Mathematical tools
M1. Introduction to tensors

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1.1 Vector spaces

Definition: set V is called **vector space** over the field $(\mathbb{K}, +, .)$ if:

- the group $(V, +)$ with $+ : V \times V \to V$ is abelian
- there exists an operation $. : \mathbb{K} \times V \to V$ such that:
  for all $\lambda, \mu \in \mathbb{K}$ and $v, w \in V$ there is
  - $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$,
  - $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$,
  - $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$,
  - $1_{\mathbb{K}} \cdot v = v$.

Example: $\mathbb{K} := \mathbb{R}$, $V := \mathbb{R}^n$ defined by

$$V := \left\{ (v_1, \ldots, v_n)^T \mid v_1, \ldots, v_n \in \mathbb{R} \right\}$$

$n = \dim(V)$ is the **dimension** of V.

Basis: there exist a **basis** $\mathcal{B} = \langle v_1, v_2, \ldots, v_n \rangle$, composed of n linearly independent **basis vectors** $v_i \in V$.

1.1 Vector spaces: Dual space

Definition: a **scalar product** on V is the application $(\cdot,\cdot) : V \times V \to \mathbb{K}$ which, for all $u, v \in V$

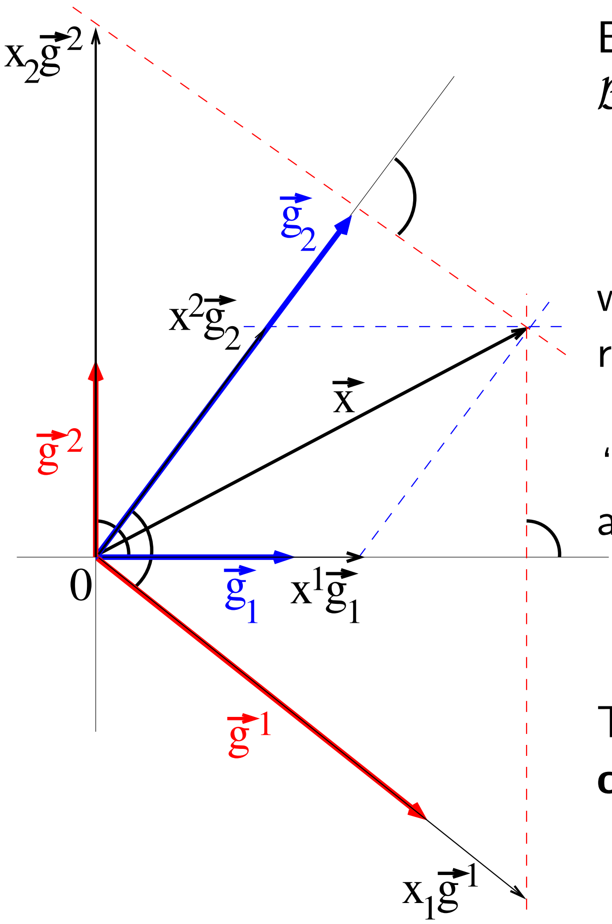
- (u, v) is bilinear
- (u, v) is conjugate-symmetric
- (u, u) is positive definite, ie. $(u, u) \geq 0$ and $(u, u) = 0$ implies $u = 0$.

Definition: **euclidean vector-space** is a vector space V together with a scalar product.

Definition: the **dual space** $V^\star$ to a vector space V is the set of all linear functions $f : V \to \mathbb{R}$. For a euclidean space $V^\star \cong V$ because of the representation of any $f(u) \in V^\star$ by $f(u) = (v, u)$.

Definition: For any vector space V and its basis $\mathcal{B} = \langle v_1, \ldots, v_n \rangle$ we can construct a **dual basis** $\mathcal{B}^\star = \langle v_1^\star, \ldots, v_n^\star \rangle$ of $V^\star$ such that $v_i^\star(v_j) = \delta_{ij}$.

1.2 Basis and dual basis for $\mathbb{R}^n$



Every $\mathbf{x} \in V$ can be represented in the basis $\mathcal{B} = \langle \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n \rangle$ by:

$$\mathbf{x} = x^i\, \mathbf{g}_i$$

where x^i are **coordinates** of vector $\mathbf{x}$ with respect to the basis $\mathcal{B} = \langle \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n \rangle$

"Other" coordinates of $\mathbf{x}$ can be obtained by a scalar product:

$$x_j = \mathbf{x} \cdot \mathbf{g}_j \quad (\neq x^j \;\text{ in general!})$$

These coordinates are with respect to the **reciprocal / dual basis** $\mathcal{B}' = \langle \mathbf{g}^1, \mathbf{g}^2, \ldots, \mathbf{g}^n \rangle$,

$$\mathbf{x} = x_j\, \mathbf{g}^j$$

1.2 Basis and dual basis: mutual relations

Starting from

$$\mathbf{x} = x^i\,\mathbf{g}_i = x_j\,\mathbf{g}^j \qquad \text{where} \quad x_i := \mathbf{x}\cdot\mathbf{g}_i \tag{1}$$

we multiply (scalar product) by $\mathbf{g}_k$ to obtain:

$$x_k = g_{ki}\,x^i \qquad \text{with} \qquad g_{ki} := \mathbf{g}_k\cdot\mathbf{g}_i. \tag{2}$$

The matrix g_{ki} is called the **covariant metric tensor** (cf. later). Substitute (2) back to (1) to obtain:

$$\mathbf{x} = x^i\,\mathbf{g}_i = x^i\,g_{ji}\,\mathbf{g}^j$$

Hence

$$\mathbf{g}_i = g_{ij}\,\mathbf{g}^j \qquad \text{and} \qquad \mathbf{g}^k = g^{ki}\mathbf{g}_i, \tag{3}$$

where g^{ki} is the **contravariant metric tensor** defined by:

$$g^{ki}\,g_{ij} = \delta^k_j \qquad \text{i.e. inverse of } g_{ij}.$$

1.2 Basis and dual basis: mutual relations

From (3) and $g_{ij} := \mathbf{g}_i\cdot\mathbf{g}_j$:

$$\mathbf{g}^k\cdot\mathbf{g}_\ell = g^{ki}\,\mathbf{g}_i\cdot\mathbf{g}_\ell = g^{ki}\,g_{i\ell} = \delta^k_\ell \tag{4}$$

Hence the dual basis is orthogonal to the basis and from (3)

$$g^{k\ell} = \mathbf{g}^k\cdot\mathbf{g}^\ell.$$

Link between $\mathcal{B}'$ of V and $\mathcal{B}^\star$ of $V^\star$: through the scalar product:

$$v_i^\star(\mathbf{x}) = (\mathbf{g}^i,\mathbf{x}) = \mathbf{g}^i\cdot\mathbf{x}$$

thanks to (4).

Basis change through a bijection $F : \mathbb{R}^n \to \mathbb{R}^n$ (matrix):
Represent F through a matrix $[F^i_j]$.
Introduce a new basis $\tilde{\mathcal{B}} = \langle \tilde{\mathbf{g}}_1, \ldots, \tilde{\mathbf{g}}_n \rangle$ through

$$\tilde{\mathbf{g}}_j := F^i_j \mathbf{g}_i \qquad \text{and hence} \qquad \mathbf{g}_j = \overline{F}^k_j \tilde{\mathbf{g}}_k.$$

where $[\overline{F}^j_k]$ is the inverse of $[F^i_j]$, ie. $\overline{F}^j_k \cdot F^i_j = \delta^i_k$.
Search for coordinates $\tilde{x}^\ell$ of $\mathbf{x}$ with respect to $\mathbf{g}_\ell$:

$$\mathbf{x} = x^i \, \mathbf{g}_i = x^i \delta^k_i \mathbf{g}_k = x^i \, \overline{F}^\ell_i \, F^k_\ell \mathbf{g}_k = \tilde{x}^\ell \, \tilde{\mathbf{g}}_\ell$$

Hence

$$\tilde{x}^\ell = x^i \, \overline{F}^\ell_i \qquad \text{and} \qquad \tilde{x}_\ell = \mathbf{x} \cdot \tilde{\mathbf{g}}_\ell = \mathbf{x} \cdot \left[F^i_\ell \mathbf{g}_i \right] = F^i_\ell \, (\mathbf{x} \cdot \mathbf{g}_i) = F^i_\ell \, x_i.$$

And also

$$\tilde{\mathbf{g}}^\ell = \overline{F}^\ell_k \, \mathbf{g}^k$$

Thus:

$$\tilde{\mathbf{g}}_j = F^i_j \, \mathbf{g}_i \qquad \text{and} \qquad \tilde{\mathbf{g}}^\ell = \overline{F}^\ell_k \, \mathbf{g}^k$$

$$\tilde{x}_\ell = F^i_\ell \, x_i \qquad \text{and} \qquad \tilde{x}^\ell = \overline{F}^\ell_k \, x^k$$

Observation:

- $\mathbf{g}_i$ and x_i transform to $\tilde{\mathbf{g}}_\ell$ and $\tilde{x}_\ell$ with F^i_ℓ
  $\Rightarrow$ "co-variant" = "with transformation F",
- $\mathbf{g}^k$ and x^k transform to $\tilde{\mathbf{g}}^\ell$ and $\tilde{x}^\ell$against F
  $\Rightarrow$ "contra-variant" = "against F" = "through an inverse mapping".

Let V be a real vector-space and $V^\star$ its dual.

Definition (tensor as a multi-linear map): A tensor is a multi-linear application $T : \underbrace{V \times \ldots \times V}_{K \text{ copies}} \times \underbrace{V^\star \times \ldots \times V^\star}_{\overline{K} \text{ copies}} \to \mathbb{R}$.

T is a K-times covariant and $\overline{K}$-times contravariant tensor of order $N = K + \overline{K}$.

For a euclidean space $V = \mathbb{R}^n$, we can represent V by the basis $\mathcal{B} = \langle \mathbf{g}_1, \ldots, \mathbf{g}_n \rangle$ and $V^\star$ by the reciprocal basis $\mathcal{B}' = \langle \mathbf{g}^1, \ldots, \mathbf{g}^n \rangle$:

Definition (tensor as a hyper-matrix): the representation hyper-matrix $\mathbf{T}$ in $\mathbb{R}^n$ of the multi-linear application T writes

$$\mathbf{T} = T_{i_1 \ldots i_K}^{j_1 \ldots j_{\overline{K}}} \, \mathbf{g}^{i_1} \otimes \ldots \otimes \mathbf{g}^{i_K} \otimes \mathbf{g}_{j_1} \otimes \ldots \otimes \mathbf{g}_{j_{\overline{K}}}$$

where $\left\{ T_{i_1 \ldots i_K}^{j_1 \ldots j_{\overline{K}}} \right\}$ are the components of the tensor with respect to the basis $\mathcal{B} = \langle \mathbf{g}_1, \ldots, \mathbf{g}_n \rangle$ (depends on this basis!).

2.1 Tensor: possible definitions

Definition (tensor as a $(K + \overline{K})$-dimensional table):
An ordered $n \times (K + \overline{K})$-tuple $T_{i_1 \ldots i_K}^{j_1 \ldots j_{\overline{K}}}$ is a (K-times covariant and $\overline{K}$-times contravariant) tensor of order $N = K + \overline{K}$ in $\mathbb{R}^n$, if for any basis-transformation $\tilde{\mathbf{g}}_j = F_j^i \, \mathbf{g}_i$ in $\mathbb{R}^n$ and its inverse $\mathbf{g}_j = \overline{F}_j^i \, \tilde{\mathbf{g}}_i$, it transforms by the following recipe

$$\tilde{T}_{\tilde{i}_1 \ldots \tilde{i}_K}^{\tilde{j}_1 \ldots \tilde{j}_{\overline{K}}} = F_{\tilde{i}_1}^{i_1} \cdots F_{\tilde{i}_K}^{i_K} \cdot \overline{F}_{j_1}^{\tilde{j}_1} \cdots \overline{F}_{j_{\overline{K}}}^{\tilde{j}_{\overline{K}}} \cdot T_{i_1 \ldots i_K}^{j_1 \ldots j_{\overline{K}}}$$

Link between the definitions:
Let us evaluate $T(\mathbf{x}_1, \ldots, \mathbf{x}_K, \mathbf{y}_1, \ldots, \mathbf{y}_{\overline{K}})$ for a special choice

$$\mathbf{x}_\ell = \mathbf{g}_{i_\ell} \qquad \text{and} \qquad \mathbf{y}_\ell = \mathbf{g}^{j_\ell}.$$

The multi-linear map gives

$$T(\mathbf{g}_{i_1}, \ldots, \mathbf{g}_{i_K}, \mathbf{g}^{j_1}, \ldots, \mathbf{g}^{j_{\overline{K}}}) = T_{i_1 \ldots i_K}^{j_1 \ldots j_{\overline{K}}}.$$

For $\mathbf{x}_\ell = x_{,\ell}^i \, \mathbf{g}_i$ and $\mathbf{y}_k = y_{j,k} \, \mathbf{g}^j$ by multi-linearity

$$T(\mathbf{x}_1, \ldots, \mathbf{x}_K, \mathbf{y}_1, \ldots, \mathbf{y}_{\overline{K}}) = T_{i_1 \ldots i_K}^{j_1 \ldots j_{\overline{K}}} \, x_{,1}^{i_1} \cdots x_{,K}^{i_K} \cdot y_{j_1,1} \cdots y_{j_{\overline{K}},\overline{K}}$$

2.2 Basic operations on tensors (on tensor components)

Let **A**, **B** be tensors (hyper-matrices).

- **addition** of tensors of the same type:
$$\mathbf{C} = \mathbf{A} + \mathbf{B}, \qquad \text{e.g. } C_{ij} = A_{ij} + B_{ij},$$
- **multiplication by** $\alpha \in \mathbb{R}$:
$$\mathbf{C} = \alpha \mathbf{A}, \qquad \text{e.g. } C_{ij} = \alpha\, A_{ij},$$
- **tensor-multiplication**:
$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B}, \qquad \text{e.g. } C^{ij}_{k\ell} = A^{ij} \cdot B_{k\ell},$$
- **contraction of indices**:
$$\mathbf{C} = \mathbf{A}\mathbf{B}, \qquad \text{e.g. } C^{\ell}_{k} = A_{ki}\, B^{i\ell},$$
$$\gamma = \mathbf{A} : \mathbf{B}, \qquad \text{e.g. } \gamma = A_{ij}\, B^{ij},$$
- **raising or lowering of an index**: contravariant $\leftrightarrow$ covariant component:
$$C^{i}_{k} = C_{jk}\, g^{ij} \quad \text{or} \quad C_{ik\ell} = C^{j}_{k\ell}\, g_{ij},$$

2.3 Special operations for 2nd-order tensors

Let **A** be a 2nd-order tensor (hyper-matrix)

$$\mathbf{A} = A^{ij}\, \mathbf{g}_i \otimes \mathbf{g}_j = A_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j = A^{i}_{j}\, \mathbf{g}_i \otimes \mathbf{g}^j = A^{j}_{i}\, \mathbf{g}^i \otimes \mathbf{g}_j$$

- **transpose:** $\mathbf{B} = \mathbf{A}^T \qquad B_{ji} = A_{ij}, \quad B^{j}_{\cdot i} = A^{i}_{\cdot j}.$
- **trace:** $\mathrm{tr}(\mathbf{A}) = A^{i}_{i} = g^{ij}\, A_{ij} = g_{ij}\, A^{ij}$
- **Frobenius norm:** $|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}} = \mathrm{tr}(\mathbf{A}^T\mathbf{A}) = \mathrm{tr}(\mathbf{A}\mathbf{A}^T) = (A_{ij}\, A^{ij})^{1/2}$
- **determinant:** $\det(\mathbf{A})$

$$\det(\mathbf{A}) = \det[A^{i}_{j}] = \det[A^{j}_{i}] = \det[g_{ij}]\, \det[A^{ij}] = \det[g^{ij}]\, \det[A_{ij}].$$

2.3 Decompositions of a 2nd-order tensor

- **symmetric and skew (antisymmetric) tensors:** $\mathbf{A} = \mathbf{S} + \mathbf{W}$,

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) \qquad \text{and} \qquad \mathbf{W} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right).$$

  - $\mathbf{S} = \mathbf{S}^T, \quad \mathbf{W} = -\mathbf{W}^T$.
  - For every $\mathbf{B}$ (2nd order tensor):

$$\begin{aligned} \mathbf{S} : \mathbf{B} &= \mathbf{S} : \mathbf{B}^T = \mathbf{S} : \frac{1}{2}\left(\mathbf{B} + \mathbf{B}^T\right) \\ \mathbf{W} : \mathbf{B} &= -\mathbf{W} : \mathbf{B}^T = \mathbf{W} : \frac{1}{2}\left(\mathbf{B} - \mathbf{B}^T\right) \\ \mathbf{S} : \mathbf{W} &= 0 \end{aligned}$$

  - $\exists \mathbf{w} \in \mathbb{R}^n$ s.t. $\forall \mathbf{u} \in \mathbb{R}^n$ there is $\mathbf{W}\mathbf{u} = \mathbf{w} \times \mathbf{u}$.

2.3 Decompositions of a 2nd-order tensor

- **spherical part $\alpha\mathbf{I}$ and deviator $\mathrm{dev}(\mathbf{A})$ of $\mathbf{A}$,**

$$\mathbf{A} = \alpha\mathbf{I} + \mathrm{dev}(\mathbf{A}) \qquad \text{with} \qquad \alpha := \frac{1}{n}\,\mathrm{tr}(\mathbf{A}) = \frac{1}{n}(\mathbf{I} : \mathbf{A})$$

  - $\mathrm{tr}(\mathrm{dev}(\mathbf{A})) = 0$

2.3 Eigenvalues, eigenvectors and principal scalar invariants

Eigenvalue problem: find all pairs $(\mathbf{u}, \lambda) \in \mathbb{C}^n \times \mathbb{C}$ such that

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u}. \tag{5}$$

Let $\mathbf{Q}$ is an orthogonal matrix (2nd-order tensor), then from (5)

$$\left(\mathbf{Q}^T \mathbf{A} \mathbf{Q}\right) \mathbf{Q}^T \mathbf{u} = \lambda \mathbf{Q}^T \mathbf{u},$$

ie. eigenvalues of $\mathbf{A}$ are equal to eigenvalues of $\mathbf{Q}^T \mathbf{A} \mathbf{Q}$.

2.3 Eigenvalues, eigenvectors and principal scalar invariants

Eigenvalues or A are invariant to rotations $\Rightarrow$ characteristic polynomial is also invariant

$$\begin{aligned} \det[\mathbf{A} - \lambda \mathbf{I}] &= 0 \\ \lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 &= 0 \end{aligned}$$

with the following principal scalar invariants:

$$\begin{aligned} I_1(\mathbf{A}) &= \mathrm{tr}(\mathbf{A}) \\ I_2(\mathbf{A}) &= \frac{1}{2}\left[\mathrm{tr}^2(\mathbf{A}) - \mathrm{tr}(\mathbf{A}^2)\right] = \mathrm{tr}(\mathbf{A}^{-1}) \det(\mathbf{A}) \\ I_3(\mathbf{A}) &= \det(\mathbf{A}) \end{aligned}$$

Cayley-Hamilton theorem: for every 2nd-order tensor $\mathbf{A}$ with its characteristic equation

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

there is

$$\mathbf{A}^3 - I_1\mathbf{A}^2 + I_2\mathbf{A} - I_3\mathbf{I} = \mathbf{0},$$

3.1 Levi-Civita permutation symbol

Definition: Levi-Civita permutation symbol e_{ijk} and e^{ijk}:

$$e_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{otherwise } (i=j \text{ or } i=k \text{ or } j=k). \end{cases}$$

$$e^{ijk} = e_{ijk} \quad , \quad e^{ijk}\, e_{ijk} = 6.$$

Matrix determinant from Linear Algebra I+II:

Determinant of a matrix $A \in \mathbb{R}^{3\times 3}$ was defined by:

$$\det(A) = \sum_{\sigma\in S_3} \mathrm{sign}(\sigma) \prod_{i=1}^{3} A_{i,\sigma(i)} \quad , \quad A = [A_{ij}].$$

Here, S_3 is the set of all permutations of $\{1,2,3\}$ and $\mathrm{sign}(\sigma) = +1$ for even permutations σ and -1 for odd ones.

Determinant with Levi-Civita permutation symbol:

$$\det(A) = e^{ijk}\, A_{1i}\, A_{2j}\, A_{3k} = \frac{1}{6}\, e^{ijk}\, e^{\ell mn}\, A_{\ell i}\, A_{mj}\, A_{nk}.$$

But attention, Levi-Civita permutation symbol is not a tensor!

3.2 Levi-Civita permutation tensor

Levi-Civita permutation symbol is not a tensor:
proof by change of basis $\tilde{\mathbf{g}}_j = F^i_j \mathbf{g}_i$:

$$F^\ell_i \, F^m_j \, F^n_k \, e_{\ell mn} = \det(F) \, e_{ijk} \neq e_{ijk}!$$

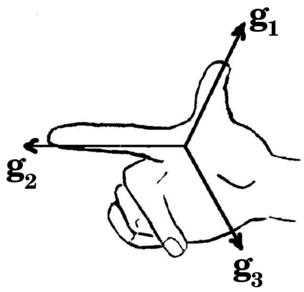
However, Levi-Civita symbol can form the following tensor.

Definition: Levi-Civita permutation tensor ε_{ijk} and ε^{ijk}:

$$\varepsilon_{ijk} = \Delta_g \sqrt{g} \, e_{ijk} \quad , \quad \varepsilon^{ijk} = \frac{\Delta_g}{\sqrt{g}} \, e^{ijk}$$

$$g = \det[g_{ij}] \quad , \quad \Delta_g = \begin{cases} +1 & \text{if } \langle \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \rangle \text{ right-handed,} \\ -1 & \text{if } \langle \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \rangle \text{ left-handed.} \end{cases}$$

Right-handedness of a basis (coordinate system) $\langle \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \rangle$:



right-handed


???

3.2 Levi-Civita permutation tensor

Proof of tensor-like behaviour: change of basis $\tilde{\mathbf{g}}_j = F^i_j \mathbf{g}_i$:

$$\begin{aligned}
\tilde{\varepsilon}_{ijk} &= F^\ell_i \, F^m_j \, F^n_k \, \varepsilon_{\ell mn} = F^\ell_i \, F^m_j \, F^n_k \, \Delta_g \sqrt{g} \, e_{\ell mk} \\
&= \Delta_g \sqrt{\det[g_{ij}]} \, \det(F) e_{ijk} \\
&= \sqrt{\frac{\det[\tilde{g}_{ij}]}{\det^2(F)}} \, \Delta_g \, \det(F) e_{ijk} \\
&= \sqrt{\det[\tilde{g}_{ij}]} \, \Delta_g \, \mathrm{sign}(\det(F)) \, e_{ijk} \\
&= \sqrt{\tilde{g}} \, \Delta_{\tilde{g}} \, e_{ijk}
\end{aligned}$$

Hence, the tensor-like transformation of covariant components $\varepsilon_{\ell mn}$ before basis-change (above) gives exactly the covariant components $\tilde{\varepsilon}_{ijk}$ of Levi-Civita tensor with respect to the transformed basis (below).

3.3 Levi-Civita permutation tensor: volume of triad $\mathbf{a}, \mathbf{b}, \mathbf{c}$

Let $V : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ by

$$V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

This application is multi-linear → function V defines a tensor.

Let us express **a**, **b** and **c** with respect to a basis $\langle \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \rangle$:

$$\mathbf{a} = a^i\, \mathbf{g}_i \quad , \quad \mathbf{b} = b^j\, \mathbf{g}_j \quad , \quad \mathbf{c} = c^k\, \mathbf{g}_k$$

The 3× covariant components ϑ_{ijk} of $\underline{\underline{\underline{V}}}$ must satisfy

$$V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \vartheta_{ijk}\, a^i\, b^j\, c^k$$

It appears that $\vartheta_{ijk} \equiv \varepsilon_{ijk}$, ie. $\underline{\underline{\underline{V}}}$ coincides with the Levi-Civita permutation tensor! Prove it! Idea: verify that it is so for canonical coordinates and use unicity of tensors for other bases

3.3 Levi-Civita permutation tensor: vector product

Let us denote $\mathbf{d} = \mathbf{a} \times \mathbf{b}$. Then

$$V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{d} \cdot \mathbf{c}$$

We know that scalar product is also multi-linear, hence tensor

$$\mathbf{d} \cdot \mathbf{c} = d_k\, c^k = d^j\, c_j.$$

We have just shown that

$$V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \varepsilon_{ijk}\, a^i\, b^j\, c^k = d_k\, c^k$$

Hence, covariant components of vector product can be easily computed at any coordinate system by

$$d_k = \varepsilon_{ijk}\, a^i\, b^j \qquad \text{and then} \qquad d^i = g^{ij}\, d_j$$

Cramer's rule: Levi-Civita tensor vs. metric tensor

$$g^{kn} = \frac{1}{2}\varepsilon^{ijk}\,\varepsilon^{\ell mn}\,g_{i\ell}\,g_{jm} \qquad , \qquad g_{kn} = \frac{1}{2}\varepsilon_{ijk}\,\varepsilon_{\ell mn}\,g^{i\ell}\,g^{jm}$$

Cramer's rule to solve linear systems $[g^{ij}] = [g_{kl}]^{-1}$!