Algorithms
A2. Finite element method for PDEs

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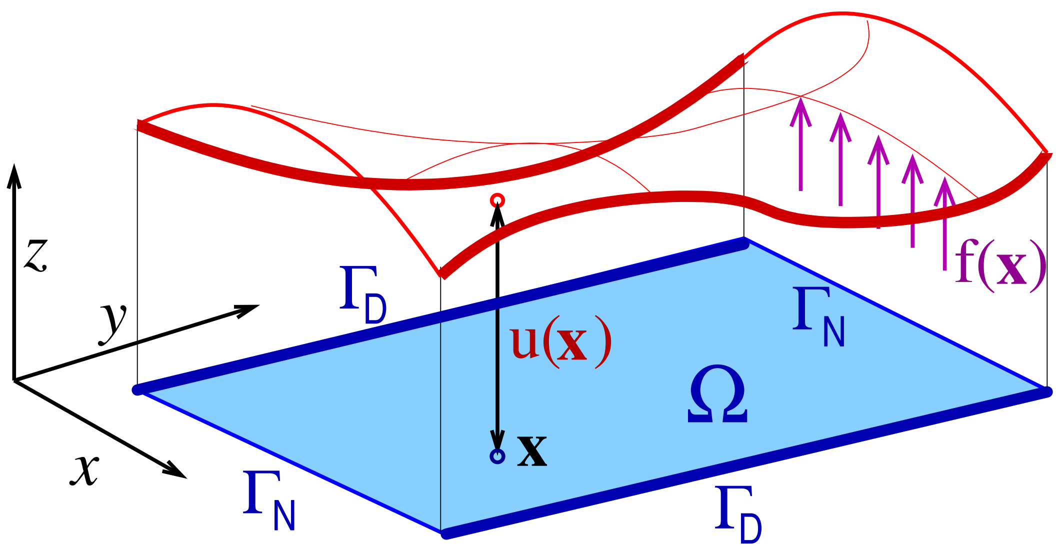
1. Model problem: elastic membrane (Poisson's equation)

Task: originally flat (2D) membrane (in blue) is deformed:

- by a surfacic and edge forces $f(x, y)$ on Ω, resp. $g_N(x, y)$ on Γ_N,
- by clamping on a part of its boundary ($\Gamma_D \subset \partial\Omega$).

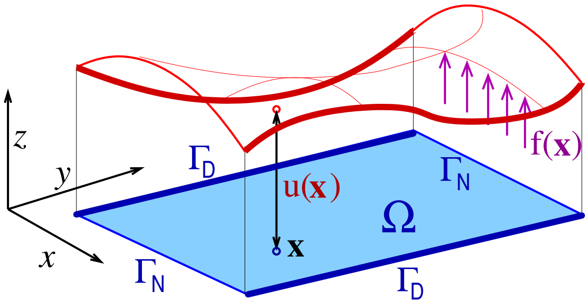
Calculate its vertical displacement $u(\mathbf{x})$, $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$.



2. Classical formulation of the problem

Vertical displacement $u(x,y)$ of elastic membrane in 3D:

$$\begin{aligned} -\operatorname{div}\left(\underline{\underline{A}}\,\operatorname{grad} u\right) &= f && \text{in } \Omega \subset \mathbb{R}^2 \\ u &= u_D && \text{on } \Gamma_D \subset \partial\Omega \\ \mathbf{n}^T \cdot \underline{\underline{A}}\,\operatorname{grad} u &= g_N && \text{on } \Gamma_N \subset \partial\Omega \end{aligned}$$



- $\underline{\underline{A}}$ is a (2×2) symmetric positive definite matrix
- Ω is the computational domain
- Γ_D and Γ_N is a disjoint covering of the border of Ω, for imposing Dirichlet resp. Neumann-type boundary conditions
- $\mathbf{n}$ is the unit normal vector on Γ_N
- u_D is a given vertical displacement on Γ_D (the membrane is clamped on Γ_D).

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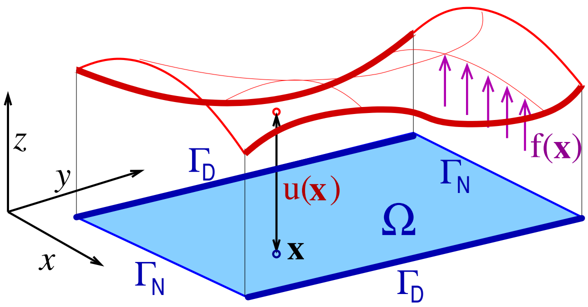


Disadvantages of the classical formulation:

- to give a meaning to the classical formulation, $u(\mathbf{x})$ needs to be twice continuously differentiable in Ω and continuous in $\overline{\Omega}$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Natural solutions are not in general *that* **regular**.
- C^2 is a vector space of **infinite dimension** (computer representation?)
- explicit analytical solution is known only for special Ω's, or it comes in the form of an **integral (expensive to calculate)**.

3. Weak formulation of the problem

Important questions to be asked (and answered):

- What are the **minimal regularity requirements** on **u** (and all other functions involved) in order to give sense to this problem? What is the least-regular (ie. largest) space of functions V in which we are allowed to look for solutions?
- How to approximate the functions $u \in V$, $\dim V = \infty$, by using a **finite-dimensional space** V_h?
- **How to choose the basis of** V_h**?**

3.1. Weak formulation of the problem: reducing regularity

Suppose we have a functional space V**,** $C^2(\Omega) \cap C^0(\overline{\Omega}) \subset V$**. define space** V_0 **by:**

$$V_0 \equiv \{\omega \in V, \omega = 0 \text{ on } \Gamma_D\}$$

Solution u **of the classical problem satisfies also**:

- $(u - u_D) \in V_0$ with V_0
- $$-\int_\Omega \mathrm{div}\,\left(\underline{\underline{A}}\,\mathrm{grad}\,u\right)\, v\, dx = \int_\Omega f\, v\, dx \qquad \forall v \in V_0$$

Apply the Green's integration by parts to get the weak problem: find $u \in V$, $(u - u_D) \in V_0$ such that

$$\int_\Omega \left(\underline{\underline{A}}\,\mathrm{grad}\,u\right) \cdot \mathrm{grad}\, v\, dx = \int_\Omega f\, v\, dx + \int_{\Gamma_N} g_N\, v\, d\Gamma \qquad \forall v \in V_0$$

3.1. Weak formulation of the problem: reducing regularity

What space V sould we take? We want:

- the least regular V,
- the weak problem has to be still meaningful for such V.

For this case we can take $V \equiv H^1(\Omega)$, the Hilbert space obtained as a closure of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|v\|_1^2 = \underbrace{\int_\Omega v^2\, dx}_{\|v\|_0} + \underbrace{\int_\Omega (\mathrm{grad}\, v)^2\, dx}_{|v|_1} .$$

3.2. How to approximate V or V_0 with V_h?

Approximation of $V \equiv H^1(\Omega)$ with a finite-dimensional space V_h: Construct a sequence of $\{V_h\}$ with different scale of resolution h (mesh-size), so that:

- **Conforming subspace:** $V_h \subset V$
- **Good approximation:** V_h must be dense in V, ie. for a given $u \in V$ and an error $\epsilon > 0$, there exist $h > 0$ such that

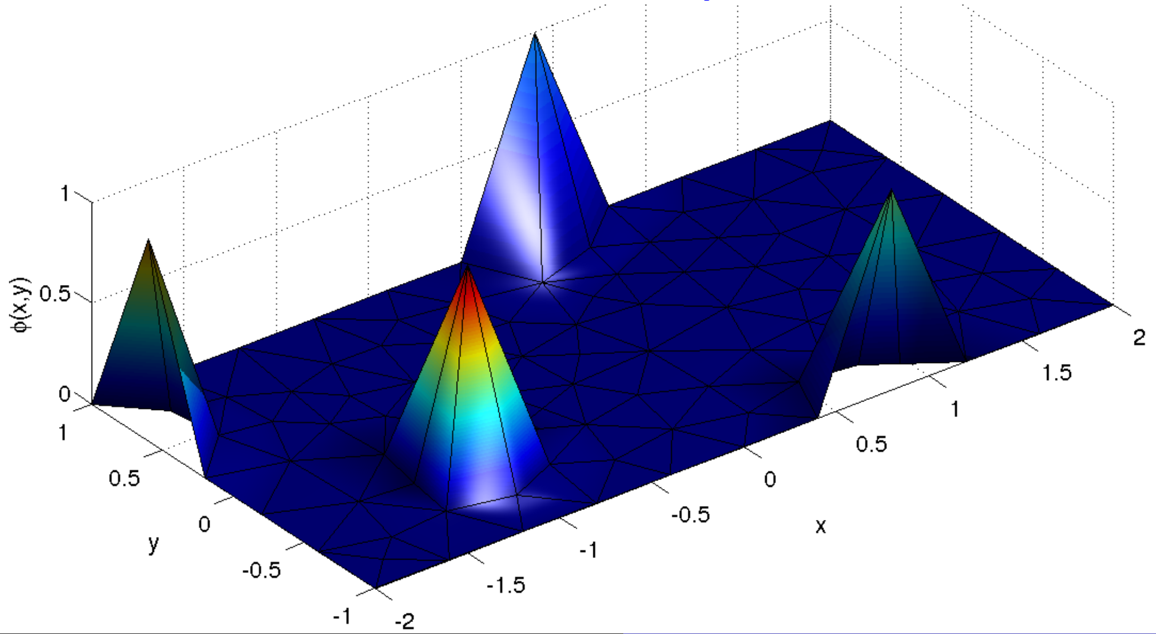
$$\inf_{u_h \in V_h} \|u - u_h\|_1 \leq \epsilon$$

  Hence, $V_h \to V$ if $h \to 0$.

3.3. How to choose a basis of $V_h = \mathrm{span}(\varphi_1, \ldots, \varphi_n)$?

- **Simplicity of construction:** for ex. piecewise polynomials
- **Computational complexity:** local support of $\varphi_k(\mathbf{x})$, $k = 1, \ldots, n$
  In order to obtain *sparse matrices* instead of full ones
- **Other considerations:** numerical stability, numerical quadrature, etc.

Good candidate: P1 finite element space



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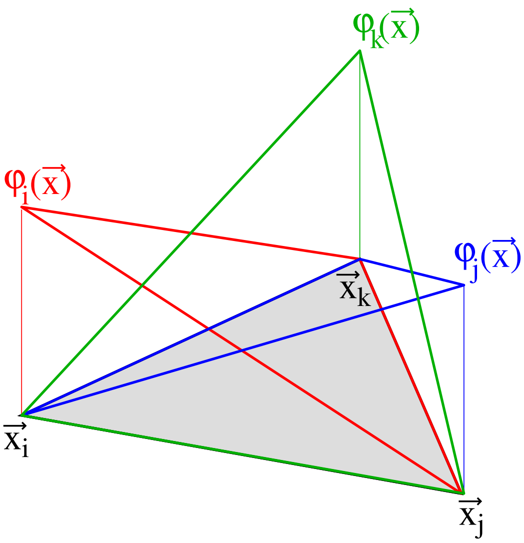

On one triangle / tetrahedron T:
$\varphi_\ell(\mathbf{x})$ is linear and

$$\varphi_\ell(\mathbf{x}_i) = \begin{cases} 1, & \text{if } \ell = i \\ 0, & \text{otherwise} \end{cases}$$

$$\varphi_\ell(\mathbf{x}_i) = a_\ell + b_\ell \cdot x_i + c_\ell \cdot y_i = \delta_\ell^i$$

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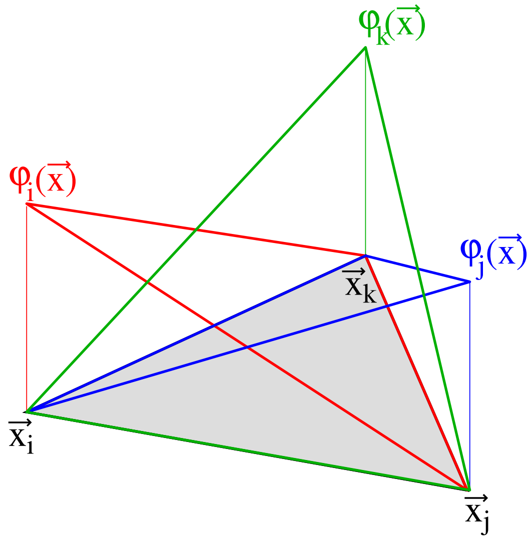

On one triangle / tetrahedron T:

$$u_h(\mathbf{x}) = u_i\,\varphi_i(\mathbf{x}) + u_j\,\varphi_j(\mathbf{x}) + u_k\,\varphi_k(\mathbf{x})$$

$$\mathrm{grad}\, u_h\Big|_T = \begin{pmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix} \cdot \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix}$$

4. Discretization of the weak problem

Weak problem in V: find $u \in V$, $u = u_D$ on Γ_D such that

$$\int_\Omega (\underline{\underline{A}}\,\mathrm{grad}\, u) \cdot \mathrm{grad}\, v\, dx = \int_\Omega f\, v\, dx + \int_{\Gamma_N} g_N\, v\, d\Gamma \qquad \forall v \in V_0$$

Discretization in $V_h \subset V$: find $u_h \in V_h$, $u_h = u_D$ on Γ_D such that

$$\int_\Omega (\underline{\underline{A}}\,\mathrm{grad}\, u_h) \cdot \mathrm{grad}\, v_h\, dx = \int_\Omega f\, v_h\, dx + \int_{\Gamma_N} g_N\, v_h\, d\Gamma \qquad \forall v_h \in V_h \cap V_0$$

We want to finish with a linear algebraic system: unknowns $\mathbf{u} = (u_i)$ are the values of u_h at each vertex of the finite element mesh.

4.1. Assembly of the system matrix K

Elementary stiffness matrices K_T: take $u_h, v_h \in V_h$, use their **piecewise linearity** on triangles T:

$$\int\limits_\Omega \left(\underline{\underline{A}}\,\nabla u_h\right)\cdot\nabla v_h\,dx \;\approx\; \sum_{\substack{T\in\Omega\\ T=\triangle(i,j,k)}} \int\limits_T \nabla v_h^T\,\underline{\underline{A}}_T\,\nabla u_h\,dx =$$

$$= \sum_{\substack{T\in\Omega\\ T=\triangle(i,j,k)}} (v_i, v_j, v_k)\; \underbrace{\mathrm{vol}(T)\begin{pmatrix} b_i & c_i\\ b_j & c_j\\ b_k & c_k\end{pmatrix}\underline{\underline{A}}_T\begin{pmatrix} b_i & b_j & b_k\\ c_i & c_j & c_k\end{pmatrix}}_{K_T\text{...elementary stiffness matrix}}\begin{pmatrix} u_i\\ u_j\\ u_k\end{pmatrix}$$

Here, $\mathrm{vol}(T)$ is the area of triangle T (in 2D), resp. the volume of the tetraheron T (in 3D). Matrix $\underline{\underline{A}}_T$ is a piecewise constant approximation of $\underline{\underline{A}}$ on the element T.

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$$= \sum_{\substack{T\in\Omega\\ T=\triangle(i,j,k)}} (v_i, v_j, v_k)\; \underbrace{\begin{pmatrix} \kappa_{ii} & \kappa_{ij} & \kappa_{ik}\\ \kappa_{ji} & \kappa_{jj} & \kappa_{jk}\\ \kappa_{ki} & \kappa_{kj} & \kappa_{kk}\end{pmatrix}}_{K_T}\begin{pmatrix} u_i\\ u_j\\ u_k\end{pmatrix}$$

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$$= \sum_{\substack{T\in\Omega\\ T=\triangle(i,j,k)}} (v_1\ldots v_i\ldots v_j\ldots v_k\ldots v_n) \begin{pmatrix} & \vdots & & \vdots & & \vdots & \\ \cdots & \kappa_{11} & \cdots & \kappa_{12} & \cdots & \kappa_{13} & \cdots \\ & \vdots & & \vdots & & \vdots & \\ \cdots & \kappa_{21} & \cdots & \kappa_{22} & \cdots & \kappa_{23} & \cdots \\ & \vdots & & \vdots & & \vdots & \\ \cdots & \kappa_{31} & \cdots & \kappa_{32} & \cdots & \kappa_{33} & \cdots \\ & \vdots & & \vdots & & \vdots & \end{pmatrix} \begin{pmatrix} u_1 \\ \cdot \\ u_i \\ \cdot \\ u_j \\ \cdot \\ u_k \\ \cdot \\ u_n \end{pmatrix}$$

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