

Algorithms

A2. Finite element method for PDEs

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March 2, 2011, Université de Fribourg

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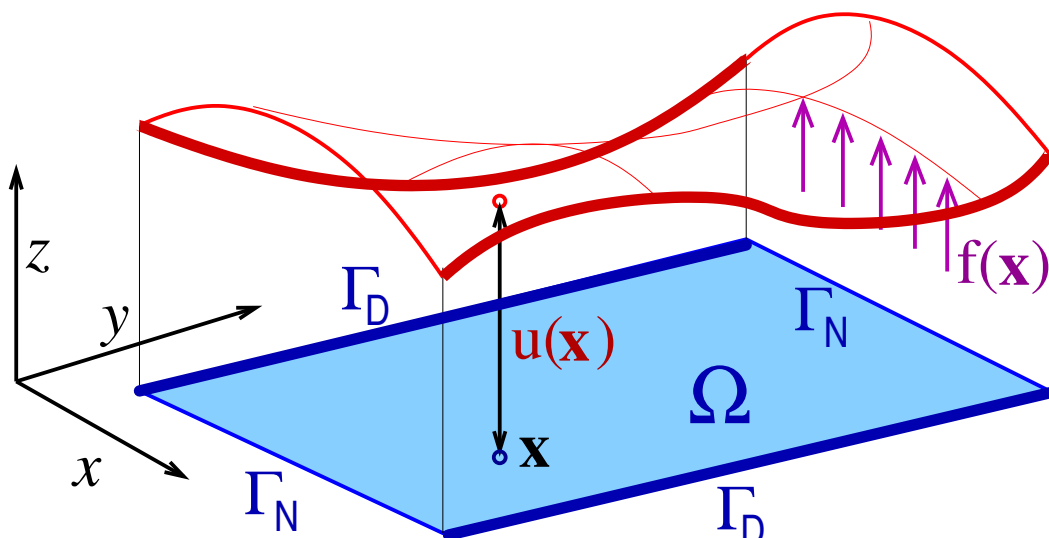
A2. Finite elements for PDEs

1. Model problem: elastic membrane (Poisson's equation)

Task: originally flat (2D) membrane (in blue) is deformed:

- by a surfacic and edge forces $f(x, y)$ on Ω , resp. $g_N(x, y)$ on Γ_N ,
- by clamping on a part of its boundary ($\Gamma_D \subset \partial\Omega$).

Calculate its vertical displacement $u(\mathbf{x})$, $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$.



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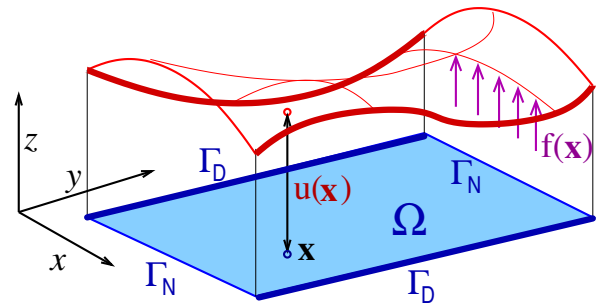
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A2. Finite elements for PDEs

2. Classical formulation of the problem

Vertical displacement $u(x, y)$ of elastic membrane in 3D:

$$\begin{aligned} -\operatorname{div}(\underline{\underline{A}} \operatorname{grad} u) &= f && \text{in } \Omega \subset \mathbb{R}^2 \\ u &= u_D && \text{on } \Gamma_D \subset \partial\Omega \\ \mathbf{n}^T \cdot \underline{\underline{A}} \operatorname{grad} u &= g_N && \text{on } \Gamma_N \subset \partial\Omega \end{aligned}$$



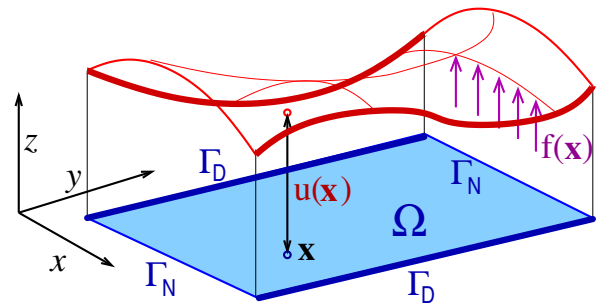
- $\underline{\underline{A}}$ is a (2×2) symmetric positive definite matrix
- Ω is the computational domain
- Γ_D and Γ_N is a disjoint covering of the border of Ω , for imposing Dirichlet resp. Neumann-type boundary conditions
- \mathbf{n} is the unit normal vector on Γ_N
- u_D is a given vertical displacement on Γ_D (the membrane is clamped on Γ_D).



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Disadvantages of the classical formulation:

- to give a meaning to the classical formulation, $u(\mathbf{x})$ needs to be twice continuously differentiable in Ω and continuous in $\overline{\Omega}$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Natural solutions are not in general **that regular**.
- C^2 is a vector space of **infinite dimension** (computer representation?)
- explicit analytical solution is known only for special Ω 's, or it comes in the form of an **integral (expensive to calculate)**.



3. Weak formulation of the problem

Important questions to be asked (and answered):

- What are the **minimal regularity requirements** on \mathbf{u} (and all other functions involved) in order to give sense to this problem? What is the least-regular (ie. largest) space of functions V in which we are allowed to look for solutions?
- How to approximate the functions $u \in V$, $\dim V = \infty$, by using a **finite-dimensional space** V_h ?
- **How to choose the basis of V_h ?**

3.1. Weak formulation of the problem: reducing regularity

Suppose we have a functional space V , $C^2(\Omega) \cap C^0(\bar{\Omega}) \subset V$.
define space V_0 by:

$$V_0 \equiv \{\omega \in V, \omega = 0 \text{ on } \Gamma_D\}$$

Solution u of the classical problem satisfies also:

- $(u - u_D) \in V_0$ with V_0
- $$-\int_{\Omega} \operatorname{div}(\underline{\underline{A}} \operatorname{grad} u) v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in V_0$$

Apply the Green's integration by parts to get the weak problem: find $u \in V$, $(u - u_D) \in V_0$ such that

$$\int_{\Omega} (\underline{\underline{A}} \operatorname{grad} u) \cdot \operatorname{grad} v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma_N} g_N v \, d\Gamma \quad \forall v \in V_0$$

3.1. Weak formulation of the problem: reducing regularity

What space V could we take? We want:

- the least regular V ,
- the weak problem has to be still meaningful for such V .

For this case we can take $V \equiv H^1(\Omega)$, the Hilbert space obtained as a closure of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|v\|_1^2 = \underbrace{\int_{\Omega} v^2 dx}_{\|v\|_0} + \underbrace{\int_{\Omega} (\text{grad } v)^2 dx}_{|v|_1}.$$



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3.2. How to approximate V or V_0 with V_h ?

Approximation of $V \equiv H^1(\Omega)$ with a finite-dimensional space V_h :

Construct a sequence of $\{V_h\}$ with different scale of resolution h (mesh-size), so that:

- **Conforming subspace:** $V_h \subset V$
- **Good approximation:** V_h must be dense in V , ie. for a given $u \in V$ and an error $\epsilon > 0$, there exist $h > 0$ such that

$$\inf_{u_h \in V_h} \|u - u_h\|_1 \leq \epsilon$$

Hence, $V_h \rightarrow V$ if $h \rightarrow 0$.



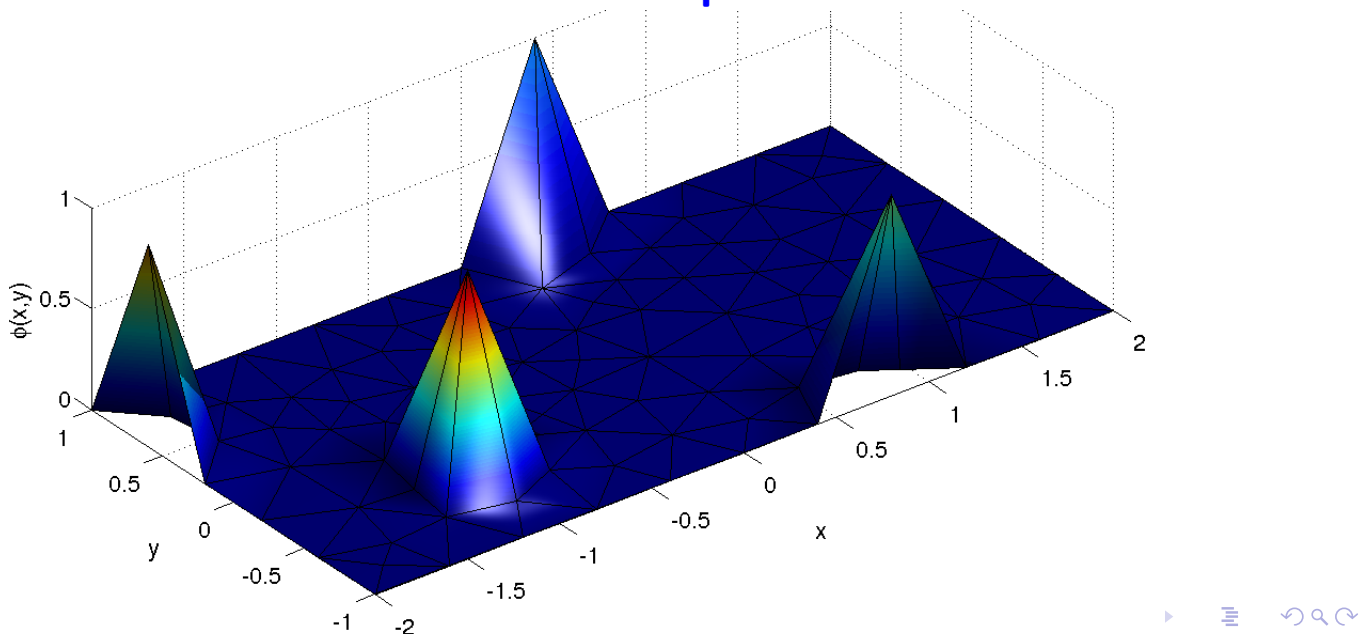
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3.3. How to choose a basis of $V_h = \text{span}(\varphi_1, \dots, \varphi_n)$?

- **Simplicity of construction:** for ex. piecewise polynomials
- **Computational complexity:** local support of $\varphi_k(\mathbf{x})$, $k = 1, \dots, n$
In order to obtain *sparse matrices* instead of full ones
- **Other considerations:** numerical stability, numerical quadrature, etc.

Good candidate: P1 finite element space



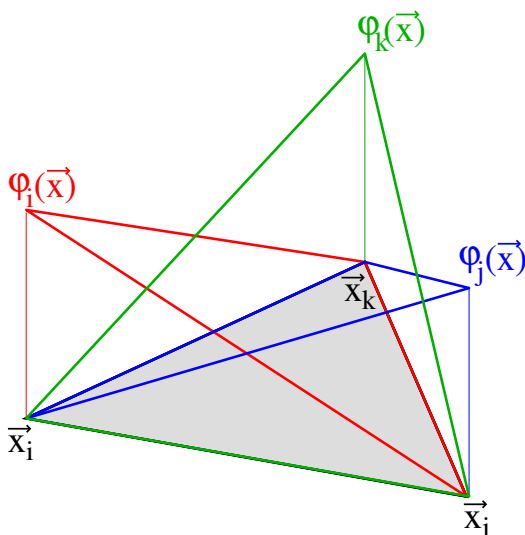
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On one triangle / tetrahedron T :

$\varphi_\ell(\mathbf{x})$ is linear and

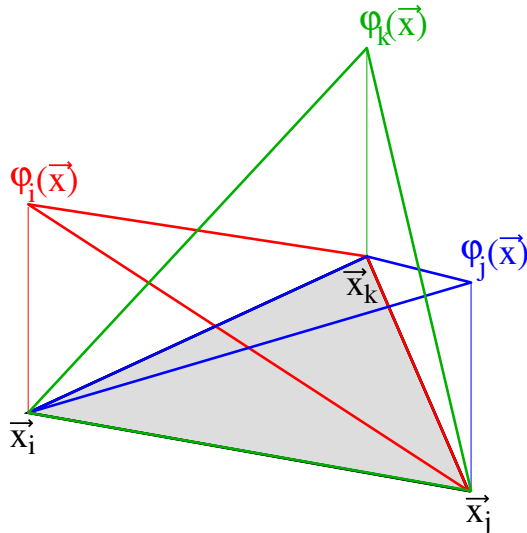
$$\varphi_\ell(\mathbf{x}_i) = \begin{cases} 1, & \text{if } \ell = i \\ 0, & \text{otherwise} \end{cases}$$

$$\varphi_\ell(\mathbf{x}_i) = a_\ell + b_\ell \cdot x_i + c_\ell \cdot y_i = \delta_\ell^i$$

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On one triangle / tetrahedron T :

$$u_h(\mathbf{x}) = u_i \varphi_i(\mathbf{x}) + u_j \varphi_j(\mathbf{x}) + u_k \varphi_k(\mathbf{x})$$

$$\text{grad } u_h \Big|_T = \begin{pmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix} \cdot \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix}$$



4. Discretization of the weak problem

Weak problem in V : find $u \in V$, $u = u_D$ on Γ_D such that

$$\int_{\Omega} (\underline{\underline{A}} \text{grad } u) \cdot \text{grad } v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma_N} g_N v \, d\Gamma \quad \forall v \in V_0$$

Discretization in $V_h \subset V$: find $u_h \in V_h$, $u_h = u_D$ on Γ_D such that

$$\int_{\Omega} (\underline{\underline{A}} \text{grad } u_h) \cdot \text{grad } v_h \, dx = \int_{\Omega} f v_h \, dx + \int_{\Gamma_N} g_N v_h \, d\Gamma \quad \forall v_h \in V_h \cap V_0$$

