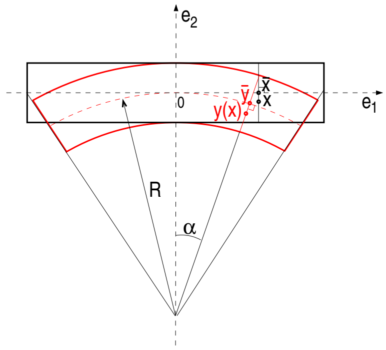
SA 2010 **Continuum mechanics I, Corrections Practice 1** Dr. Aleš Janka

Practice class of Nov 3, 2010



0) The components y^j of $\vec{y}(\vec{x})$ are given by the following formula:

$$y^1 = (R + x^2)\,\sin\left(\frac{x^1}{R}\right) \qquad (1)$$

$$y^2 = (R + x^2)\,\cos\left(\frac{x^1}{R}\right) - R. \qquad (2)$$

Indeed, the length of the mid-line segment $(0, \bar{x})$ is x^1. This segment is deformed into an arc $(0, \bar{y})$ whose length is also x^1 (rule: mid-line of the beam is not stretched), or $\alpha\, R$ (length of a circular arc). Hence $\alpha = \frac{x^1}{R}$.

The inverse functions $x^1 = x^1(y^1, y^2)$, $x^2 = x^2(y^1, y^2)$ can be obtained by summing or/and dividing the equations:

$$y^1 = (R + x^2)\,\sin\left(\frac{x^1}{R}\right)$$

$$y^2 + R = (R + x^2)\,\cos\left(\frac{x^1}{R}\right).$$

We obtain:

$$x^1 = R\,\arctan\left(\frac{y^1}{y^2 + R}\right), \qquad (3)$$

$$x^2 = \sqrt{(y^1)^2 + (y^2 + R)^2}. \qquad (4)$$

a) All components of the Green strain tensor in the cartesian coordinate system are given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} + \sum_{k=1}^{2}\frac{\partial u^k}{\partial x^i}\frac{\partial u^k}{\partial x^j}\right) \quad , \quad u^i = y^i - x^i \quad , \quad i, j = 1, 2.$$

After substitution from (1)-(2) and simplification we get

$$\varepsilon_{11} = \frac{x^2\,(2\,R + x^2)}{2\,R^2} \quad , \quad \varepsilon_{12} = 0 \quad , \quad \varepsilon_{22} = 0.$$

b) All components of the Cauchy strain tensor in the cartesian coordinate system are given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) \quad , \quad u^i = y^i - x^i \quad , \quad i, j = 1, 2.$$

After substitution from (1)-(2) and simplification we get

$$e_{11} = \frac{R + x^2}{R}\cos\left(\frac{x^1}{R}\right) \quad , \quad e_{12} = -\frac{x^2}{2R}\sin\left(\frac{x^1}{R}\right) \quad , \quad e_{22} = \cos\left(\frac{x^1}{R}\right) - 1.$$

c) All components of the Almansi strain tensor in the cartesian coordinate system are given by

$$\mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial y^j} + \frac{\partial u^j}{\partial y^i} - \sum_{k=1}^{2}\frac{\partial u^k}{\partial y^i}\frac{\partial u^k}{\partial y^j}\right) \quad , \quad u^i = y^i - x^i \quad , \quad i, j = 1, 2.$$

After substitution from (3)-(4) and simplification we get

$$\mathcal{E}_{11} = \frac{\left(R+y^2\right)^2\,\left[(y^1)^2+(y^2)^2+2\,R\,y^2\right]}{2\,\left[R^2+2\,R\,y^2+(y^1)^2+(y^2)^2\right]^2} \quad , \quad \mathcal{E}_{12} = -\frac{y^1\,\left(R+y^2\right)\,\left[(y^1)^2+(y^2)^2+2\,R\,y^2\right]}{2\,\left[R^2+2\,R\,y^2+(y^1)^2+(y^2)^2\right]^2}$$

$$\mathcal{E}_{22} = \frac{1}{2}\left(\frac{R+y^2}{\sqrt{\left(R+y^2\right)^2+(y^1)^2}}-1\right)^2 - \frac{R+y^2}{\sqrt{\left(R+y^2\right)^2+(y^1)^2}} + \frac{R^2\,(y^1)^2}{2\,\left[R^2+2\,R\,y^2+(y^1)^2+(y^2)^2\right]^2} + 1.$$

e) Translations and rotations of a continuum are in the kernel of the Green and of the Almansi strains (as functions of the displacement $\vec{u}$). These modes can be described (in 2D) by a rotation angle α and a vector of translation $\vec{a} = (a^1, a^2)^T$:

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}. \tag{5}$$

The inverse of (5) is

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \left[\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} - \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}\right].$$

For Cauchy (linearized) strain, however, these modes are no longer zero-deformation modes, since

$$e_{11} = e_{22} = \cos\alpha - 1 \neq 0 \quad ! \text{ (if } \alpha \neq 0\text{)}$$

What are the zero-deformation (rigid-body) modes for Cauchy strain, then?

Let us look for functions $u^1(x^1, x^2)$ and $u^2(x^1, x^2)$ such that

$$\frac{\partial u^1}{\partial x^1} = 0, \tag{6}$$

$$\frac{\partial u^1}{\partial x^2} + \frac{\partial u^2}{\partial x^2} = 0, \tag{7}$$

$$\frac{\partial u^2}{\partial x^2} = 0. \tag{8}$$

By integrating the equations (6) and (8) we get

$$u^1(x^1, x^2) = f(x^2) \qquad \text{and} \qquad u^2(x^1, x^2) = g(x^1).$$

Pluging-in this result into (7) gives

$$\frac{\partial f}{\partial x^2} = -\frac{\partial g}{\partial x^1} = C \quad \text{constant},$$

because the term $\frac{\partial f}{\partial x^2}$ is a function of x^2 only, while the term $\frac{\partial g}{\partial x^1}$ depends only on x^1. Hence, $f(x^2) = C\,x^2 + a^1$ and $g(x^1) = -C\,x^1 + a^2$, ie. all zero-deformation modes can be written as:

$$\begin{aligned} u^1 &= C\,x^2 + a^1, \\ u^2 &= -C\,x^1 + a^2, \end{aligned}$$

with some real constants C, a^1 and a^2.