Continuum mechanics
VII. Typical problems of elasto-dynamics and fluid dynamics

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0. Force equilibria for elasto-dynamics

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y^j}(\rho\, v^j) = \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0 \qquad \text{in } \Omega$$

Force equilibria (statics) → momentum conservation (dynamics)

Analogy for $\sum_j \mathbf{F}_j = \frac{d}{dt}\left(\sum_i m_i \mathbf{v}_i\right)$:

$$\int_\omega \rho\, f^i dy + \int_{\partial\omega} \tau^{ij}\, n_j\, dS = \frac{d}{dt}\left(\int_\omega \rho\, v^i dy\right)$$

Time derivative of an integral in Euler formulation (cf. III. sect.1)

$$\frac{d}{dt}\left(\int_\omega \rho\, v^i dy\right) = \int_\omega \left[\frac{\partial(\rho v^i)}{\partial t} + v^j \frac{\partial(\rho v^i)}{\partial y^j} + \rho\, v^i\, \frac{\partial v^j}{\partial y^j}\right] dy$$

$$= \int_\omega \Bigg[\rho \frac{\partial v^i}{\partial t} + \underbrace{v^i \frac{\partial \rho}{\partial t} + v^i\, v^j\, \frac{\partial \rho}{\partial y^j} + \rho\, v^i\, \frac{\partial v^j}{\partial y^j}}_{v^i\left(\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v})\right) = 0} + \rho\, v^j\, \frac{\partial v^i}{\partial y^j}\Bigg] dy$$

0. Force equilibria for elasto-dynamics

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y^j}(\rho\, v^j) = \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0 \qquad \text{in } \Omega$$

Force equilibria (statics) → momentum conservation (dynamics)

$$\int_\omega \rho\, f^i dy + \int_\omega \frac{\partial}{\partial y^j} \tau^{ij}\, dy = \int_\omega \rho \left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial y^j} \right) dy = \int_\omega \rho \frac{Dv^i}{Dt} dy$$

Hence, for $|\omega| \to 0$ we get

$$\rho \frac{D\mathbf{v}}{Dt} - \mathrm{div}\, \underline{\underline{\tau}} = \underbrace{\rho\, \mathbf{f}}_{\mathbf{F}} \qquad \text{in } \Omega$$

1. Elasto-dynamics in small deformations, linear material

Momentum conservation:

$$\rho \frac{D^2 \mathbf{u}}{Dt^2} - \nabla_j \tau^{ij} = F^i \qquad \text{ie.} \qquad \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}\, \underline{\underline{\tau}} = \mathbf{F} \qquad \text{in } \Omega$$

Constitutive law: Hooke's law, $\tau^{ij} = E^{ijk\ell} e_{k\ell}$, e.g.

$$\underline{\underline{\tau}} = 2\mu \underline{\underline{e}} + \lambda\, \mathrm{tr}(\underline{\underline{e}})\, \underline{\underline{Id}} \qquad \text{where} \qquad \lambda = K - \frac{2\mu}{3}$$

Kinematic equation: Cauchy strain tensor $\underline{\underline{e}}$:

$$e_{k\ell} = \frac{1}{2}\left[\nabla_k u_\ell + \nabla_\ell u_k\right] \quad \text{ie.} \quad \underline{\underline{e}} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right] \qquad \text{in } \Omega$$

Boundary conditions: for time $t \in [0, \infty)$:

$$\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}}(t) && \text{on } \Gamma_D \subset \partial\Omega && \text{(Dirichlet-type)},\\ \underline{\underline{\tau}} \cdot \mathbf{n} &= \mathbf{g}(t) && \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D && \text{(Neumann-type)}. \end{aligned}$$

Initial condition: for all $x \in \Omega$:

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x) \qquad \text{and} \qquad \frac{\partial \mathbf{u}}{\partial t}(x, 0) = \mathbf{v}_0$$

2. Fluid dynamics (constitutive law for Newton's fluid)

Elastic solid: material reversibly deformed under applied stress
Hooke's law: hydrostatic pressure $p = -s$:

$$\begin{aligned} \tilde{\tau}^{im} &= 2\,\mu\,\tilde{e}_{im} \\ -p = s &= K\,e_{\ell\ell} = (3\,\lambda + 2\,\mu)\,e = \frac{E}{1-2\,\nu}\,e \end{aligned}$$

Cauchy stress: $\tau^{ij} = \tilde{\tau}^{ij} + s\,\delta^{ij} = \tilde{\tau}^{ij} - p\,\delta^{ij}$

Fluid: continually deforming (in time) under applied shear stress
Newton's fluid: modification of Hooke's law for fluids:

$$\begin{aligned} \tilde{\tau}^{ij} &= 2\,\eta\,\dot{\tilde{e}}_{ij} = 2\,\eta\,\dot{e}_{ij} + \eta'\,\dot{e}_{\ell\ell}\,\delta_{ij} \\ -p = s &= (3\,\lambda' + 2\,\eta)\,\dot{e} = 3\,K\dot{e} \end{aligned}$$

Dynamic viscosity η and **second viscosity** $\eta' = -\frac{2}{3}\eta$
Note: we have replaced $\tilde{e}_{ij}$ and $\tilde{e}$ by their time-derivatives

3. Viscosity and visco-elasticity

Distinction solid vs. fluid is not obvious:

Some materials behave both like a solid and like a fluid
depending on the observation period (asphalt, glass, some plastics)

Example: University of Queensland pitch-drop experiment:
(set up in 1927, won the Ig-Nobel Price in October 2005)

Dec 1938	1st drop fell
Feb 1947	2nd drop fell
Apr 1954	3rd drop fell
May 1962	4th drop fell
Aug 1970	5th drop fell
Apr 1979	6th drop fell
Jul 1988	7th drop fell
Nov 28, 2000	8th drop fell

Digression: General visco-elastic materials

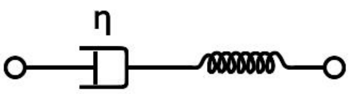
Generalization of constitutive laws for viscoelastic materials:

$$\sum_{k=1}^{M} \alpha_k \frac{\partial^k s}{\partial t^k} = \sum_{\ell=1}^{N} \beta_\ell \frac{\partial^\ell e}{\partial t^\ell}$$

$$\sum_{k=1}^{M} \tilde{\alpha}_k \frac{\partial^k \tilde{\tau}^{ij}}{\partial t^k} = \sum_{\ell=1}^{N} \tilde{\beta}_\ell \frac{\partial^\ell \tilde{e}_{ij}}{\partial t^\ell}$$

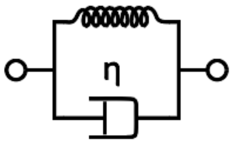
Maxwell material:

$$\frac{\dot{\tau}(t)}{E} + \frac{\tau(t)}{\eta} = \dot{e}(t)$$



Kelvin-Voigt material:

$$\tau = E\, e(t) + \eta\, \dot{e}(t)$$



4. Fluids: Newton's fluid

Force equilibrium:

$$\rho \frac{Dv^i}{Dt} - \frac{\partial}{\partial y^j} \tau^{ij} = \rho f^i \qquad \text{in } \Omega$$

Mass conservation / continuity equation:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0 \qquad \text{in } \Omega$$

Kinematic equation:

$$\dot{e}_{ij} = \frac{1}{2}\left(\frac{\partial v^i}{\partial y^j} + \frac{\partial v^j}{\partial y^i}\right)$$

Constitutive law:

$$\tau^{ij} = 2\,\eta\, \dot{e}_{ij} + \eta'\, \dot{e}_{\ell\ell}\, \delta_{ij} - p\, \delta_{ij}$$

4. Fluids: incompressible Newton's fluid

Force equilibrium:

$$\rho \frac{Dv^i}{Dt} - \frac{\partial}{\partial y^j}\tau^{ij} = \rho f^i \qquad \text{in } \Omega$$

Mass conservation / continuity equation: particle density is const. in time:

$$\underbrace{\frac{D\rho}{Dt}}_{=0} + \rho \operatorname{div}(\mathbf{v}) = 0 \qquad \text{ie.} \qquad \operatorname{div}(\mathbf{v}) = 0 \qquad \text{in } \Omega$$

Kinematic equation:

$$\dot{e}_{ij} = \frac{1}{2}\left(\frac{\partial v^i}{\partial y^j} + \frac{\partial v^j}{\partial y^i}\right)$$

Constitutive law:

$$\tau^{ij} = 2\,\eta\,\dot{e}_{ij} - p\,\delta_{ij}$$

2. Fluids: Navier-Stokes equation

Force equilibria: assembling all to Navier-Stokes equation:

$$-\frac{\partial}{\partial y^j}\left(\eta\left(\frac{\partial v^i}{\partial y^j} + \frac{\partial v^j}{\partial y^i}\right) - p\,\delta_{ij}\right) = \rho\, f^i - \rho\,\frac{Dv^i}{Dt}$$

Mass conservation / continuity equation:

$$\frac{\partial v^j}{\partial y^j} = 0$$

2. Fluids: Navier-Stokes equation

Force equilibria: assembling all to Navier-Stokes equation:

$$-\eta\left(\frac{\partial^2 v^i}{\partial y^j \partial y^j}+\frac{\partial^2 v^j}{\partial y^j \partial y^i}\right)+\frac{\partial p}{\partial y^i}=\rho f^i-\rho\frac{Dv^i}{Dt}$$

Mass conservation / continuity equation:

$$\frac{\partial v^j}{\partial y^j}=0$$

2. Fluids: Navier-Stokes equation

Force equilibria: assembling all to Navier-Stokes equation:

$$\rho\frac{\partial v^i}{\partial t}+\rho\frac{\partial v^i}{\partial y^j}v^j-\eta\left(\frac{\partial^2 v^i}{\partial y^j \partial y^j}\right)+\frac{\partial p}{\partial y^i}=\rho f^i$$

Mass conservation / continuity equation:

$$\frac{\partial v^j}{\partial y^j}=0$$