that $|S\bar{r}^i_j|_\varepsilon$ is as small as possible. Because the discrete form of $|S\bar{r}^i_j|_\varepsilon$ is $\langle AS\hat{\bar{r}}^{ij}, S\hat{\bar{r}}^{ij}\rangle \leq \varrho(S^2A)\langle \hat{\bar{r}}^i_j, \hat{\bar{r}}^i_j\rangle$, this means that we wish to employ such $\mathcal{S}$ which minimizes $\varrho(S^2A)$ ($\hat{\bar{r}}^i_j$ is given and constant).

Choosing $\mathcal{S}$ to be a polynomial in A is not only computationally the easiest option, but also the polynomial smoother $\mathcal{S}$ with an absolute term equal to one automatically assures that, for decent d_s, the property of decomposition of local kernel (RBM's) on Ω^{int} (satisfied by unsmoothed $\bar{r}^i_j$) will not be spoiled by smoothing. Following [3] and [2] we can define the smoother $\mathcal{S}$ recursively as follows:

Algorithm 5.4 (prolongator smoother, version 1) For the desired degree d_s of the prolongator smoother $\mathcal{S}$ and for the estimate $\bar{\varrho}$ of the spectral radius of A such that

$$\varrho(A) \leq \bar{\varrho} \leq C_\varrho \varrho(A)$$

we define the prolongator smoother $\mathcal{S}$ by

1. Let $K = \lfloor \log_3(2d_s + 1) \rfloor$, where $\lfloor \cdot \rfloor$ is the truncation to the nearest smaller integer.
2. Set $A_1 = A$, $A_0 = I$, $\bar{\varrho}_1 = \bar{\varrho}$.
3. Define for $i = 1 \dots K$ polynomial smoothers S_i by
$$S_i = I - \frac{\omega}{\bar{\varrho}_i} A_i \ , \quad \omega = \frac{4}{3},$$
where
$$A_{i+1} = S_i^2 A_i \quad , \quad \bar{\varrho}_{i+1} = \frac{1}{9} \bar{\varrho}_i$$
The choice of $\omega = \frac{4}{3}$ will be justified in the Remark 5.5
4. $\mathcal{S} = \prod_{i=1}^{K} S_i$.

Remark 5.5 The idea behind the Algorithm 5.4 is following: Our goal is to reduce $\varrho(S^2 A)$ as much as possible. This is done gradually in K steps (for $i = 1 \dots K$); having $\varrho(A_i)$ we would like to construct a linear smoother S_i in the form of $S_i = (I - \frac{\omega}{\bar{\varrho}_i} A_i)$ such that it maximally reduces $\varrho(A_{i+1}) = \varrho(S_i^2 A_i)$. So, we have:

$$\begin{aligned} \varrho(A_{i+1}) = \varrho((I - \frac{\omega}{\bar{\varrho}_i} A_i)^2 A_i) &= \max_{\lambda \in \sigma(A_i)} (1 - \frac{\omega}{\bar{\varrho}_i} \lambda)^2 \lambda \\ &\leq \max_{\lambda \in (0, \bar{\varrho}_i]} (1 - \frac{\omega}{\bar{\varrho}_i} \lambda)^2 \lambda \end{aligned}$$

When we minimize the right-hand side of the latter inequality with respect to $\omega \in (0, 2)$, we get

$$\min_{\omega \in (0,2)} \left(\max_{\lambda \in (0, \bar{\varrho}_i]} (1 - \frac{\omega}{\bar{\varrho}_i} \lambda)^2 \lambda \right) = \max_{\lambda \in (0, \bar{\varrho}_i]} (1 - \frac{4}{3} \bar{\varrho}_i^{-1} \lambda)^2 \lambda = \frac{1}{9} \bar{\varrho}_i$$

With $\omega = \frac{4}{3}$, we have under the prescribed conditions minimized the bound for $\varrho(A_{i+1})$, but this bound might be still high - in that case we apply the same algorithm recursively several times (for $i = 1 \dots K$).

Remark 5.6 The Algorithm 5.4 is capable of generating smoother $\mathcal{S}$ of certain degrees only (eg. $\deg(\mathcal{S}) \in \{1, 4, 13, 40, \dots\}$). The choice of K in step 1 gives $\mathcal{S}$ of a degree closest to d_s. As

$$\deg(S_i) = \deg(A_i) = \deg((I - \frac{4}{3} \bar{\varrho}_i^{-1} A_{i-1})^2 A_{i-1}) = 3 \deg(A_{i-1}) = 3^{i-1}$$

we have

$$\deg(\mathcal{S}) = \deg(\prod_{i=1}^{K} S_i) = \sum_{i=1}^{K} \deg(A_i) = \sum_{i=1}^{K} 3^{i-1} = \frac{3^K - 1}{2},$$

where $K = \lfloor \log_3(2d_s + 1) \rfloor$. Further, there exists $\xi \geq 1$ such that

1. $\lfloor \log_3(2d_s+1) \rfloor = \log_3(\frac{2d_s}{\xi}+1)$
2. $0 \le \log_3(2d_s+1) - \log_3(\frac{2d_s}{\xi}+1) < 1$

and from the two inequalities in point 2, it follows that $1 \le \xi < 3$, which means that

$$\frac{d_s}{3} < \deg(\mathcal{S}) \le d_s.$$

As Remark 5.6 says, the Algorithm 5.4 is capable to generate smoothers only of certain degrees, which might be very distant from d_s. This is a slight disadvantage of the algorithm. However, if we use the following lemma proposed by Mandel together with the knowledge of minimax properties of Chebyshev polynomials, we are able to devise a new algorithm capable of generating polynomial smoothers $\mathcal{S}$ of any degree:

Lemma 5.7 *Let $\mathcal{P}_n$ be a set of polynomials of degree n such that $P_n(0) = 1$ for all $P_n \in \mathcal{P}_n$. Then for any $L > 0$ and any integer $n > 0$, there is a unique polynomial $T_n(\lambda) \in \mathcal{P}_n(\lambda)$, such that*

$$\min_{P_n \in \mathcal{P}_n} \left(\max_{\lambda \in [0,L]} P_n^2(\lambda)\lambda \right) = \max_{\lambda \in [0,L]} T_n^2(\lambda)\lambda. \tag{37}$$

The polynomial T_n is given by

$$T_n(\lambda) = \left(1 - \frac{\lambda}{r_1}\right) \cdots \left(1 - \frac{\lambda}{r_n}\right), \tag{38}$$

where the roots r_k of T_n are

$$r_k = \frac{L}{2}\left(1 - \cos\frac{2k\pi}{2n+1}\right), \quad k = 1, \ldots, n. \tag{39}$$

In addition, the polynomial T_n satisfies

$$\max_{\lambda \in [0,L]} T_n^2(\lambda)\lambda = \frac{L}{(2n+1)^2} \tag{40}$$

and

$$\max_{\lambda \in [0,L]} |T_n(\lambda)| = 1 \tag{41}$$

Proof Let us have a polynomial R_{2n+1} of degree $(2n+1)$ such that

$$\|R_{2n+1}\|_\infty = \max_{x \in [-1,1]} R_{2n+1}(x) = -\min_{x \in [-1,1]} R_{2n+1}(x)$$
$$\|R_{2n+1}\|_\infty = R_{2n+1}(1),$$

where $\|R_{2n+1}\|_\infty = \max_{x \in [-1,1]} |R_{2n+1}(x)|$. Then we can write the following:

$$P_n^2(\lambda)\lambda = Q_{2n+1}(\lambda) = \frac{c}{2} \frac{\left(\|R_{2n+1}\|_\infty - R_{2n+1}(1 - \frac{2\lambda}{L})\right)}{\|R_{2n+1}\|_\infty}\ , \tag{42}$$

where $c = \max_{\lambda \in [0,L]} Q_{2n+1}(\lambda)$. Because of the constraint $P_n(0) = 1$, we also have

$$1 = Q'_{2n+1}(0) = \frac{c}{L} \frac{R'_{2n+1}(1)}{\|R_{2n+1}\|_\infty} \tag{43}$$

And hence

$$c = L\frac{\|R_{2n+1}\|_\infty}{R'_{2n+1}(1)} \tag{44}$$

Now, our goal is to minimize c. Let us assume for the moment that we have "normalized" R_{2n+1}, so that $R'_{2n+1}(1) = 1$. Then we claim that among all polynomials of degree $(2n+1)$ with the first derivative in 1 equal to one, and achieving their maxima on the interval $x \in [-1, 1]$ for $x = 1$, the multiples of Chebyshev polynomials are those minimizing $\|R_{2n+1}\|_\infty$.

Indeed, by contradiction: let us have some multiple of Chebyshev polynomial R_{2n+1}, $R'_{2n+1}(1) = 1$ and suppose we have found polynomial $\tilde{R}_{2n+1}$ of the same degree, $\tilde{R}'_{2n+1}(1) = 1$ such that $\|\tilde{R}_{2n+1}\|_\infty < \|R_{2n+1}\|_\infty$. It is clear, that $\Delta R_{2n+1}(x) = R_{2n+1}(x) - \tilde{R}_{2n+1}(x)$ changes its sign at least $(2n+2)$ times, ie. it has at least $(2n+1)$ roots, ie. there are at least $2n$ local extremes between these roots. Also, $\Delta R'_{2n+1}(1) = 0$, which means that the first derivative of ΔR_{2n+1} vanishes at least at $(2n+1)$ points, and hence the contradiction, because also $\deg(\Delta R_{2n+1}) \le 2n+1$.

Now, let us choose such a multiple of the Chebyshev polynomial $R_{2n+1}(x)$ for which $\|R_{2n+1}\|_\infty = 1$, then $R'_{2n+1}(1) = (2n+1)^2$, and we can write (42) in the form

$$T_n^2(\lambda)\lambda = Q_{2n+1}(\lambda) = \frac{c}{2}\Big(1 - R_{2n+1}(1 - \frac{2\lambda}{L})\Big), \tag{45}$$

where

$$c = \max Q_{2n+1}(\lambda) = \frac{L}{R'_{2n+1}(1)} = \frac{L}{(2n+1)^2}.$$

This proves (40) and (37).

The polynomial $Q_{2n+1}(\lambda)$ vanishes at the points λ where $T_{2n+1}(1 - 2\lambda/L) = 1$, it means that $1 - 2\lambda/L = \cos(2k\pi/2n+1)$. The value $k = 0$ gives the simple root $\lambda = 0$ of Q_{2n+1}, while $k = 1, \ldots, n$ yield double roots of Q_{2n+1}, given by (39). This proves that T_n is indeed the polynomial (38).

It remains to prove (41). First, (41) is equivalent to $T_n^2(\lambda) \le 1$, for all $\lambda \in [0, L]$. Using (45), this is equivalent to

$$\frac{L\Big(1 - R_{2n+1}(1 - 2\lambda/L)\Big)}{2\lambda R'_{2n+1}(1)} \le 1, \quad \forall \lambda \in [0, L] \tag{46}$$

Using substitution $(1 - 2\lambda/L) = x$, (46) becomes by a simple manipulation

$$R_{2n+1}(x) \ge 1 + R'_{2n+1}(1)(x-1), \quad \forall x \in [-1, 1],$$

which is the well known fact that the graph of a Chebyshev polynomial lies above its tangent at $x = 1$.

∎

Algorithm 5.8 (prolongator smoother, version 2) For the desired degree d_s of the prolongator smoother $\mathcal{S}$ and for the estimate $\bar{\varrho}$ of the spectral radius of A such that

$$\varrho(A) \le \bar{\varrho} \le C_\varrho \varrho(A)$$

we define the prolongator smoother $\mathcal{S}$ by

$$\mathcal{S} = \prod_{k=1}^{d_s}\Big(I - r_k^{-1}A\Big),$$

where

$$r_k = \frac{\bar{\varrho}}{2}\left(1 - \cos\frac{2k\pi}{2d_s+1}\right), \quad k = 1, \dots, d_s. \tag{47}$$

Remark 5.9 Note, that if d_s is such that $K = \log_3(2d_s+1)$ is an integer, then Algorithm 5.4 and Algorithm 5.8 generate the same smoother $\mathcal{S}$. Indeed, for such d_s both polynomials $\mathcal{S}$ have the same degree and the same roots: rather tedious proof rests on the gradual evaluation of roots of the polynomial generated by Algorithm 5.4 and comparing them with the roots (47) of the second polynomial.

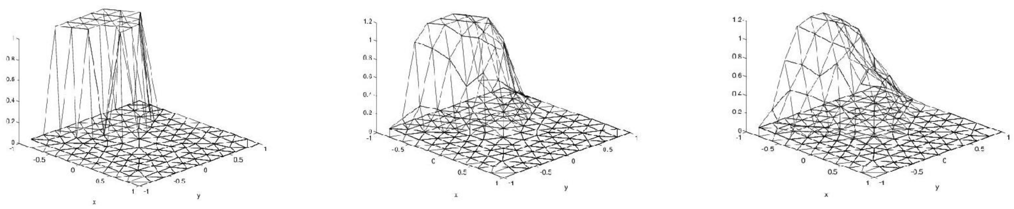

Figure 5: Successive smoothing by a smoother of degree 2, from left to right

5.2 Coarse space and computational subdomains

Having obtained the smoothed prolongator P, we are going to define the coarse space through the coarse-space basis functions. The nonzero structure of the prolongator P determines the supports of our coarse-space basis functions in the following way:

$$\Phi^i_j = \Pi_h P \hat{e}^i_j.$$

Here, $\Pi_h \hat{x} = \sum_{i=1}^n x_i \varphi_i$, $x_i \in \mathbb{R}^d$, $\hat{x} \in \mathbb{R}^{dn}$ is the finite element interpolator, $\{\varphi_i\}$ is the finite element basis and $\hat{e}^i_j$ is the $[n_k(i-1)+j]$-th vector of the canonical basis of $\mathbb{R}^{n_k m}$, $i = 1 \dots m$, $j = 1 \dots n_k$.

The computational subdomains Ω_i are derived from the nonzero structure of the matrix P^{symb}, which is obtained in the same way as the prolongator P (ie. $P = \mathcal{S}\tilde{P}$), except the matrix operations involved are performed only symbolically, ie. using the arithmetic of

$$1 + \alpha = 1, \quad 0 + \alpha = \alpha, \quad 1 \cdot \alpha = \alpha, \quad 0 \cdot \alpha = 0, \qquad \alpha \in \{0, 1\}$$

Then, defining

$$\Omega_i = \mathrm{supp}(\Pi_h P^{\mathrm{symb}} \hat{e}^i_j), \quad j = 1 \dots n_k \tag{48}$$

we have

$$\mathrm{supp}(\Phi^i_j) \subset \Omega_i, \quad j = 1 \dots n_k. \tag{49}$$

Note that if sparse matrix operations are implemented so that numerical zeroes are never dropped, the results of these symbolic operations are obtained for free as a side benifit of the computation, so the symbolic operations need not be performed at all.

Algorithmically, this can be accomplished as follows: first, let us define for each block of columns of the smoothed prolongator P^{symb} associated with one subdomain Ω_j the list of nonzeros

$$\mathcal{N}_j = \{i : P^{\text{symb}}_{ik} \neq 0;\ k = n_k(j-1)+1 \ldots n_k(j-1)+n_k\}, \quad n_j = \text{card}(\mathcal{N}_j)$$

and the $d \cdot n$ by n_j 0/1 matrix N_j resulting from selecting the columns with indices in $\mathcal{N}_j$ from the $d \cdot n$ by $d \cdot n$ identity matrix. Further, we define local matrices A_i and local correction operators R_i by

$$A_i = N_i^T A N_i, \quad R_i = N_i (A_i)^{-1} N_i^T, \quad i = 1, \ldots, m. \tag{50}$$

Analogously, for the coarse level we set

$$A_0 = P^T A P, \quad R_0 = P(A_0)^{-1} P^T. \tag{51}$$

For a positive i, $R_i A$ is the A-orthogonal projection onto the local space

$$\hat{V}_i = \{\hat{x} \in \mathbb{R}^{dn} : x_j = 0 \text{ for } j \notin \mathcal{N}_i\}.$$

Note that $(\hat{V}_i, \|\cdot\|_A)$ is the vector space isometrically isomorphic to the space of finite element functions $(V_i \equiv \{\Pi_h \hat{x},\ \hat{x} \in \hat{V}_i\}, |\cdot|_\varepsilon)$.

For the sake of parallelism, we need a disjoint covering $\{\mathcal{C}_i\}_{i=1}^{n_c}$ of the set $\{1, \ldots, m\}$ satisfying

$$\cos(\hat{V}_j, \hat{V}_k) = 0 \text{ for every } j, k \in \mathcal{C}_i\ ,\ i = 1, \ldots, n_c \tag{52}$$

where the cosine is measured in A-scalar product. For unstructured meshes, such a decomposition can be created using a simple greedy algorithm (cf. [2] and Algorithm 8.6), as the information about the orthogonality of spaces $\hat{V}_i$ is available. Trivially, spaces $\hat{V}_i$ and $\hat{V}_j$ are orthogonal if

$$a_{kl} = 0 \text{ for all } k \in \mathcal{N}_j, l \in \mathcal{N}_j, \quad \text{ie. } A^B_{ij} = 0^B,$$

where a_{kl} are entries of the stiffness matrix A. The disjoint covering $\{\mathcal{C}_i\}_{i=1}^{n_c}$ of the set of V_i's satisfying (52) can be created using the Algorithm 8.4.

Having implemented the coloring $\{\mathcal{C}_i\}_{i=1}^{n_c}$, we are able to solve the subdomain problems in parallel in n_c "waves", solving all subdomains in one particular $\mathcal{C}_j$ in parallel (by aditive algorithm), and taking all $\mathcal{C}_i$'s one after another multiplicatively.

6 Convergence Estimates for Smoothed Aggregation

In this section, we are going to connect the general convergence estimates with the way how the coarse space is derived (Section 3 and Section 5, respectively). In order to prove convergence of the method, we only have to verify that Assumptions 3.1 and 3.9 are satisfied.

Definition 6.1 *The systems of possible non-zeros in the stiffness matrix $A = \{A^B_{ij}\}_{i,j=1}^n$ can be represented by an undirected graph*

$$\mathcal{G} = \{\mathcal{V}, \mathcal{E}\},$$

where vertices $\mathcal{V} = \{1, \ldots, n\}$ are indices of all unconstrained nodes and edges $\mathcal{E}$ are given by

$$\mathcal{E} = \{[i, j] \in \mathcal{V}\ :\ A^B_{ij} \neq 0^B\}.$$

Definition 6.2 *Let us for $i \in \mathcal{V}$ and a positive integer r define the graph r-neighbourhood of i by*

$$\mathcal{B}(i,r) = \{j \in \mathcal{V} \ : \ \mathrm{dist}(i,j) \le r\}.$$

Here, the distance of two vertices i, j is the minimal length of path connecting i, j measured in the number of edges on the path.

Assumption 6.3 (shape and size of aggregates, cf. [2]) There are positive integer constants c, C, C_1, C_2 and a positive integer α characterizing the graph size of aggregates such that

1. In each aggregate $\mathcal{A}_i$ there is a node j satisfying

$$\mathcal{B}(j, c\alpha) \subset \mathcal{A}_i, \quad \text{and} \quad \mathrm{dist}(k,j) \le C\alpha \quad \text{for every} \quad k \in \mathcal{A}_i.$$

2. For the degree d_s of the prolongator smoother $\mathcal{S}$ there is

$$C_1\alpha \le d_s \le C_2\alpha.$$

The decomposition satisfying Assumption 6.3 can be easily generated by a simple greedy algorithm (Algorithm 8.6).

The following lemma is a key to verification of assumptions on energy of the coarse-space basis functions (Assumption 3.9 3)).

Lemma 6.4 *For the prolongator smoother created by Algorithms 5.4 and 5.8 it holds that*

$$\varrho(\mathcal{S}^2 A) \le C \deg(\mathcal{S})^{-2} \varrho(A) \qquad \textit{and} \qquad \varrho(\mathcal{S}) \le 1.$$

Proof The second estimate follows immediately from the fact, that $\mathcal{S}$ is a product of terms of the form $I - \frac{4}{3}\bar{\varrho}_j^{-1} A_j$.

The first inequality follows from the Remark 5.5:

$$\varrho(A_i) \le \left(\frac{1}{9}\right)^{i-1} \bar{\varrho} \equiv \bar{\varrho}$$

Further, it is easy to show for Algorithm 5.4 that

$$\mathcal{S}^2 A = A_K, \quad \text{where} \quad K = \lfloor \log_3(2d_S + 1) \rfloor$$

Therefore, using Remark 5.6 (ie. $C\deg(\mathcal{S}) \le C d_S \le 3^{K-1}$), and taking into account that $\bar{\varrho} \le C_\varrho \varrho(A)$, we arrive at

$$\varrho(\mathcal{S}^2 A) \le \left(\frac{1}{9}\right)^{K-1} \bar{\varrho} \le C \deg^{-2}(\mathcal{S}) \varrho(A).$$

For Algorithm 5.8 this inequality follows directly from Lemma 5.7.

■

The following lemma demonstrates validity of Assumption 3.9 3a) and 3b):

Lemma 6.5 *Under the Assumption 6.3, for coarse space basis functions Φ_j^i and the rigid body modes $r_j^i \in \mathrm{RBM}(\Omega_i)$ on any subdomain Ω_i, and for all $\gamma_j^i \in \mathbb{R}$, $j = 1, \ldots, 6$ there is*

$$\| \sum_{j=1}^{6} \gamma_j^i \Phi_j^i \|_{[L^2(\Omega_i)]^d} \leq \| \sum_{j=1}^{6} \gamma_j^i r_j^i \|_{[L^2(\Omega_i)]^d} \tag{53}$$

$$| \sum_{j=1}^{6} \gamma_j^i \Phi_j^i |_{\varepsilon(\Omega_i)} \leq C H^{-1} \| \sum_{j=1}^{6} \gamma_j^i r_j^i \|_{[L^2(\Omega_i)]^d} \tag{54}$$

where $d = 2, 3$ is the dimension of the problem.

Proof Let $\hat{\hat{r}}_j^i$ denote the restriction of $\hat{r}_j^i$ (of the discrete counterpart of $r_j^i \in \mathrm{RBM}(\Omega_i)$, also $\hat{\hat{r}}_j^i = \tilde{P} \hat{e}_j^i$) to the nodal aggregate Ω_i^A, $\Omega_i^A \subset \Omega_i$. Also, let us denote the finite element interpolator on the fine mesh by Π_h. Then from the linearity of Π_h and of $\mathcal{S}$ we have for (53):

$$\begin{aligned}
\| \sum_{j=1}^{6} \gamma_j^i \Phi_j^i \|^2_{[L^2(\Omega_i)]^d} = \| \sum_{j=1}^{6} \gamma_j^i \Pi_h \mathcal{S} \hat{\hat{r}}_j^i \|^2_{[L^2(\Omega_i)]^d} &= \| \Pi_h \mathcal{S} \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big) \|^2_{[L^2(\Omega_i)]^d} \\
&\leq \text{using } l^2 \text{ discrete norm instead of } L^2 \\
&\leq C h^d \Big\langle \mathcal{S} \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big), \mathcal{S} \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big) \Big\rangle \\
&\leq C h^d \varrho(\mathcal{S}^2) \Big\langle \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big), \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big) \Big\rangle \\
&\leq \text{Lemma 6.4, changing } l^2 \text{ to } L^2, \text{ extending } \hat{\hat{r}}_j^i \text{ to } \hat{r}_j^i \\
&\leq C \| \Pi_h \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big) \|^2_{[L^2(\Omega_i^A)]^d} \leq C \| \Pi_h \Big(\sum_{j=1}^{6} \gamma_j^i \hat{r}_j^i \Big) \|^2_{[L^2(\Omega_i)]^d} \\
&\leq C \| \sum_{j=1}^{6} \gamma_j^i r_j^i \|^2_{[L^2(\Omega_i)]^d}
\end{aligned}$$

Similarly, for (54) we have, when using the Gershgorin's estimate of the spectral radius $\varrho(A) \leq C h^{d-2}$:

$$\begin{aligned}
| \sum_{j=1}^{6} \gamma_j^i \Phi_j^i |^2_{\varepsilon(\Omega_i)} = | \sum_{j=1}^{6} \gamma_j^i \Pi_h \mathcal{S} \hat{\hat{r}}_j^i |^2_{\varepsilon(\Omega_i)} &= | \Pi_h \mathcal{S} \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big) |^2_{\varepsilon(\Omega_i)} \\
&\leq \text{using } l^2 \text{ discrete norm instead of } L^2, \text{ using Lemma 6.4} \\
&\leq C \Big\langle A \mathcal{S} \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big), \mathcal{S} \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big) \Big\rangle \\
&\leq C \deg(\mathcal{S})^{-2} \varrho(A) \Big\langle \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big), \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big) \Big\rangle \\
&\leq \deg(\mathcal{S}) \leq C\alpha = C \tfrac{H}{h}, \text{ Gershgorin, changing } l^2 \text{ to } L^2 \text{ norm} \\
&\leq C \Big(\frac{H}{h} \Big)^{-2} h^{d-2} \Big\langle \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big), \Big(\sum_{j=1}^{6} \gamma_j^i \hat{\hat{r}}_j^i \Big) \Big\rangle
\end{aligned}$$

$$\leq CH^{-2}\|\Pi_h\Big(\sum_{j=1}^{6}\gamma_j^i\hat{\tilde{r}}_j^i\Big)\|^2_{[L^2(\Omega_i^A)]^d}$$

$$\leq CH^{-2}\|\Pi_h\Big(\sum_{j=1}^{6}\gamma_j^i\hat{r}_j^i\Big)\|^2_{[L^2(\Omega_i)]^d} \leq CH^{-2}\|\sum_{j=1}^{6}\gamma_j^i r_j^i\|^2_{[L^2(\Omega_i)]^d}$$

which we had to prove.

■

Lemma 6.6 *(verification of Assumption 3.9) Under the Assumption 6.3 the coarse space basis functions Φ_j^i generated by smoothed aggregation satisfy the Assumption 3.9.*

Proof The Assumption 3.9 3) follows form Lemma 6.5. The Assumption 3.9 4) is satisfied straightforwardly (see (49)). It remains to verify Assumptions 3.9 1) and 2):

The basis functions $r_j^i \in \mathrm{RBM}(\Omega_i)$, $r_j^i = \Pi_h \tilde{P}\hat{e}_j^i$ satisfy the decomposition of local kernel of A (ie. decomposition of all RBM's) everywhere on $\Omega \setminus \Big(\bigcup_{\forall j} B_{\Gamma_j}\Big)$, where B_{Γ_j} is the union of elements $\mathcal{T}_k$ such that $\partial\mathcal{T}_k \cup \Gamma_j \neq \emptyset$.

Let us choose one particular Γ_i and suppose that we have a stiffness matrix A_{Γ_i} corresponding to our problem with Dirichlet's constraints just on Γ_i and let us denote the appropriate prolongator smoother by $\mathcal{S}_{\Gamma_i}$. Let $\mathcal{D}_{\Gamma_i}$ be the index set of all finite element nodes $v_j \in B_{\Gamma_i}$.

First, let us choose $r_{\Gamma_i} \in \mathrm{RBM}(\Omega)$ so that

$$\Pi_h\hat{r}_{\Gamma_i} = r_{\Gamma_i} \in \mathrm{RBM}(\Omega) \setminus \Big(\mathrm{RBM}(\Omega) \cap V_{\Gamma_i}(\Omega)\Big), \tag{55}$$

where $V_{\Gamma_i}(\Omega)$ is as in Definition 3.8, ie. we take some $\mathrm{RBM}(\Omega)$ such that it is influenced by the Dirichlet's boundary condition on Γ_i.

Then for such r_{Γ_i} it holds that

$$\Big(A_{\Gamma_i}\hat{r}_{\Gamma_i}\Big)_n = 0 \quad \text{for every} \quad n \notin \mathcal{D}_{\Gamma_i}, \tag{56}$$

where the zero on the right-hand side represents a block of zeros for all dofs on the n-th node, and the dereference $(\cdot)_n$ is taken with respect to nodes and not to dofs.

Furthermore, for a positive integer p there is

$$\Big(A_{\Gamma_i}^p\hat{r}_{\Gamma_i}\Big)_n = 0 \quad \text{for every node } n \text{ such that} \quad \mathrm{dist}(n, \mathcal{D}_{\Gamma_i}) \geq p+1 \tag{57}$$

where $\mathrm{dist}(\cdot,\cdot)$ is the graph distance of Definition 6.2.

On the other hand, when we choose r_{Γ_i} such that

$$\Pi_h\hat{r}_{\Gamma_i} = r_{\Gamma_i} \in \Big(\mathrm{RBM}(\Omega) \cap V_{\Gamma_i}(\Omega)\Big), \tag{58}$$

ie. we take some $\mathrm{RBM}(\Omega)$ that is not influenced by the Dirichlet's boundary condition on Γ_i, then for a positive integer p there is

$$\Big(A_{\Gamma_i}^p\hat{r}_{\Gamma_i}\Big)_n = 0 \quad \text{for every node } n \tag{59}$$

Let us now set

$$\hat{w}_{\Gamma_i} = \hat{r}_{\Gamma_i} - \mathcal{S}_{\Gamma_i}\hat{r}_{\Gamma_i} \tag{60}$$

Using $\hat{\bar{r}}_j^i$ (ie. discrete RBM's on the nodal aggregates Ω_i^A) we have

$$\hat{r}_{\Gamma_i} = \sum_{k \in I_1} \sum_{j=1}^{6} \alpha_j^k(\hat{r}_{\Gamma_i}) \hat{\bar{r}}_j^k, \tag{61}$$

where the existence of $\{\alpha_j^k\}$ is given by the fact that $\Pi_h \hat{\bar{r}}_j^k \in \mathrm{RBM}(\Omega_j^A)$ and that $\hat{\bar{r}}_j^k$ and $\hat{\bar{r}}_m^i$ are orthogonal for $k \neq i$, $j, m = 1 \ldots 6$.

Using linearity of $\mathcal{S}_{\Gamma_i}$ in (60) we arrive at

$$\begin{aligned} \hat{w}_{\Gamma_i} = \hat{r}_{\Gamma_i} - \mathcal{S}_{\Gamma_i} \hat{r}_{\Gamma_i} &= \hat{r}_{\Gamma_i} - \mathcal{S}_{\Gamma_i} \sum_{k \in I_1} \sum_{j=1}^{6} \alpha_j^k(\hat{r}_{\Gamma_i}) \hat{\bar{r}}_j^k \\ &= \hat{r}_{\Gamma_i} - \sum_{k \in I_1} \sum_{j=1}^{6} \alpha_j^k(\hat{r}_{\Gamma_i}) \mathcal{S}_{\Gamma_i} \hat{\bar{r}}_j^k \end{aligned} \tag{62}$$

Following (57) and (59) and applying the fact that $\mathcal{S}_{\Gamma_i}$ is a polynomial with the absolute term equal to 1 and that $\deg(\mathcal{S}_{\Gamma_i}) = \deg(\mathcal{S})$ we get

$$(\hat{w}_{\Gamma_i})_n = 0 \begin{cases} \forall n \ : \ \mathrm{dist}(n, \mathcal{D}_{\Gamma_i}) \geq \deg(\mathcal{S}) & \text{if } \hat{r}_{\Gamma_i} \text{ is from (55)} \\ \forall n & \text{if } \hat{r}_{\Gamma_i} \text{ is from (58)} \end{cases} \tag{63}$$

This shows, that a natural choice of the patches G_i from the Assumption 3.9 2) would be an envelope of such nodes n which violate (63) for one particular Γ_i. Indeed, such G_i's would consist of at most $\deg(\mathcal{S})$ strips of elements surrounding Γ_i. And because of $\deg(\mathcal{S}) \leq C\alpha = CH/h$ and because of the quasiuniformity of the mesh we have for our choice of G_i's (cf. Assumption 3.9 2)):

$$\mathrm{dist}(x, \Gamma_i) \leq CH \quad \forall x \in G_i, \tag{64}$$

where $\mathrm{dist}(\cdot, \cdot)$ is the Euclidean distance.

Now, considering (63) just on $\hat{\Omega}_i$'s, where $\hat{\Omega}_i$ is the set of nodes in Ω_i, and taking all the Γ_i's together we get

$$\Big(\hat{r} - \sum_{k \in \mathcal{N}_i} \sum_{j=1}^{6} \alpha_j^k(\hat{r}) \mathcal{S} \hat{\bar{r}}_j^k \Big)_m = 0, \tag{65}$$

for all nodes $m \in \hat{\Omega}_i$, if

$$\Pi_h \hat{r} = r \in \mathrm{RBM}(\Omega_i) \cap \Big(\bigcap_{j \in \mathcal{G}_i} V_{\Gamma_i}(\Omega) \Big), \tag{66}$$

where $\mathcal{G}_i$ is the index set of locally "active" Dirichlet constraints Γ_i (cf. Assumption 3.9 2)). In fact, (65) is a discrete counterpart of

$$r = \sum_{k \in \mathcal{N}_i} \sum_{j=1}^{6} \alpha_j^k(r) \Phi_j^k \tag{67}$$

for r as in (66), as $\mathcal{S} \hat{\bar{r}}_j^k = \hat{\Phi}_j^k$. This verifies the Assumption 3.9 2)).

Following (61) we have for $r \in \mathrm{RBM}(\Omega)$

$$r = \sum_{k \in I_1} \sum_{j=1}^{6} \alpha_j^k(r) \bar{r}_j^k \tag{68}$$

By restricting the latter equation to Ω_i and extending $\bar{r}^i_j \in \mathrm{RBM}(\Omega^A_i)$ to $r^i_j \in \mathrm{RBM}(\Omega_i)$ we get

$$r = \sum_{j=1}^{6} \alpha^i_j(r) r^i_j \quad \text{restricted on } \Omega_i \tag{69}$$

which together with (67) verifies the Assumption 3.9 3)).

■

Lemma 6.7 *(verification of Assumption 3.1, cf. [2]) Under the Assumption 6.3, the computational subdomains Ω_i defined by (48) satisfy the Assumption 3.1.*

Proof Computational subdomains are defined by $\Omega_i = \mathrm{supp}(\Pi_h P^{\mathrm{symb}} \hat{e}^i_j)$, where P^{symb} is the symbolic smoothed prolongator with ones in place of possible non-zeros. It has been created by symbolical smoothing of the tentative prolongator $\tilde{P}$ by the smoother $\mathcal{S}$. The degree of $\mathcal{S}$ satisfies

$$c\alpha \leq \deg(\mathcal{S}) \leq C\alpha$$

Further, the support of basis functions derived from the tentative prolongator

$$\mathrm{supp}(\Pi_h \tilde{P} \hat{e}^i_j)$$

is formed by all elements $\mathcal{T}_j$ such that at least one vertex of $\mathcal{T}_j$ belongs to the aggregate $\mathcal{A}_i$. The smoothing by $\mathcal{S}$ adds $\deg(\mathcal{S})$ layers of surrounding finite elements.

Taking into account the quasiuniformity of the mesh and the fact that $H = h\alpha \approx h{\cdot}\deg(\mathcal{S})$, the measure of the added $\deg(\mathcal{S})$ layers of elements itself is greater or equal to CH^d. So,

$$\mathrm{meas}(\Omega_i) \geq CH^d,$$

which proves the Assumption 3.1 4). Also, due to Assumption 6.3 1), we similarly obtain $\mathrm{diam}(\Omega_i) \leq CH$. Hence 1) is also verified.

Let us prove 2). The supports of basis functions $\Pi_h \tilde{P} \hat{e}^i_j$ cover the domain Ω. As Ω_i is created by adding $\deg(\mathcal{S}) \geq c\alpha$ layers of elements to $\mathrm{supp}(\Pi_h \tilde{P} \hat{e}^i_j)$, we have

$$\mathrm{dist}(x, \partial\Omega_i) \geq cH \quad \forall x \in \mathrm{supp}(\Pi_h \tilde{P} \hat{e}^i_j),$$

where $\mathrm{dist}(\cdot,\cdot)$ is the Euclidean distance. As every point $x \in \Omega$ belongs to some $\mathrm{supp}(\Pi_h \tilde{P} \hat{e}^i_j)$, 2) is proved.

It remains to verify 3). The first part of the Assumption 6.3 1) says that each aggregate $\mathcal{A}_i$ contains a "graph ball" of radius $r \geq c\alpha$, where c is a positive integer constant. Let us interpret this assumption geometrically in $\mathbb{R}^d$.

For each aggregate of vertices $\mathcal{A}_i$ let us define the cluster C_i consisting of all finite elements $\mathcal{T}_i$ so that all vertices of $\mathcal{T}_i$ belong to $\mathcal{A}_i$. From the first part of the Assumption 6.3 1), it follows that there is a ball $B_i \subset C_i$ such that $\mathrm{diam}(B_i) \geq cH$. As the aggregates $\mathcal{A}_i$ are disjoint, the clusters C_i are disjoint and balls B_i are disjoint as well. Summing up, we have proved the following properties for subdomains Ω_i:

- $\mathrm{diam}(\Omega_i) \leq CH$
- For each Ω_i there is a ball $B_i \subset \Omega_i$ such that $\mathrm{diam}(B_i) \geq cH$ and balls B_i, $i = 1 \dots m$ are mutually disjoint.

From here, the Assumption 3.1 3) follows.

■

7 Discrete Coarse Space Estimates

In this section, we are going to deal with a modification of the convergence theory devised in Section 3.2. Unlike the already presented continuous estimates for approximation property and stability of Q_0, this discrete theory is completely meas(Γ_i)-independent (see Remark 3.19). All assumptions here on the coarse space are done through the properties of the smoother $\mathcal{S}$ and the tentative prolongator $\tilde{P}$.

Recall that $\|\cdot\|_A = \langle A\cdot,\cdot\rangle$ is the discrete $|\cdot|_\varepsilon$ norm and let $\|\cdot\|_{[l^2]^d}$ be the discrete l^2-norm (Euclidean norm in $\mathbb{R}^{d\cdot n}$). Further, we still use H, h to denote the resolutions of the coarse space, or of the fine mesh, respectively.

Definition 7.1 *Let us define a subdomain Ω_i^A, $\Omega_i^A \subset \Omega$ as a union of finite element triangles $\mathcal{T}_j$ such that all nodes of $\mathcal{T}_j$ are in $\mathcal{A}_i$, where $\mathcal{A}_i$ is the aggregation of nodes as in Definition 5.1. Also,*

$$\Omega_i^A \subset \mathrm{supp}(\Pi_h \tilde{P}\hat{e}_j^i)$$

where $\tilde{P}$ is the tentative prolongator.

Definition 7.2 *(interpolators $\tilde{Q}_0$ and Q_0) Being given the tentative prolongator $\tilde{P}$ we can define a tentative coarse space $\tilde{V}_0$ by $\tilde{V}_0 = \mathrm{span}\{\Pi_h\tilde{P}\hat{e}_j^i\}_{j=1\ldots6}^{i\in I_1}$, where $\hat{e}_j^i$ is a vector of the canonical basis for $\mathbb{R}^{n_k m}$, $i \in I_1$, $j = 1\ldots6$.*

Also, let us define subspaces $\tilde{V}_0^i \subset \tilde{V}_0$ by $\tilde{V}_0^i = \mathrm{span}\{\Pi_h\tilde{P}\hat{e}_j^i\}_{j=1\ldots6}$.

Let us define the linear interpolator $\tilde{Q}_0$ from the fine finite element space onto the tentative coarse space, $\tilde{Q}_0 : V \to \tilde{V}_0$ by

$$\tilde{Q}_0 u = \sum_{i\in I_1}\sum_{j=1}^{6} \alpha_j^i(u)\bar{r}_j^i \quad , \quad \bar{r}_j^i = \Pi_h\tilde{P}\hat{e}_j^i$$

where $\{\alpha_j^i(u)\}$'s are such that $\sum_{j=1}^6 \alpha_j^i(u)\bar{r}_j^i$ is a L^2-orthogonal projection of u onto $\tilde{V}_0^i$ for all $i \in I_1$.

Also, let us define the coarse space $V_0 \subset V$ as in Definition 3.12, that is by:

$$V_0 = \mathrm{span}\{\Phi_j^i\}_{j=1\ldots6}^{i\in I_1}, \quad \Phi_j^i = \mathcal{S}\bar{r}_j^i$$

. Now, we can define the interpolator Q_0 onto the coarse space $Q_0 : V \to V_0$ by

$$Q_0 u = \mathcal{S}\tilde{Q}_0 u$$

for all $u \in V$. Let us further denote the discrete counterpart of $\tilde{Q}_0$ for simplicity also by $\tilde{Q}_0$, ie. we will take the following as vectors

$$\tilde{Q}_0\hat{u} = \sum_{i\in I_1}\sum_{j=1}^{6} \alpha_j^i(\hat{u})\hat{\bar{r}}_j^i$$

where $\Pi_h\hat{\bar{r}}_j^i = \bar{r}_j^i \in \tilde{V}_0^i$, and $\Pi_h\hat{u} = u$, with $\{\alpha_j^i\}$'s such that $\Pi_h\tilde{Q}_0\hat{u} = \tilde{Q}_0 u$.

Assumption 7.3 (tentative coarse-space basis) For all $r \in \mathrm{RBM}(\Omega)$ there exist $\{\alpha_j^i\}$'s such that

$$\sum_{i=1}^{n}\sum_{j=1}^{6}\alpha_j^i \bar{r}_j^i = r \quad \text{on } \Omega \setminus \Big(\bigcup_{\forall j} B_{\Gamma_j}\Big) \tag{70}$$

where $\{\bar{r}_j^i\}$ is the tentative coarse-space basis, $\bar{r}_j^i = \Pi_h \tilde{P} \hat{e}_j^i$, and $\bigcup_{\forall j} B_{\Gamma_j}$ is the union of elements $\mathcal{T}_k$ adjacent to all Γ_j's. In another words, we require that $\bar{r}_j^i|_{\Omega_i^A} \in \mathrm{RBM}(\Omega_i^A)$.

Lemma 7.4 *(weak approxim. property for $\tilde{Q}_0$) Under the Assumption 7.3 we are able to prove for $\tilde{Q}_0$ from the Definition 7.2 that*

$$\|u - \tilde{Q}_0 u\|_{[L^2(\Omega)]^d} \le CH|u|_{\varepsilon(\Omega)}, \tag{71}$$

or, equivalently, in discrete norms

$$\|\hat{u} - \tilde{Q}_0 \hat{u}\|_{[l^2(\Omega)]^d} \le C\frac{H}{h}\|\hat{u}\|_A$$

for all $u \in V$, $\hat{u} = \Pi_h^{-1} u$. The constant C is independent of H and h.

Proof

$$\begin{aligned}
\|u - \tilde{Q}_0 u\|^2_{[L^2(\Omega)]^d} &\le Ch^2\|\hat{u} - \tilde{Q}_0\hat{u}\|^2_{[l^2(\Omega)]^d} = Ch^2\sum_i \|\hat{u} - \tilde{Q}_0\hat{u}\|^2_{[l^2(\Omega_i^A)]^d} \\
&= Ch^2\sum_i \|\hat{u}_i - \tilde{Q}_0^i\hat{u}\|^2_{[l^2(\Omega_i^A)]^d},
\end{aligned}$$

where $u_i = u|_{\Omega_i^A}$ and $\tilde{Q}_0^i u = \tilde{Q}_0 u|_{\Omega_i^A}$.

Further, set $\hat{\bar{u}}_i = \hat{u}_i - k$, $k \in \mathrm{RBM}(\Omega_i^A)$ such that $(\hat{\bar{u}}_i, \mathrm{RBM}(\Omega_i^A))_{l^2(\Omega_i^A)} = 0$. Then

$$\begin{aligned}
\|\hat{u}_i - \tilde{Q}_0^i\hat{u}\|_{[l^2(\Omega_i^A)]^d} &= \text{ using Assumption 7.3} \\
&= \|\hat{\bar{u}}_i - \tilde{Q}_0^i\hat{\bar{u}}_i\|_{[l^2(\Omega_i^A)]^d} \\
&\le \|\hat{\bar{u}}_i\|_{[l^2(\Omega_i^A)]^d} + \|\tilde{Q}_0^i\hat{\bar{u}}_i\|_{[l^2(\Omega_i^A)]^d}
\end{aligned}$$

where using the property of l^2-orthogonal projection onto $\mathrm{RBM}(\Omega_i^A)$ gives

$$\begin{aligned}
\|\tilde{Q}_0^i\hat{u}\|_{[l^2(\Omega_i^A)]^d} &= \|\sum_{j=1}^{6}\alpha_j^i(\hat{\bar{u}}_i)\hat{\bar{r}}_j^i\|_{[l^2(\Omega_i^A)]^d} \\
&\le C\|\hat{\bar{u}}_i\|_{[l^2(\Omega_i^A)]^d}
\end{aligned}$$

Thus,

$$\begin{aligned}
h\|\hat{u} - \tilde{Q}_0\hat{u}\|_{[l^2(\Omega_i^A)]^d} &\le Ch\|\hat{\bar{u}}_i\|_{[l^2(\Omega_i^A)]^d} \le C\|\bar{u}_i\|_{[L^2(\Omega_i^A)]^d} = C\|u - k\|_{[L^2(\Omega_i^A)]^d} \\
&\le \text{using Poincaré - Korn} \\
&\le CH|u|_{\varepsilon(\Omega_i^A)}
\end{aligned}$$

and summing squares of the latter over i completes the proof.

■

The main idea of this discrete approach is to verify the needed properties of the coarse space, namely ε-stability and weak approximation property with the help of just algebraic assumptions on the basis functions of the tentative coarse-space basis, ie. unsmoothed $\{r^i_j\}$'s and on the polynomial smoother.

Under the algebraic assumption on the tentative basis (Assumption 7.3) we are able to prove the approximation property of the tentative coarse-space basis. Combining this result with assumptions on the smoother $\mathcal{S}$ (Assumption 7.5) we are able to verify weak approximation property and ε-stability also for smoothed coarse-space basis.

It is very important to realize, that by Assumptions 7.3 and 7.5 replacing Assumption 3.9 we gain the independence on Dirichlet boundary conditions with potentially small measure.

Assumption 7.5 (smoother $\mathcal{S}$) Let us assume that for the smoother $\mathcal{S}$ there exist positive constants C_1, C_2, and C_3 such that

1. $$\|\mathcal{S}\hat{u}\|_A \leq C_1\|\hat{u}\|_A \quad \text{and} \quad \|\mathcal{S}\hat{u}\|_{[l^2]^d} \leq C_1\|u\|_{[l^2]^d}$$
2. $$\|(I-\mathcal{S})\hat{u}\|_{[l^2]^d} \leq C_2\tfrac{H}{h}\|\hat{u}\|_A$$
3. $$\varrho(S^TAS) \leq C_3\left(\tfrac{h}{H}\right)^2\varrho(A)$$

for all $\hat{u} \in \mathbb{R}^{dn}$.

Lemma 7.6 *(discrete approximation property) Under the Assumption 7.5 we can say that if we have the weak approximation property for $\tilde{Q}_0$, ie*

$$\|\hat{u}-\tilde{Q}_0\hat{u}\|_{[l^2]^d} \leq C\frac{H}{h}\|\hat{u}\|_A, \tag{72}$$

for all $\hat{u} = \Pi_h^{-1}u$, $u \in V$, then we also have the weak approximation property for $Q_0 = \mathcal{S}\tilde{Q}_0$:

$$\|\hat{u}-\mathcal{S}\tilde{Q}_0\hat{u}\|_{[l^2]^d} \leq C'\frac{H}{h}\|\hat{u}\|_A, \tag{73}$$

where the constants do not depend on h and H.

Proof

$$\begin{aligned}\|\hat{u}-\mathcal{S}\tilde{Q}_0\hat{u}\|_{[l^2]^d} &= \|\mathcal{S}(\hat{u}-\tilde{Q}_0\hat{u}) + (I-\mathcal{S})\hat{u}\|_{[l^2]^d}\\ &\leq \|\mathcal{S}(\hat{u}-\tilde{Q}_0\hat{u})\|_{[l^2]^d} + \|(I-\mathcal{S})\hat{u}\|_{[l^2]^d}\\ &\leq \text{using the Assumption 7.5 1) for the first term}\\ &\leq \|\hat{u}-\tilde{Q}_0\hat{u}\|_{[l^2]^d} + \|(I-\mathcal{S})\hat{u}\|_{[l^2]^d}\end{aligned}$$

Now, we apply the Assumption 7.5 2) and (72) and get:

$$\begin{aligned}\|\hat{u}-\mathcal{S}\tilde{Q}_0\hat{u}\|_{[l^2]^d} &\leq \|\hat{u}-\tilde{Q}_0\hat{u}\|_{[l^2]^d} + \|(I-\mathcal{S})\hat{u}\|_{[l^2]^d}\\ &\leq C\frac{H}{h}\|\hat{u}\|_A + C_2\frac{H}{h}\|\hat{u}\|_A = C'\frac{H}{h}\|\hat{u}\|_A.\end{aligned}$$

Which is what we had to prove.

■

Lemma 7.7 *(discrete stability) Under the Assumption 7.5 1) and the Assumption 7.3 we have for all* $\hat{u} = \Pi_h^{-1} u$, $u \in V$ *that*

$$\|Q_0 \hat{u}\|_A \leq C \|\hat{u}\|_A,$$

where the constant C *is independent of* h *nad* H.

Proof

$$\begin{aligned} \|Q_0 \hat{u}\|_A &= \|\mathcal{S}\tilde{Q}_0 \hat{u}\|_A = \|\hat{u} - \mathcal{S}\tilde{Q}_0 \hat{u} + \hat{u}\|_A \\ &\leq \|\hat{u} - \mathcal{S}\tilde{Q}_0 \hat{u}\|_A + \|\hat{u}\|_A \\ &= \|\mathcal{S}(\hat{u} - \tilde{Q}_0 \hat{u}) + (I - \mathcal{S})\hat{u}\|_A + \|\hat{u}\|_A \\ &\leq \|\mathcal{S}(\hat{u} - \tilde{Q}_0 \hat{u})\|_A + \|(I - \mathcal{S})\hat{u}\|_A + \|\hat{u}\|_A \\ &\leq \|\mathcal{S}(\hat{u} - \tilde{Q}_0 \hat{u})\|_A + \|\hat{u}\|_A + \|\mathcal{S}\hat{u}\|_A + \|\hat{u}\|_A \\ &\leq \text{using the Assumption 7.5 for the third term} \\ &\leq \|\mathcal{S}(\hat{u} - \tilde{Q}_0 \hat{u})\|_A + C\|\hat{u}\|_A \end{aligned} \tag{74}$$

From

$$\|\mathcal{S}(\hat{u} - \tilde{Q}_0 \hat{u})\|_A \leq \sqrt{\varrho(S^2 A)} \|\hat{u} - \tilde{Q}_0 \hat{u}\|_{[l^2]^d}$$

we get by using the Assumption 7.5 3):

$$\|\mathcal{S}(\hat{u} - \tilde{Q}_0 \hat{u})\|_A \leq C \frac{h}{H} \sqrt{\varrho(A)} \|\hat{u} - \tilde{Q}_0 \hat{u}\|_{[l^2]^d}$$

which together with (71) and (74) proves the lemma.

■

Now, we have to verify the properties of the smoother $\mathcal{S}$:

Lemma 7.8 *(verification of Assumption 7.5) The smoother* $\mathcal{S}$ *generated by the Algorithm 5.4 satisfies the Assumption 7.5.*

Proof The verification of Assumption 7.5 1) and 3) follows straight from the Lemma 6.4. It remains to show 2):

Let us define $\mathcal{S}_k$ by $\mathcal{S}_k = \prod_{j=1}^k S_j$, $\mathcal{S} = \mathcal{S}_K$, where S_i's are as in Algorithm 5.4. Then for all $\hat{u} \in R^{dn}$ we have

$$\begin{aligned} \|(I - \mathcal{S})\hat{u}\|_{[l^2]^d} &= \|(I - \mathcal{S}_K)\hat{u}\|_{[l^2]^d} = \|(I - \mathcal{S}_{K-1} S_K)\hat{u}\|_{[l^2]^d} \\ &= \|[I - (I - \frac{\omega}{\bar{\varrho}_K} A_K)\mathcal{S}_{K-1}]\hat{u}\|_{[l^2]^d} \\ &\leq \|(I - \mathcal{S}_{K-1})\hat{u}\|_{[l^2]^d} + \frac{\omega}{\bar{\varrho}_K} \|A_K \mathcal{S}_{K-1} \hat{u}\|_{[l^2]^d} \\ &\leq \text{ using that } \varrho(\mathcal{S}_k) \leq 1 \text{ for all } k \\ &\leq \|(I - \mathcal{S}_{K-1})\hat{u}\|_{[l^2]^d} + \frac{\omega}{\bar{\varrho}_K} \|A_K \hat{u}\|_{[l^2]^d} \end{aligned}$$

where

$$\|A_K \hat{u}\|_{[l^2]^d} = \|\mathcal{S}_{K-1}^2 A \hat{u}\|_{[l^2]^d} \leq \|A\hat{u}\|_{[l^2]^d} \leq \sqrt{\varrho(A)} \|\hat{u}\|_A.$$

Using the same resursively for $\|(I-\mathcal{S}_k)u\|_{[l^2]^d}$ gives

$$\|(I-\mathcal{S})\hat{u}\|_{[l^2]^d} \leq \sum_{i=1}^{K} \frac{\omega}{\bar{\varrho}_i}\sqrt{\varrho(A)}\|\hat{u}\|_A.$$

and because $\bar{\varrho}_k = (\frac{1}{9})^{k-1}\bar{\varrho}$ and $\varrho(A) \leq \bar{\varrho} \leq C_\varrho \varrho(A)$ we have:

$$\begin{aligned}\|(I-\mathcal{S})\hat{u}\|_{[l^2]^d} &\leq C\frac{3^K-1}{2}\frac{\sqrt{\varrho(A)}}{\varrho(A)}\|\hat{u}\|_A \leq C\deg(\mathcal{S})\frac{1}{\sqrt{\varrho(A)}}\|\hat{u}\|_A \\ &\leq C\frac{H}{h}\frac{1}{\sqrt{\varrho(A)}}\|\hat{u}\|_A\end{aligned}$$

which was to be proved.

■

Lemma 7.9 *(verification of Assumption 7.3) The tentative basis $\{\bar{r}^i_j\}$ of the coarse space constructed as in the Section 5 satisfies the Assumption 7.3.*

Proof The existence of such coefficients $\{\alpha^i_j\}$ simply follows from the fact that

$$\hat{\bar{r}}^i_j = \Pi_h^{-1}\bar{r}^i_j, \quad \bar{r}^i_j|_{\Omega^A_i} \in \mathrm{RBM}(\Omega^A_i),$$

and that $\hat{\bar{r}}^i_j$ is orthogonal to $\hat{\bar{r}}^k_m$, $j,m = 1\ldots 6$, $k \neq i$.

■

To summarize the results, Lemma 7.6 and Lemma 7.7 say that our interpolator $Q_0 : V \to V_0$ satisfies the Assumption 3.2 needed for the estimates on the fine level and for proving the h and H independance for the constant in Lions' lemma (Lemma 2.3). The remaining Assumption 3.1 can be verified in the same way as in Lemma 6.7.

Having verified these two Assumptions, we can state the main convergence theorem for the method (Theorem 2.11) about the H and h independent rate of convergence of the method.

Corollary 7.10 (convergence theorem) Assumptions 7.5, 7.3 and Lemmas 7.6 and 7.7 and Lemma 3.6 imply that for all $v \in V$ there exists a decomposition $\{v_i\}$, $v_i \in V_i$, $i \in I$ such, that $v = \sum_i v_i$ and $\sum_i |v_i|^2_\varepsilon \leq C_L|v|^2_\varepsilon$. Thus, we can use the Lions' lemma (Lemma 2.3) and for $\varrho(\epsilon)$ as in Definition 2.4 and for the energy norm of an error contractor E of the Schwarz overlapping domain decomposition (see Definition 2.2) we can give a uniform upper bound

$$\|E\|^2_\varepsilon \leq 1 - \frac{1}{C_L[1+\varrho(\epsilon)]^{\mathbf{2}}}$$

8 Computer implementation

8.1 Algorithms

Algorithm 8.1 (prolongator smoother, version 1, cf. Algorithm 5.4) For the desired degree d_s of the prolongator smoother $\mathcal{S}$ and for the estimate $\bar{\varrho}$ of the spectral radius of A such that

$$\varrho(A) \leq \bar{\varrho} \leq C_\varrho \varrho(A)$$

we define the prolongator smoother $\mathcal{S}$ by

1. Let $K = \lfloor \log_3(2d_s + 1) \rfloor$, where $\lfloor \cdot \rfloor$ is the truncation to the nearest smaller integer.
2. Set $A_1 = A$, $A_0 = I$, $\bar{\varrho}_1 = \bar{\varrho}$.
3. Define for $i = 1 \dots K$ polynomial smoothers S_i by

$$S_i = I - \frac{\omega}{\bar{\varrho}_i} A_i \ , \quad \omega = \frac{4}{3},$$

   where

$$A_{i+1} = S_i^2 A_i \quad , \quad \bar{\varrho}_{i+1} = \frac{1}{9} \bar{\varrho}_i$$

   The choice of $\omega = \frac{4}{3}$ will be justified in the Remark 5.5

4. $\mathcal{S} = \prod_{i=1}^{K} S_i$.

Algorithm 8.2 (prolongator smoother, improved version 1) Smoothing the tentative prolongator $\tilde{P}$ gradually already during the construction of $\mathcal{S}$ gives the Algorithm 5.4 in this form:

1. $K := \lfloor \log_3(2d_s + 1) \rfloor$
2. $S_0 := I$, $A_0 := A$, $P_0 := \tilde{P}$, $\bar{\varrho}_1 := \bar{\varrho}$
3. for $i = 1 \dots K$ do
   (a) $A_i := S_{i-1}^2 A_{i-1}$
   (b) $S_i := I - \frac{\omega}{\bar{\varrho}_i} A_i \ , \quad \omega = \frac{4}{3} \ , \quad \bar{\varrho}_{i+1} = \frac{1}{9} \bar{\varrho}_i$
   (c) $P_i := S_i P_{i-1}$

   end for.
4. $\mathcal{S}\tilde{P} := P_K$

Let us stress out that we do not need to assemble the polynomials S_i and matrices A_i in the steps 3a) and 3b), because the expression $P_i = S_i P_{i-1}$ can be evaluated more cheaply using Horner's scheme. In this way, we avoid the huge fill-in which would have occurred when making higher powers of A.

Algorithm 8.3 (prolongator smoother, version 2, cf. Algorithm 5.8) For the desired degree d_s of the prolongator smoother $\mathcal{S}$ and for the estimate $\bar{\varrho}$ of the spectral radius of A such that

$$\varrho(A) \leq \bar{\varrho} \leq C_\varrho \varrho(A)$$

we define the prolongator smoother $\mathcal{S}$ by

$$\mathcal{S} = \prod_{k=1}^{d_s} \left(I - r_k^{-1} A \right),$$

where

$$r_k = \frac{\bar{\varrho}}{2} \left(1 - \cos \frac{2k\pi}{2d_s + 1} \right), \quad k = 1, \ldots, d_s. \tag{75}$$

Algorithm 8.4 (disjunct covering $\{\mathcal{C}_i\}_{i=1}^{n_c}$, cf. [2])

set $\mathcal{R} = \{1, \ldots, m\}$ and $i = 0$

repeat

set $i := i + 1$,
set $\mathcal{R}_i := \mathcal{R}$, $\mathcal{C}_i = \emptyset$,
repeat
choose $j \in \mathcal{R}_i$,
set $\mathcal{C}_i := \mathcal{C}_i \cup \{j\}$,
for each k: if $\cos(\hat{V}_j, \hat{V}_k) \neq 0$ then set $\mathcal{R}_i := \mathcal{R}_i \setminus \{k\}$,
until $\mathcal{R}_i$ is empty,
set $\mathcal{R} := \mathcal{R} \setminus \mathcal{C}_i$,

until $\mathcal{R}$ is empty,

set $n_c := i$.

Now we have all components needed to write down the implementation of the Schwarz method with the error propagation operator E:

Algorithm 8.5 (Schwarz overlapping domain decomposition, cf. [2])
Given a vector x^i, the method returns x^{i+1} computed as follows:

1. set $z^0 := x^i$.
2. for $i = 1, \ldots, n_c$ do local corrections:
$$z^i := z^{i-1} + \sum_{j \in \mathcal{C}_i} R_j d^i \ , \quad \text{where } d^i := b - Az^{i-1}$$
and R_j is the correction operator defined by (50).
3. coarse-level correction:
$$z^0 := z^{n_c} + R_0(b - Az^{n_c}),$$
where R_0 is the coarse-level correction given by (51).

4. (optional) set $z^{n_c} := z^0$
   for $i = n_c, \ldots, 1$ do

$$z^{i-1} := z^i + \sum_{j \in \mathcal{C}_i} R_j d^i \ , \quad \text{where } d^i := b - Az^i$$

5. set $x^{i+1} = z^0$.

Note that if the optional post-smoothing step is used, then the algorithm can be used as a symmetric preconditioner of conjugate gradients.

Algorithm 8.6 (greedy algorithm) Let us first extend the definition of graph neighbourhood of a node in Definition 6.2 to the graph neighbourhood of a set $X \subset \{1, \ldots, n\}$ of nodes:

$$\mathcal{B}(X, \alpha) = \{i : \text{dist}(i, X) \leq \alpha\}.$$

Then for the stiffness matrix A and a positive integer α creates the following algorithm a system of aggregates $\{\mathcal{A}_i\}$:

1. set $\mathcal{R} := \{1, \ldots, n\}$, $j := 0$.
2. for $i = 1, \ldots, n$ do

   if $i \in \mathcal{R}$ then

   if $\mathcal{B}(\{i\}, \alpha) \setminus \mathcal{R} = \emptyset$ then
   set $j := j + 1$,
   set $\mathcal{A}_j := \mathcal{B}(\{i\}, \alpha)$,
   set $\mathcal{R} := \mathcal{R} \setminus \mathcal{A}_j$,
   end if

   end if

   end for

3. set $m := j$,
4. for $i = 1, \ldots, m$

   set $\mathcal{A}_i := \mathcal{A}_i \cup \Big(\mathcal{B}(\mathcal{A}_i, \alpha) \cap \mathcal{R}\Big)$,
   set $\mathcal{R} := \mathcal{R} \setminus \mathcal{A}_i$

   end for

Remark 8.7 Algorithm 8.6 creates a disjoint covering $\{\mathcal{A}_i\}$ of $\mathcal{R}$. The first for loop creates big lumps of nodes $\mathcal{A}_i$, which do not cover $\mathcal{R}$ fully, but their distance from each other is less than α. In the second part, the remaining non-aggregated nodes are divided into the nearest aggregates, such that none of them is left uncovered.

8.2 Computational complexity

Following Vaněk and Brezina in [2], we will now give an asymptotic bound on the amount of floating point operations needed to carry out the iteration to reduce the error to the truncation level. We will give the estimates for implementation on both serial and parallel architectures.

Let N_{es} denote the typical number of elements per subdomain, d be the dimension of the space on which the continuous problem is cast, and n be the number of all degrees of freedom in the whole system.

Let us first compute the amount of work needed for the setup. On a machine with a single CPU, we need $\mathcal{O}(\deg(\mathcal{S})n)$ operations to compute the prolongator $P = \mathcal{S}\tilde{P}$. Taking into account that $\deg(\mathcal{S}) \approx \frac{H}{h} \approx N_{es}^{1/d}$, this becomes $\mathcal{O}(N_{es}^{1/d} n)$. Further, we need $\mathcal{O}(\frac{n}{N_{es}} N_{es}^{\frac{3d-2}{d}})$ and $\mathcal{O}((\frac{n}{N_{es}})^{\frac{3d-2}{d}})$ operations to compute Cholesky factorizations of the local and coarse level matrices, respectively. We also need $\mathcal{O}(n)$ operations to compute the coarse level matrix, but this number can be taken out of the considerations, as it is dominated by other expenditures.

Each step of the iteration requires $\mathcal{O}(\frac{n}{N_{es}} N_{es}^{\frac{2d-1}{d}})$ and $\mathcal{O}((\frac{n}{N_{es}})^{\frac{2d-1}{d}})$ operations to compute the back substitutions in the local and coarse spaces, respectively. The amount of work required to compute the defect, the corrections and restriction is $\mathcal{O}(n)$ and hence negligible.

Taking into account all the above listed expenditures, we use trivial calculus to conclude that the optimal value of number of elements per subdomain is $N_{es} = n^{\frac{2d-2}{5d-4}}$. That is, $N_{es}^{opt} = n^{\frac{1}{3}}$ for 2D problems and $N_{es}^{opt} = n^{\frac{4}{11}}$ for 3D problems. The total amount of work involved in the setup and iterations for these optimal values is $\mathcal{O}(n^{\frac{4}{3}})$ and $\mathcal{O}(n^{\frac{49}{33}})$ in 2D and 3D, respectively.

The reason why we introduced the coloring classes $\mathcal{C}_i$ in the algorithm was to facilitate the use of modern parallel architecture computers. For simplicity, we assume that we have at least $n^{\frac{1}{2}}$ processors. Then most of the procedures can take advantage of parallel implementation. In the evaluations of computational work we omit all operations costing $\mathcal{O}(n)$ operations.

The setup will then require $\mathcal{O}(\deg(\mathcal{S})n^{\frac{1}{2}})$ operations to compute the prolongator $P = \mathcal{S}\tilde{P}$. If we assume that the local Cholesky decompositions are performed in parallel, we need $\mathcal{O}(N_{es}^{\frac{3d-2}{d}})$ and $\mathcal{O}((\frac{n}{N_{es}})^{\frac{3d-2}{d}})$ operations to compute Cholesky factorizations of the local and coarse level matrices, respectively.

Each step of the iteration will require $\mathcal{O}(N_{es}^{\frac{2d-1}{d}})$ and $\mathcal{O}((\frac{n}{N_{es}})^{\frac{2d-1}{d}})$ operations to compute the back substitutions in the local and coarse spaces, respectively.

Balancing these values, we obtain that the optimal size of a subdomain is about $n^{\frac{1}{2}}$ in both 2D and 3D. The resulting computational complexity can be then bounded by $\mathcal{O}(n)$ in 2D and by $\mathcal{O}(n^{\frac{7}{6}})$ in 3D.

The above estimates show that the amount of work required to complete the whole iterative process (including its setup) is asymptotically lower that even just the back substitution step of direct methods based on matrix factorization, which would be $\mathcal{O}(n^{\frac{3}{2}})$ and $\mathcal{O}(n^{\frac{5}{3}})$ in 2D and 3D, respectively.

9 Numerical experiments

In this section, we are giving numerical experiments with some real-life problems. We have used the symmetric Schwarz overlapping domain decomposition algorithm as a preconditioner of the conjugate gradients method. We are stopping the CGM iteration loop once the relative preconditioned residual satisfies

$$\frac{\langle CAe_i, Ae_i \rangle}{\langle CAe_0, Ae_0 \rangle} \mathrm{cond}(C, A) \leq \epsilon^2, \tag{76}$$

where C denotes the domain decomposition preconditioner, $\mathrm{cond}(C, A)$ is the condition number estimate computed at runtime and e_i is the error after the i-th iteration. In all cases we have used $\epsilon = 1.0 \cdot 10^{-4}$. All experiments have been performed by the courtesy of the Supercomputer Centre at the University of West Bohemia in Plzeň, Czech Republic on Digital AlphaServer 8400, `kirke.zcu.cz`, with 2GB of RAM, using all of its 8 processors, each at 3330-4800 GFlops/s.

name	no. of nodes	dofs on node	$\|A\|_\infty$	input data	mesh	no. of eqs.
test1	$12,125$	3	$4.35 \cdot 10^8$	16.5 MB	n.a.	$36,375$
solid1	$25,058$	3	$2.88 \cdot 10^8$	27.7 MB	Fig. 6	$75,174$
solid2	$40,329$	3	$1.49 \cdot 10^7$	75.9 MB	Fig. 7	$120,987$
shell	$9,915$	6	$3.18 \cdot 10^5$	20.0 MB	Fig. 8	$59,490$

Table 1: Description of experimental input data

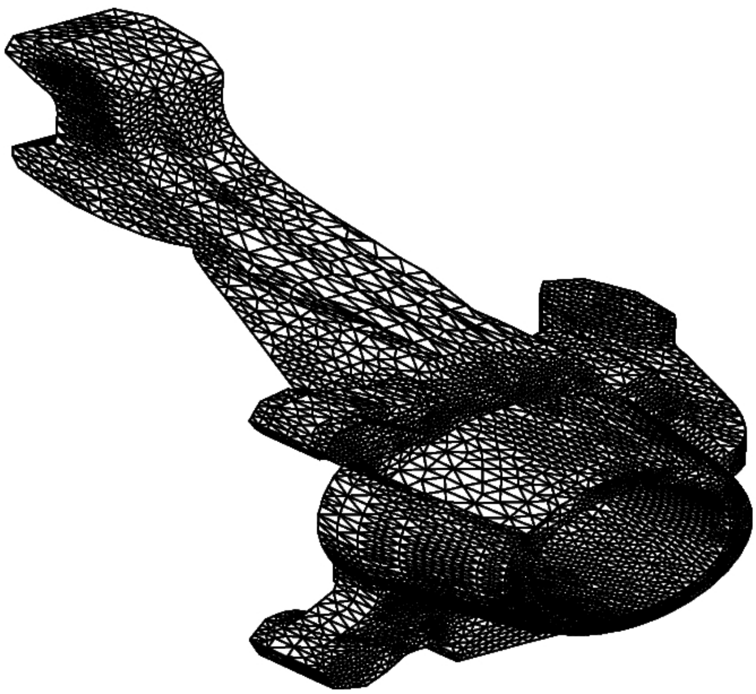

Figure 6: solid1

aggregates		domains				CPU/wall time		
radius	overlap	number	colors	iter.	cond	setup	iterations	memory
1	1	620	18	13	8.08	193.4/302 s	26.9/133 s	47.6 MB
2	2	206	14	12	7.56	45.27/93 s	36.8/166 s	83.0 MB

Table 2: test1

aggregates		domains				CPU/wall time		
radius	overlap	number	colors	iter.	cond	setup	iterations	memory
1	1	1,433	25	6	1.83	192.4/279 s	46.8/143 s	212 MB
2	2	346	21	6	2.28	228.0/360 s	67.8/260 s	445 MB
3	3	140	18	7	3.04	272.9/585 s	146.5/542 s	765 MB
3	1	140	11	10	4.95	55.6/87 s	48.7/175 s	165 MB

Table 3: solid1

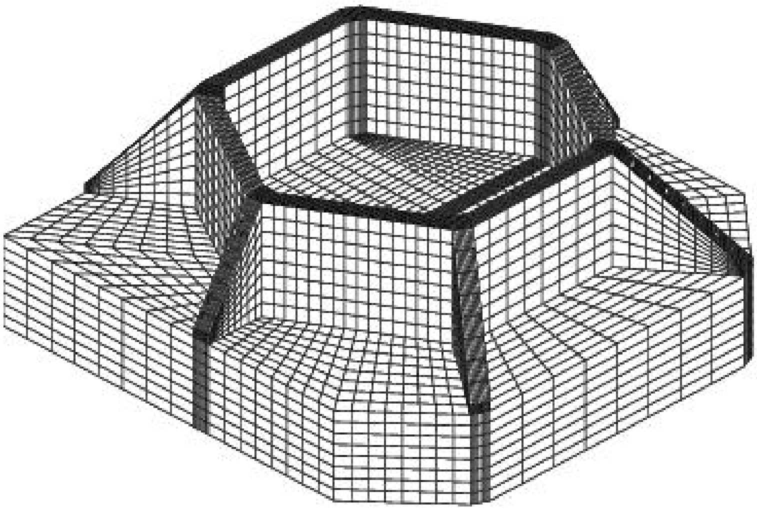

Figure 7: solid2

aggregates		domains				CPU/wall time		
radius	overlap	number	colors	iter.	cond	setup	iterations	memory
1	1	1,082	20	13	7.04	1201/6215 s	209/527 s	465 MB
2	2	295	20	13	8.80	850/3297 s	582/2080 s	1,285 MB

Table 4: solid2

aggregates		domains				CPU/wall time		
radius	overlap	number	colors	iter.	cond	setup	iterations	memory
1	1	1,215	9	9	3.47	652.0/793 s	25.5/84 s	78.6 MB
2	2	450	12	9	4.02	108.7/393 s	22.1/161 s	84.8 MB
3	3	235	11	9	4.57	49.6/95 s	28.9/117 s	125 MB

Table 5: shell

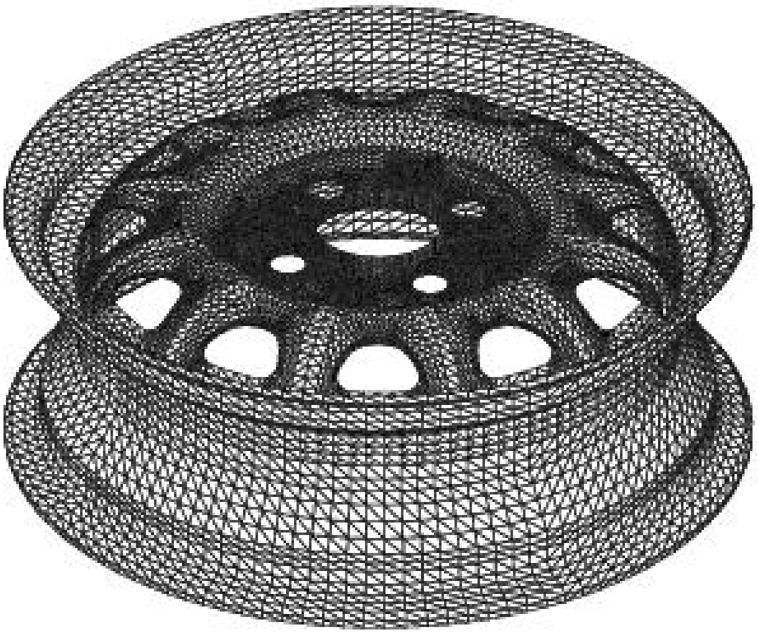

Figure 8: shell

References

[1] P. Vaněk, J. Mandel, and M. Brezina: Algebraic multigrid by smoothed aggregation for second and fourth order elliptic problems, Computing, 56 (1996), pp. 179-196.

[2] Marian Brezina and Petr Vaněk: One Black-Box Iterative Solver, University of Colorado, Denver 1997, to appear.

[3] Petr Vaněk and Jitka Křížková: Two-level Method On Unstructured Meshes With Convergence Rate Independent Of The Coarse-Space Size, University of West Bohemia, Plzeň, Preprint no. 70, Jan 1995.

[4] James H. Bramble, Joseph E. Pasciak, Junping Wang and Jinchao Xu: Convergence Estimates for Product Iterative Methods with Applications to Domain Decomposition and Multigrid, 1989

[5] Jinchao Xu: Iterative Methods by Space Decomposition and Subspace Correction, Siam Review, vol. 34, 4 (1992), pp. 581-613

[6] Jinchao Xu: An Introduction to Multilevel Methods, VII. Numerical Analysis Summer School, University of Leicester, UK, to be published by Oxford University Press

[7] Robert A. Adams: Sobolev Spaces, Academic Press, London 1975

[8] J. Nečas, I. Hlaváček: Úvod do matematické teorie pružných a pružně plastických těles, TKI, SNTL Praha 1983, in Czech

[9] K. Rektorys: Variační metody v inženýrských problémech a v problémech matematické fyziky, TKI, SNTL Praha 1974, in Czech

[10] R. Tezaur, P. Vaněk, M. Brezina: Two-level method for solids on unstructured meshes, 1996. To appear.

[11] V. A. Konratiev, O. A. Oleinik: Hardy's and Korn's Type Inequalities and Their Applications, Rendiconti di Matematica, Serie VII, 10 (1990), pp. 641-666.

[12] C. M. Dafermos: Some Remarks on Korn's Inequality, ZAMP, 19 (1968), pp. 913-920.