Mathematical tools
M3. Tensor fields in curvilinear coordinate systems

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1. Curvilinear coordinate system

Position vector of point M (with respect to the origin): $\overrightarrow{OM} = \mathbf{x}$

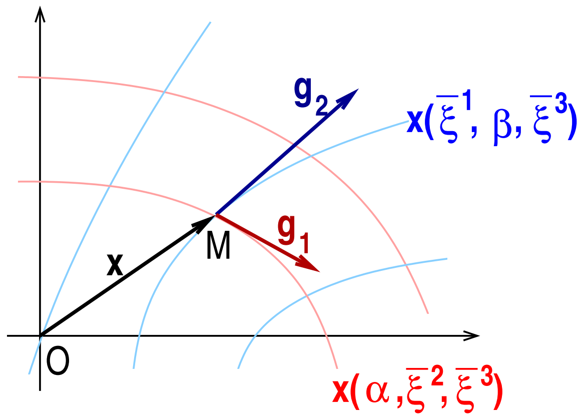
Let $\mathbf{x} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth bijective mapping:

$$\mathbf{x} : (\xi^1, \xi^2, \xi^3)^T \mapsto \mathbf{x}(\xi^1, \xi^2, \xi^3)$$

Curvilinear coordinates of a point: ξ^1, ξ^2, ξ^3.

Coordinate curves through a point: parametric curves given by

$$\begin{aligned} \mathbf{x}_1(\alpha) &: \quad \alpha \mapsto \mathbf{x}(\alpha, \overline{\xi}^2, \overline{\xi}^3) \\ \mathbf{x}_2(\beta) &: \quad \beta \mapsto \mathbf{x}(\overline{\xi}^1, \beta, \overline{\xi}^3) \\ \mathbf{x}_3(\gamma) &: \quad \gamma \mapsto \mathbf{x}(\overline{\xi}^1, \overline{\xi}^2, \gamma) \end{aligned}$$



all 3 curves pass through $\mathbf{x}(\overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3)$ (=point M with $\overline{\xi}^1$, $\overline{\xi}^2$, $\overline{\xi}^3$ fixed).

1. Curvilinear coordinate system: local basis

Local basis: composed of tangents to the coordinate curves:

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}$$

We suppose here that $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ are linearly independent in $\mathbb{R}^3$.

Covariant local basis: differential of the position vector $\mathbf{x}$:

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \xi^i} \, d\xi^i = \mathbf{g}_i \, d\xi^i$$

"How much $\mathbf{x}(\xi^1, \xi^2, \xi^3)$ changes if we perturb ξ^i by $d\xi^i$."
Contravariant basis is induced as before so that $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$

Metric tensor: $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ ... (as before)

Huge difference with what we have seen so far:

$$\mathbf{g}_i = \mathbf{g}_i(\mathbf{x}(\xi^1, \xi^2, \xi^3))$$

ie. the basis is not constant for all points $\mathbf{x} \in \mathbb{R}^3$!

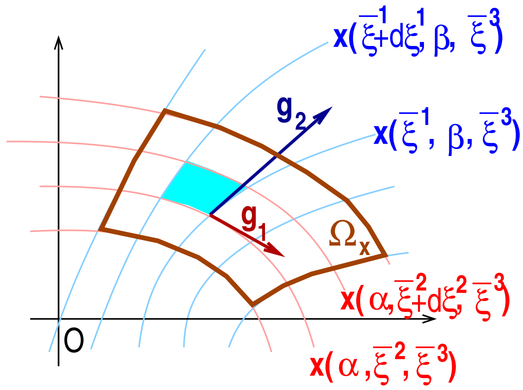
1.1 Curvilinear coordinate system: infinitensimal volume

Infinitensimal volume due to coordinate change:

$$dV = \underbrace{|(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3|}_{\sqrt{g}} \, d\xi^1 \, d\xi^2 \, d\xi^3$$

with $g = \det[g_{ij}] = \det(\mathbf{F}^T \mathbf{F}) = \det^2(\mathbf{F})$, where $\mathbf{F} = (\mathbf{g}_1 | \mathbf{g}_2 | \mathbf{g}_3)$.

In cartesian coordinates: $d\mathbf{x} = dx^i \, \mathbf{e}_i$ and $dV = dx^1 \, dx^2 \, dx^3$.



Volume integral of a scalar field $f(\mathbf{x})$ in curvilinear coords:

$$\int_{\Omega_x} f(\mathbf{x}) \underbrace{dx^1 \, dx^2 \, dx^3}_{dV} = \int_{\Omega_\xi} f(\mathbf{x}(\xi^1, \xi^2, \xi^3)) \underbrace{\sqrt{g} \, d\xi^1 \, d\xi^2 \, d\xi^3}_{dV}$$

with $\Omega_x = \mathbf{x}(\Omega_\xi)$.

2. Differential (resp. gradient) of a scalar field

Scalar field: $f : \mathbf{x} \in \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}$

Gradient of $f(\mathbf{x})$ (with respect to the position $\mathbf{x}$):
it is a vector $\underline{\nabla} f \in \mathbb{R}^3$ such that for all $d\mathbf{x} \in \mathbb{R}^3$:

$$f(\mathbf{x} + d\mathbf{x}) = f(\mathbf{x}) + \underbrace{\underline{\nabla} f(\mathbf{x}) \cdot d\mathbf{x}}_{df} + o(d\mathbf{x})$$

here, df is the differential of $f(\mathbf{x})$ along $d\mathbf{x}$.

Gradient and the directional derivative of $f(\mathbf{x})$
The gradient of $f(\mathbf{x})$ is such a vector $\nabla f \in \mathbb{R}^3$ for which

$$[\nabla f(\mathbf{x})] \cdot \mathbf{d} = \left[\frac{d}{d\alpha} f(\mathbf{x} + \alpha \mathbf{d})\right]_{\alpha=0} \qquad \forall \mathbf{d} \in \mathbb{R}^3.$$

The definition is independent of the choice of basis $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$
Hence, $\nabla f(\mathbf{x})$ **is a field of tensors** of order $N = 1$ (vector field).

2.1 Coordinates of $\underline{\nabla} f$ in the local basis

$$\begin{aligned}
f(\mathbf{x}+d\mathbf{x}) &= f(\mathbf{x}) + \underline{\nabla} f(\mathbf{x}) \cdot d\mathbf{x} + o(d\mathbf{x}) && \ldots \text{ definition of gradient} \\
&= f(\mathbf{x}) + \frac{\partial f}{\partial \xi^i} d\xi^i + \sum_k o(d\xi^k) && \ldots\ f(\mathbf{x}) \text{ as a function of } \xi^i \\
&= f(\mathbf{x}) + \frac{\partial f}{\partial \xi^i} \delta^i_j \, d\xi^j + \sum_k o(d\xi^k) \\
&= f(\mathbf{x}) + \underbrace{\frac{\partial f}{\partial \xi^i} \mathbf{g}^i}_{\underline{\nabla} f(\mathbf{x})} \cdot \underbrace{\mathbf{g}_j \, d\xi^j}_{d\mathbf{x}} + o(d\mathbf{x}) && \ldots \text{ cf. the first line}
\end{aligned}$$

Hence, $\underline{\nabla} f = \dfrac{\partial f}{d\xi^i} \mathbf{g}^i \Rightarrow$ **Covariant components** of $\underline{\nabla} f(\mathbf{x})$ are:

$$(\underline{\nabla} f)_i = \frac{\partial f}{\partial \xi^i} \qquad \ldots \textbf{ They coincide with } \frac{\partial f}{\partial \xi^i}\textbf{!}$$

$\nabla_i f = (\underline{\nabla} f)_i = \frac{\partial f}{\partial \xi^i}$ is named "the **covariant derivative** of scalar field f"
The differential df then expressed **"in coordinates":** $df = \nabla_i f \, d\xi^i$

3. Differential (resp. gradient) of a vector field

Vector field u(x) : (e.g. velocity, displacements, el. current, ...) field of tensors of order $N = 1$:

$$\mathbf{u} : \mathbf{x} \in \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$$

Components of u in the local basis $\langle \mathbf{g}_i \rangle$:

$$\mathbf{u}(\mathbf{x}) = u^i\, \mathbf{g}_i \qquad \text{...both } u^i \text{ and } \mathbf{g}_i \text{ depend on } \mathbf{x}!$$

Differential $d\mathbf{u}$: change in $\mathbf{u}$ going from $\mathbf{x}$ to $\mathbf{x}+d\mathbf{x}$ (up to $o(d\mathbf{x})$):

$$\mathbf{u}(\mathbf{x}+d\mathbf{x}) \approx \mathbf{u}(\mathbf{x}) + d\mathbf{u} = \mathbf{u}(\mathbf{x}) + \frac{\partial \mathbf{u}}{\partial \xi^j}\, d\xi^j = \mathbf{u}(\mathbf{x}) + \underline{\underline{\nabla \mathbf{u}}} \cdot d\mathbf{x}$$

From $\mathbf{u} = u^i\, \mathbf{g}_i$, by chain rule (both u^i and $\mathbf{g}_i$ depend on $\mathbf{x}$, ie. ξ^i!):

$$d\mathbf{u} = \frac{\partial u^i}{\partial \xi^j}\, d\xi^j\, \mathbf{g}_i + u^i\, \frac{\partial \mathbf{g}_i}{\partial \xi^k}\, d\xi^k \qquad (1)$$

Change due to changing local coordinates u^i of $\mathbf{u}$

Change due to the curvature of the coordinate system

3.1 Differential of a vector field: contravariant components

Contravariant components of $d\mathbf{u}$: $du^\ell = d\mathbf{u} \cdot \mathbf{g}^\ell, \quad d\mathbf{u} = du^\ell\, \mathbf{g}_\ell$ with respect to the local basis $\langle \mathbf{g}_i \rangle$ at the point $\mathbf{x}$ (not at $\mathbf{x}+d\mathbf{x}$!).

$$d\mathbf{u} = \frac{\partial u^i}{\partial \xi^j}\, d\xi^j\, \mathbf{g}_i + u^i\, \frac{\partial \mathbf{g}_i}{\partial \xi^k}\, d\xi^k \qquad \Big|\ \cdot \mathbf{g}^\ell$$

Hence, the contravariant components of $d\mathbf{u}$:

$$\begin{aligned} du^\ell &= d\mathbf{u} \cdot \mathbf{g}^\ell = \frac{\partial u^i}{\partial \xi^j}\, d\xi^j\, \underbrace{\mathbf{g}_i \cdot \mathbf{g}^\ell}_{\delta_i^\ell} + u^i\, \frac{\partial \mathbf{g}_i}{\partial \xi^k} \cdot \mathbf{g}^\ell\, d\xi^k \\ &= \frac{\partial u^\ell}{\partial \xi^j}\, d\xi^j + u^i\, \underbrace{\frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}^\ell}_{\Gamma_{ij}^\ell}\, d\xi^j = \left(\frac{\partial u^\ell}{\partial \xi^j} + \Gamma_{ij}^\ell\, u^i \right) d\xi^j \end{aligned}$$

where we define

$$\Gamma_{ij}^\ell = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}^\ell,$$

the **Christoffel symbols of the second kind (not a tensor!)**.

3.2 Differential of a vector field: covariant components

Analogously to (1), from $\mathbf{u} = u_i\, \mathbf{g}^i$, by chain rule:

$$d\mathbf{u} = \frac{\partial u_i}{\partial \xi^j}\, d\xi^j\, \mathbf{g}^i + u_i\, \frac{\partial \mathbf{g}^i}{\partial \xi^k}\, d\xi^k$$

Change due to changing local coordinates u_i of $\mathbf{u}$
Change due to the curvature of the coordinate system

Aside differentiation to get rid of the contravariant basis $\mathbf{g}^i$ (which is less used):

$$\mathbf{g}^i \cdot \mathbf{g}_\ell = \delta^i_\ell \qquad \Big|\ \frac{\partial}{\partial \xi^j}$$

$$\frac{\partial \mathbf{g}^i}{\partial \xi^j} \cdot \mathbf{g}_\ell + \mathbf{g}^i \cdot \frac{\partial \mathbf{g}_\ell}{\partial \xi^j} = 0$$

Hence,
$$\frac{\partial \mathbf{g}^i}{\partial \xi^j} \cdot \mathbf{g}_\ell = -\mathbf{g}^i \cdot \frac{\partial \mathbf{g}_\ell}{\partial \xi^j}$$

3.2 Differential of a vector field: covariant components

Covariant components of $d\mathbf{u}$: $du_\ell = d\mathbf{u} \cdot \mathbf{g}_\ell$, $\quad d\mathbf{u} = du_\ell\, \mathbf{g}^\ell$:

$$d\mathbf{u} = \frac{\partial u_i}{\partial \xi^j}\, d\xi^j\, \mathbf{g}^i + u_i\, \frac{\partial \mathbf{g}^i}{\partial \xi^k}\, d\xi^k \qquad \Big|\ \cdot \mathbf{g}_\ell$$

Hence, by applying the aside differentiation:

$$\begin{aligned} du_\ell &= d\mathbf{u} \cdot \mathbf{g}_\ell = \frac{\partial u_i}{\partial \xi^j}\, d\xi^j\, \underbrace{\mathbf{g}^i \cdot \mathbf{g}_\ell}_{\delta^i_\ell} + u_i\, \frac{\partial \mathbf{g}^i}{\partial \xi^k} \cdot \mathbf{g}_\ell\, d\xi^k \\ &= \frac{\partial u_\ell}{\partial \xi^j}\, d\xi^j - u_i\, \underbrace{\frac{\partial \mathbf{g}_\ell}{\partial \xi^k} \cdot \mathbf{g}^i}_{\Gamma^i_{\ell k}}\, d\xi^k = \left(\frac{\partial u_\ell}{\partial \xi^j} - u_i\, \Gamma^i_{\ell j}\right) d\xi^j \end{aligned}$$

3.3 Differential of a vector field and covariant derivatives

Perturbations of the position $\mathbf{x}(\xi^1, \xi^2, \xi^3)$ by $d\xi^1, d\xi^2, d\xi^3 \longrightarrow d\mathbf{u}$:

$$d\mathbf{x} = d\xi^i\, \mathbf{g}_i \qquad , \qquad d\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \xi^j}\, d\xi^j = \underline{\underline{\nabla \mathbf{u}}} \cdot d\mathbf{x} = du^\ell\, \mathbf{g}_\ell = du_\ell\, \mathbf{g}^\ell$$

Contravariant and covariant coordinates of the differential $d\mathbf{u}$:

$$du^\ell = \underbrace{\left(\frac{\partial u^\ell}{\partial \xi^j} + \Gamma^\ell_{ij}\, u^i \right)}_{\nabla_j u^\ell} d\xi^j$$

$$du_\ell = \underbrace{\left(\frac{\partial u_\ell}{\partial \xi^j} - \Gamma^i_{\ell j}\, u_i \right)}_{\nabla_j u_\ell} d\xi^j$$

where $\nabla_j u^\ell$ is the **covariant derivative of contravariant tensor**
and $\nabla_j u_\ell$ is the **covariant derivative of covariant tensor**

3.4 Covariant derivatives and gradient of a vector field

Differential $d\mathbf{u}$ using covariant derivatives resp. $d\mathbf{u} = \underline{\underline{\nabla \mathbf{u}}} \cdot d\mathbf{x}$:

$$d\mathbf{u} = du^\ell\, \mathbf{g}_\ell = \underbrace{\left(\frac{\partial u^\ell}{\partial \xi^j} + \Gamma^\ell_{ij}\, u^i \right)}_{\nabla_j u^\ell} \mathbf{g}_\ell\, d\xi^j$$

$$= \nabla_j u^\ell\, \mathbf{g}_\ell \underbrace{\delta^j_k}_{\mathbf{g}^j \cdot \mathbf{g}_k} d\xi^k = \underbrace{\nabla_j u^\ell \left[\mathbf{g}_\ell \otimes \mathbf{g}^j \right]}_{\underline{\underline{\nabla \mathbf{u}}}} \cdot \underbrace{\mathbf{g}_k\, d\xi^k}_{d\mathbf{x}}$$

Hence, **the gradient $\underline{\underline{\nabla \mathbf{u}}}$** is a **2nd order tensor** with:

$$\underline{\underline{\nabla \mathbf{u}}} = \left(\frac{\partial u^\ell}{\partial \xi^j} + \Gamma^\ell_{ij}\, u^i \right) \left[\mathbf{g}_\ell \otimes \mathbf{g}^j \right] = \nabla_j u^\ell \left[\mathbf{g}_\ell \otimes \mathbf{g}^j \right]$$

$\Rightarrow$ **the covariant derivative $\nabla_j u^\ell$ is in fact the tensor $\underline{\underline{\nabla \mathbf{u}}}$ in mixed components $\left(\underline{\underline{\nabla \mathbf{u}}}\right)^\ell_{\ ,j}$!!**

$\Rightarrow$ **Similarly, $\nabla_j u_\ell$ are the $2\times$covariant components $\left(\underline{\underline{\nabla \mathbf{u}}}\right)_{\ell,j}$ of $\underline{\underline{\nabla \mathbf{u}}}$:**

$$\underline{\underline{\nabla \mathbf{u}}} = \left(\frac{\partial u_\ell}{\partial \xi^j} - \Gamma^i_{\ell j}\, u_i \right) \left[\mathbf{g}^\ell \otimes \mathbf{g}^j \right] = \nabla_j u_\ell \left[\mathbf{g}^\ell \otimes \mathbf{g}^j \right]$$

4. Covariant derivatives of higher order tensor $\underline{\underline{T}}$, $N \geq 2$

Remember: covariant deriv. of scalar field = its partial deriv.:

$$\nabla_k f = \frac{\partial f}{\partial \xi^k}$$

We can exploit it:

- Multiply $\underline{\underline{T}}$ by N arbitrary vector-fields $\underline{a}$, $\underline{b}$, ... to form a scalar f. Example for 2nd-order tensor $\underline{\underline{T}}$:
$$f = T^{ij} \, a_i \, b_j$$
- Apply the "covariant = partial" trick on f:
$$\nabla_k \left(T^{ij} \, a_i \, b_j\right) = \frac{\partial}{\partial \xi^k} \left(T^{ij} \, a_i \, b_j\right)$$
$$= \frac{\partial T^{ij}}{\partial \xi^k} a_i \, b_j + T^{ij} \frac{\partial a_i}{\partial \xi^k} b_j + T^{ij} \, a_i \frac{\partial b_j}{\partial \xi^k}$$
- For the arbitrary vector fields $\underline{a}$, $\underline{b}$, ... we know how to make a covariant derivative:
$$\nabla_j a_\ell = \frac{\partial a_\ell}{\partial \xi^j} - \Gamma^i_{\ell j} \, a_i \qquad \text{i.e.} \qquad \frac{\partial a_i}{\partial \xi^k} = \nabla_k a_i + \Gamma^m_{ik} \, a_m$$

4. Covariant derivatives of higher order tensor $\underline{\underline{T}}$, $N \geq 2$

In $\nabla_k(T^{ij} \, a_i \, b_j)$, replace $\frac{\partial}{\partial \xi^k}$ of $\underline{a}$, $\underline{b}$,... by terms containing ∇_k:

$$\nabla_k \left(T^{ij} \, a_i \, b_j\right) = \frac{\partial T^{ij}}{\partial \xi^k} a_i \, b_j + T^{ij} \frac{\partial a_i}{\partial \xi^k} b_j + T^{ij} \, a_i \frac{\partial b_j}{\partial \xi^k}$$
$$= \frac{\partial T^{ij}}{\partial \xi^k} a_i \, b_j + T^{ij} \left(\nabla_k a_i + \Gamma^m_{ik} \, a_m\right) b_j +$$
$$+ T^{ij} \, a_i \left(\nabla_k b_j + \Gamma^m_{jk} \, b_m\right)$$
$$= \left(\frac{\partial T^{ij}}{\partial \xi^k} + \Gamma^i_{\ell k} \, T^{\ell j} + \Gamma^j_{\ell k} \, T^{i\ell}\right) a_i \, b_j +$$
$$+ T^{ij} \, \nabla_k a_i \, b_j + T^{ij} \, a_i \, \nabla_k b_j$$

Here, we re-indexed conveniently dummy indices in order to regroup terms with "$a_i \, b_j$". By analogy with $(a \cdot b \cdot c)' = a'bc + ab'c + abc'$, the term in brackets is $\nabla_k T^{ij}$:

$$\boxed{\nabla_k T^{ij} = \frac{\partial T^{ij}}{\partial \xi^k} + \Gamma^i_{\ell k} \, T^{\ell j} + \Gamma^j_{\ell k} \, T^{i\ell}}$$

5. Divergence of tensor fields

Divergence of a vector field u(x): definition

$$\mathrm{div} : C(\mathbb{R}^3, \mathbb{R}^3) \rightarrow \mathbb{R} \qquad \mathrm{div}\, \mathbf{u} = \lim_{\Omega \rightarrow 0} \frac{\int_{\partial\Omega} \mathbf{u} \cdot d\mathbf{s}}{|\Omega|}$$

In cartesian coordinates:

$$\mathrm{div}\, \mathbf{u} = \mathrm{tr}\,(\nabla \mathbf{u}) = \frac{\partial u^i}{\partial x^i}$$

In curvilinear coordinates: must replace $\frac{\partial}{\partial x^i}$ by ∇_i:

$$\mathrm{div}\, \mathbf{u} = \mathrm{tr}\,(\nabla \mathbf{u}) = \nabla_i u^i = \delta_i^k \; \nabla_k u^i = g^{ki} \; \nabla_i u_k$$

Generalization to higher-order tensor-fields:

$$\left(\mathrm{div}\, \underline{\underline{T}}\right)^i = \nabla_k T^{ki}$$

6. How to calculate Christoffel symbols?

Two ways of calculating Christoffel symbols of 2nd kind:

- By definition:

$$\Gamma_{ij}^{\ell} = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}^{\ell}$$

- Through the metric tensor:

$$\Gamma_{ij}^{\ell} = \frac{1}{2}\, g^{k\ell} \underbrace{\left[\frac{\partial g_{ki}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k} \right]}_{2 \cdot \Gamma_{k,ij}}$$

$\Gamma_{k,ij} = \dfrac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}_k$ are the **Christoffel symbols of the 1st kind**.

Notable property of Christoffel symbols: symmetry in (i, j):

$$\Gamma_{ij}^{\ell} = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}^{\ell} = \left| \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i} \right| = \frac{\partial^2 \mathbf{x}}{\partial \xi^j \, \partial \xi^i} \cdot \mathbf{g}^{\ell} = \frac{\partial^2 \mathbf{x}}{\partial \xi^i \, \partial \xi^j} \cdot \mathbf{g}^{\ell} = \Gamma_{ji}^{\ell}$$

6.1 Christoffel symbols through the metric tensor

$$\Gamma^{\ell}_{ik} = \frac{\partial \mathbf{g}_i}{\partial \xi^k} \cdot \mathbf{g}^{\ell} = \left| \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i} \right| = \frac{\partial^2 \mathbf{x}}{\partial \xi^k \, \partial \xi^i} \cdot \mathbf{g}^{\ell} = \underbrace{\frac{\partial^2 \mathbf{x}}{\partial \xi^k \, \partial \xi^i} \cdot \mathbf{g}_m}_{\Gamma_{m,ki}} g^{m\ell}$$

$$\Gamma_{m,ki} = \frac{\partial^2 \mathbf{x}}{\partial \xi^k \, \partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^m} = \frac{\partial}{\partial \xi^k}\left(\frac{\partial \mathbf{x}}{\partial \xi^i}\right) \cdot \frac{\partial \mathbf{x}}{\partial \xi^m} = \frac{\partial}{\partial \xi^i}\left(\frac{\partial \mathbf{x}}{\partial \xi^k}\right) \cdot \frac{\partial \mathbf{x}}{\partial \xi^m}$$

Hence, by combining half-and-half:

$$\Gamma_{m,ki} = \frac{1}{2}\left[\frac{\partial}{\partial \xi^k}\underbrace{\left(\frac{\partial \mathbf{x}}{\partial \xi^i}\right)}_{\mathbf{g}_i} \cdot \underbrace{\frac{\partial \mathbf{x}}{\partial \xi^m}}_{\mathbf{g}_m} + \frac{\partial}{\partial \xi^i}\underbrace{\left(\frac{\partial \mathbf{x}}{\partial \xi^k}\right)}_{\mathbf{g}_k} \cdot \underbrace{\frac{\partial \mathbf{x}}{\partial \xi^m}}_{\mathbf{g}_m}\right]$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial \xi^k}\underbrace{(\mathbf{g}_m \cdot \mathbf{g}_i)}_{g_{mi}} + \frac{\partial}{\partial \xi^i}\underbrace{(\mathbf{g}_m \cdot \mathbf{g}_k)}_{g_{km}} - \frac{\partial \mathbf{g}_m}{\partial \xi^k} \cdot \mathbf{g}_i - \frac{\partial \mathbf{g}_m}{\partial \xi^i} \cdot \mathbf{g}_k\right]$$

6.1 Christoffel symbols through the metric tensor

$$\Gamma_{m,ki} = \frac{1}{2}\left[\frac{\partial}{\partial \xi^k}\underbrace{(\mathbf{g}_m \cdot \mathbf{g}_i)}_{g_{mi}} + \frac{\partial}{\partial \xi^i}\underbrace{(\mathbf{g}_m \cdot \mathbf{g}_k)}_{g_{km}} - \left(\frac{\partial \mathbf{g}_m}{\partial \xi^k} \cdot \mathbf{g}_i + \frac{\partial \mathbf{g}_m}{\partial \xi^i} \cdot \mathbf{g}_k\right)\right]$$

and

$$\frac{\partial \mathbf{g}_m}{\partial \xi^k} \cdot \mathbf{g}_i + \frac{\partial \mathbf{g}_m}{\partial \xi^i} \cdot \mathbf{g}_k = \frac{\partial^2 \mathbf{x}}{\partial \xi^m \, \partial \xi^k} \cdot \mathbf{g}_i + \frac{\partial^2 \mathbf{x}}{\partial \xi^m \, \partial \xi^i} \cdot \mathbf{g}_k$$

$$= \frac{\partial \mathbf{g}_k}{\partial \xi^m} \cdot \mathbf{g}_i + \frac{\partial \mathbf{g}_i}{\partial \xi^m} \cdot \mathbf{g}_k = \frac{\partial}{\partial \xi^m}(\mathbf{g}_i \cdot \mathbf{g}_k) = \frac{\partial g_{ik}}{\partial \xi^m}$$

Hence,

$$\boxed{\Gamma^{\ell}_{ik} = g^{m\ell}\, \Gamma_{m,ik} = \frac{1}{2} g^{m\ell}\left[\frac{\partial g_{mi}}{\partial \xi^k} + \frac{\partial g_{km}}{\partial \xi^i} - \frac{\partial g_{ik}}{\partial \xi^m}\right]}$$