Continuum mechanics
VI. Typical problems of elasto-statics

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1. Elasto-statics in small deformations, linear material

Let us formulate the simplest problem of elasticity in 3D: given a computational domain $\Omega \subset \mathbb{R}^3$, find a displacement field $\mathbf{u} : \Omega \to \mathbb{R}^3$ such that:

Force equilibria: symmetric Euler stress-tensor $\underline{\underline{\tau}}$,

$$-\nabla_j \tau^{ij} = F^i \qquad \text{ie.} \qquad -\operatorname{div} \underline{\underline{\tau}} = \mathbf{F} \qquad \text{in } \Omega$$

Constitutive law: Hooke's law, $\tau^{ij} = E^{ijk\ell} e_{k\ell}$, e.g.

$$\underline{\underline{\tau}} = 2\mu \underline{\underline{e}} + \lambda \operatorname{tr}(\underline{\underline{e}})\, \underline{\underline{Id}} \qquad \text{where} \qquad \lambda = K - \frac{2\mu}{3}$$

Kinematic equation: Cauchy strain tensor $\underline{\underline{e}}$:

$$e_{k\ell} = \frac{1}{2}\left[\nabla_k u_\ell + \nabla_\ell u_k\right] \quad \text{ie.} \quad \underline{\underline{e}} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right] \qquad \text{in } \Omega$$

Boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}} \qquad \text{on } \Gamma_D \subset \partial\Omega \qquad \text{(Dirichlet-type)},$$
$$\underline{\underline{\tau}} \cdot \mathbf{n} = \mathbf{g} \qquad \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D \quad \text{(Neumann-type)}.$$

a) Non-homogenous Dirichlet condition $\rightarrow$ homogeneous:
Instead of finding $\mathbf{u} \in [H^1(\Omega)]^3$ with $\mathbf{u} = \bar{\mathbf{u}}$ on Γ_D, let us offset $\mathbf{u}$ by the known $\bar{\mathbf{u}}$ (extended somehow onto the whole Ω).

$$\mathbf{u} = \bar{\mathbf{u}} + \delta\mathbf{u} \quad \text{in } \Omega, \qquad \text{with} \qquad \delta\mathbf{u} = 0 \quad \text{on } \Gamma_D$$

Define the appropriate functional subspace of $[H^1(\Omega)]^3$ by

$$V_0 = \{\mathbf{w} \in [H^1(\Omega)]^3 \ : \ \mathbf{w} = 0 \text{ on } \Gamma_D\}.$$

Instead of finding $\mathbf{u}$, find $\delta\mathbf{u} \in V_0$, $\mathbf{u} = \bar{\mathbf{u}} + \delta\mathbf{u}$ and

b) Multiply force equilibria by any test function $\mathbf{v} \in V_0$ to get force equilibria in weak form

$$-\int_\Omega \operatorname{div} \underline{\underline{\tau}} \cdot \mathbf{v}\, dx = \int_\Omega \mathbf{F} \cdot \mathbf{v}\, dx \qquad \forall \mathbf{v} \in V_0.$$

ie. (component-wise)

$$\int_\Omega -\nabla_j \tau^{ij}\, v_i\, dx = \int_\Omega F^i\, v_i\, dx \qquad \forall \mathbf{v} \in V_0.$$

1.1. Weak formulation of elasto-statics: howto

c) Apply Green's Theorem (integration by parts)

$$\int_\Omega \tau^{ij} \cdot \nabla_j v_i\, dx - \int_{\partial\Omega} \tau^{ij}\, n_j\, v_i\, d\Gamma = \int_\Omega F^i\, v_i\, dx \qquad \forall \mathbf{v} \in V_0$$

d) Split $\int_{\partial\Omega}$ to $\int_{\Gamma_D}$ and $\int_{\Gamma_N}$ and apply boundary conditions:

$$\begin{aligned} \mathbf{v} &= 0 \qquad \text{on } \Gamma_D, \\ \tau^{ij}\, n_j &= g^i \qquad \text{on } \Gamma_N. \end{aligned}$$

to get $$\int_\Omega \tau^{ij} \cdot \nabla_j v_i\, dx = \int_\Omega F^i\, v_i\, dx + \int_{\Gamma_N} g^i\, v_i\, d\Gamma.$$

e) Use symmetry of $\tau^{ij} = \frac{1}{2}\left(\tau^{ij} + \tau^{ji}\right)$ to get

$$\begin{aligned} \int_\Omega \tau^{ij}(\mathbf{u}) \cdot \nabla_j v_i\, dx &= \frac{1}{2}\int_\Omega \tau^{ij}(\mathbf{u}) \cdot \nabla_j v_i\, dx + \frac{1}{2}\int_\Omega \underbrace{\tau^{ji}(\mathbf{u}) \cdot \nabla_j v_i}_{\tau^{ij}(\mathbf{u}) \cdot \nabla_i v_j}\, dx \\ &= \int_\Omega \tau^{ij}(\mathbf{u}) \cdot \underbrace{\frac{1}{2}\left(\nabla_j v_i + \nabla_i v_j\right)}_{e_{ij}(\mathbf{v})} dx \end{aligned}$$

1.1. Weak formulation of elasto-statics: howto

f) Use constitutive law to express $\underline{\underline{\tau}}$:
find $\delta\mathbf{u} \in V_0$, so that $\mathbf{u} = \bar{\mathbf{u}} + \delta\mathbf{u}$ and

$$\int_\Omega E^{ijk\ell}\, e_{k\ell}(\mathbf{u})\, e_{ij}(\mathbf{v})\, dx = \int_\Omega F^i\, v_i\, dx + \int_{\Gamma_N} g^i\, v_i\, d\Gamma, \qquad \forall \mathbf{v} \in V_0$$

g) Use linearity of $\underline{\underline{e}}(\mathbf{u})$ to separate $\bar{\mathbf{u}}$ from the unknown $\delta\mathbf{u}$:
find $\delta\mathbf{u} \in V_0$ so that

$$\int\limits_\Omega E^{ijk\ell} e_{k\ell}(\delta\mathbf{u})\, e_{ij}(\mathbf{v})\, dx = \int\limits_\Omega F^i\, v_i\, dx + \int\limits_{\Gamma_N} g^i\, v_i\, d\Gamma - \int\limits_\Omega E^{ijk\ell}\, e_{k\ell}(\bar{\mathbf{u}})\, e_{ij}(\mathbf{v})\, dx$$

2. Nonlinear elasto-statics: classification

Origins of nonlinearity of elasticity problems:

- **Geometrical nonlinearity:** also called **kinematic nonlinearity** for large displacements and/or large deformations
- **Material nonlineraity:** also called **physical nonlinearity** due to constitutive laws (nonlinear materials).

Classification:

- **Material nonlinearity + geometrical linearity:** use "small deformations" theory, iterate on material nonlinearity
- **Large displacements + small deformations:** big shifts and rotations as a rigid body, but small deformations. Use nonlinear kinematics (Green or Almansi strain). Linear constitutive law (e.g. Hooke's law) can be used.
- **Large displacements + large deformations:** need nonlinear kinematics (Green or Almansi strain) and nonlinear constitutive law (non-linear material).

2. Nonlinear elasto-statics (Lagrange formulation)

Force equilibria:

$$-\nabla_\ell \left[T^{j\ell} \left(\nabla_j u^k + \delta_j^k \right) \right] = F_0^k \qquad \text{in } \Omega_0$$

Kinematic equation: Green strain

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left(\nabla_j u_i + \nabla_i u_j + \nabla_i u^k \cdot \nabla_j u_k \right)$$

Nonlinear constitutive law: nonlinear material

$$T^{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = T^{ij}(\varepsilon)$$

Boundary conditions

$$\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}} \quad \text{on } \Gamma_D \subset \partial\Omega_0 && \text{(Dirichlet-type)},\\ \left(\underline{\underline{Id}} + \underline{\underline{\nabla \mathbf{u}}} \right) \underline{\underline{T}} \cdot \mathbf{n}_0 = \underline{\underline{\sigma}} \cdot \mathbf{n}_0 &= \mathbf{g} \quad \text{on } \Gamma_N = \partial\Omega_0 \setminus \Gamma_D && \text{(Neumann-type)}. \end{aligned}$$

2.1 Weak formulation of elasto-statics (Lagrange formul.)

Same steps as in Section 1.1.:

a)-b) find $\delta\mathbf{u} \in V_0$ **s.t.** $\mathbf{u} = \bar{\mathbf{u}} + \delta\mathbf{u}$ and

$$\int_{\Omega_0} -\nabla_\ell \left[T^{j\ell} \left(\nabla_j u^k + \delta_j^k \right) \right] v_k \, dx = \int_{\Omega_0} F_0^k \, v_k \, dx \qquad , \qquad \forall \mathbf{v} \in V_0.$$

c) Apply Green's Theorem (integration by parts):

$$\int_{\Omega_0} T^{j\ell} \left(\nabla_j u^k + \delta_j^k \right) \nabla_\ell v_k \, dx - \int_{\partial\Omega_0} T^{j\ell} \left(\nabla_j u^k + \delta_j^k \right) n_\ell^0 \, v_k \, d\Gamma = \int_{\Omega_0} F_0^k \, v_k \, dx$$

d) Split $\int_{\partial\Omega_0}$ **to** $\int_{\Gamma_D}$ **and** $\int_{\Gamma_N}$, apply boundary conditions $\mathbf{v} \in V_0$ on Γ_D and $\underline{\underline{\sigma}} \cdot \mathbf{n}_0 = \mathbf{g}$ on Γ_N to get

$$\int_{\Omega_0} T^{j\ell} \left(\nabla_j u^k + \delta_j^k \right) \nabla_\ell v_k \, dx = \int_{\Omega_0} F_0^k \, v_k \, dx + \int_{\Gamma_N} g^k \, v_k \, d\Gamma \qquad \forall \mathbf{v} \in V_0$$

2.1 Weak formulation of elasto-statics (Lagrange formul.)

e) Use symmetry of $T^{j\ell} = \frac{1}{2}\left(T^{j\ell} + T^{\ell j}\right)$

$$\int\limits_{\Omega_0} T^{j\ell}\left(\nabla_j u^k + \delta_j^k\right)\nabla_\ell v_k\,dx = \int\limits_{\Omega_0} T^{j\ell}\,\nabla_j u^k\,\nabla_\ell v_k\,dx + \int\limits_{\Omega_0} T^{k\ell}\,\nabla_\ell v_k\,dx$$

$$= \int\limits_{\Omega_0} T^{j\ell}\,\frac{\nabla_j u^k\,\nabla_\ell v_k + \nabla_j v_k\,\nabla_\ell u^k}{2}\,dx + \int\limits_{\Omega_0} T^{k\ell}\,\frac{\nabla_\ell v_k + \nabla_k v_\ell}{2}\,dx$$

$$= \int\limits_{\Omega_0} T^{ij}\,\underbrace{\frac{\nabla_i v_j + \nabla_j v_i + \nabla_i u^k\,\nabla_j v_k + \nabla_j u^k\,\nabla_i v_k}{2}}_{D\varepsilon_{ij}(\mathbf{u},\mathbf{v})}\,dx$$

Here, we have introduced

$$D\varepsilon_{ij}(\mathbf{u},\mathbf{v}) = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i + \nabla_i u^k\,\nabla_j v_k + \nabla_j u^k\,\nabla_i v_k\right)$$

How does it relate to $\varepsilon_{ij}(\mathbf{u})$?

2.1 Weak formulation of elasto-statics (Lagrange formul.)

f) Use constitutive law to express T^{ij}**:** $T^{ij} = \frac{\partial\Psi(\mathbf{u})}{\partial\varepsilon_{ij}}$
find $\delta\mathbf{u} \in V_0$ so that $\mathbf{u} = \bar{\mathbf{u}} + \delta\mathbf{u}$ and

$$\int_{\Omega_0} \frac{\partial\Psi(\mathbf{u})}{\partial\varepsilon_{ij}}\,D\varepsilon_{ij}(\mathbf{u},\mathbf{v})\,dx = \int_{\Omega_0} F_0^k\,v_k\,dx + \int_{\Gamma_N} g^k\,v_k\,d\Gamma$$

f') $D\varepsilon_{ij}(\mathbf{u},\mathbf{v})$ to $\varepsilon_{ij}(\mathbf{u})$: directional (Gâteaux) derivative: What is the variation of $\varepsilon_{ij}(\mathbf{u}+\alpha\,\mathbf{v})$, seen as a function of $\alpha \in \mathbb{R}$? Get the slope at $\mathbf{u}$, ie. at $\alpha = 0$:

$$\frac{d}{d\alpha}\left[\varepsilon_{ij}(\mathbf{u}+\alpha\,\mathbf{v})\right]_{\alpha\to 0} =$$

$$= \frac{d}{d\alpha}\left[\frac{1}{2}\left(\nabla_i(u_j+\alpha v_j) + \nabla_j(u_i+\alpha v_i) + \underbrace{\nabla_i(u_k+\alpha v_k)\cdot\nabla_j(u^k+\alpha v^k)}_{\nabla_i u_k\cdot\nabla_j u^k + \alpha(\nabla_j u^k\cdot\nabla_i v_k + \nabla_i u^k\cdot\nabla_j v_k) + \alpha^2\nabla_j v^k\cdot\nabla_i v_k}\right)\right]_{\alpha\to 0}$$

$$= \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i + \nabla_i u^k\cdot\nabla_j v_k + \nabla_j u^k\cdot\nabla_i v_k\right) = D\varepsilon_{ij}(\mathbf{u},\mathbf{v})$$

f") Directional derivative $D\Psi(\mathbf{u},\mathbf{v})$ of $\Psi(\mathbf{u})$ at $\mathbf{u}$ along the direction $\mathbf{v}$: use chain rule to differentiate $\Psi(\underline{\underline{\varepsilon}}(\mathbf{u}))$:

$$\frac{d}{d\alpha}\left[\Psi(\mathbf{u}+\alpha\mathbf{v})\right]_{\alpha\to 0} = \underbrace{\frac{\partial\Psi(\mathbf{u})}{\partial\varepsilon_{ij}}}_{T^{ij}} D\varepsilon_{ij}(\mathbf{u},\mathbf{v})$$

f*) Introduce the total potential energy:

$$\Pi(\mathbf{u}) = \underbrace{\int_{\Omega_0} \Psi(\mathbf{u})\, dx}_{\text{strain energy}} - \underbrace{\left(\int_{\Omega_0} F_0^k\, u_k\, dx + \int_{\Gamma_N} g^k\, u_k\, d\Gamma\right)}_{\text{work by external forces}}$$

2.1 Weak formulation of elasto-statics (Lagrange formul.)

Weak formulation and its equivalent minimization problem:
The weak formulation: find $\mathbf{u} \in \bar{\mathbf{u}} + V_0$ such that

$$\int_{\Omega_0} \frac{\partial\Psi(\mathbf{u})}{\partial\varepsilon_{ij}} D\varepsilon_{ij}(\mathbf{u},\mathbf{v})\, dx = \int_{\Omega_0} F_0^k\, v_k\, dx + \int_{\Gamma_N} g^k\, v_k\, d\Gamma \qquad \forall \mathbf{v} \in V_0$$

simplifies to finding $\mathbf{u} \in \bar{\mathbf{u}} + V_0$ such that

$$D\Pi(\mathbf{u},\mathbf{v}) = 0 \qquad \forall \mathbf{v} \in V_0$$

This is in fact the optimality condition for an equivalent minimization problem: find $\mathbf{u} \in \bar{\mathbf{u}} + V_0$ such that

$$\Pi(\mathbf{u}) \to \min .$$

g) Linearize around the known state $\bar{\mathbf{u}}$, express the problem in the unknown $\delta\mathbf{u} \in V_0$: find $\mathbf{u} \in \bar{\mathbf{u}} + V_0$ such that

$$\int_{\Omega_0} \frac{\partial \Psi(\mathbf{u})}{\partial \varepsilon_{ij}} D\varepsilon_{ij}(\mathbf{u},\mathbf{v})\, dx = \int_{\Omega_0} F_0^k v_k\, dx + \int_{\Gamma_N} g^k v_k\, d\Gamma \qquad \forall \mathbf{v} \in V_0$$

with limited Taylor expansion around $\bar{\mathbf{u}}$:

$$\frac{\partial \Psi(\mathbf{u})}{\partial \varepsilon_{ij}} = \underbrace{\frac{\partial \Psi(\bar{\mathbf{u}})}{\partial \varepsilon_{ij}}}_{T^{ij}(\bar{\mathbf{u}})} + \underbrace{\frac{\partial^2 \Psi(\bar{\mathbf{u}})}{\partial \varepsilon_{ij}\,\partial \varepsilon_{k\ell}}}_{E^{ijk\ell}(\bar{\mathbf{u}})} \cdot D\varepsilon_{k\ell}(\bar{\mathbf{u}},\delta\mathbf{u}) + o(\delta\mathbf{u}).$$

and

$$D\varepsilon_{ij}(\mathbf{u},\mathbf{v}) = D\varepsilon_{ij}(\bar{\mathbf{u}},\mathbf{v}) + D^2\varepsilon_{ij}(\bar{\mathbf{u}},\mathbf{v},\delta\mathbf{u}) + o(\delta\mathbf{u})$$

where

$$\begin{aligned} D^2\varepsilon_{ij}(\bar{\mathbf{u}},\mathbf{v},\delta\mathbf{u}) &= \frac{d}{d\alpha}\left[D\varepsilon_{ij}(\bar{\mathbf{u}}+\alpha\delta\mathbf{u},\mathbf{v})\right]_{\alpha\to 0} \\ &= \frac{1}{2}\left(\nabla_i \delta u^k \cdot \nabla_j v_k + \nabla_j \delta u^k \cdot \nabla_i v_k\right) \end{aligned}$$

2.1 Weak formulation of elasto-statics (Lagrange formul.)

g) Linearize around the known state $\bar{\mathbf{u}}$, express the problem in the unknown $\delta\mathbf{u} \in V_0$: find $\delta\mathbf{u} \in V_0$ such that

$$\int_{\Omega_0} \left[T^{ij}(\bar{\mathbf{u}}) + E^{ijk\ell}(\bar{\mathbf{u}})\, D\varepsilon_{k\ell}(\bar{\mathbf{u}},\delta\mathbf{u})\right] \cdot \left[D\varepsilon_{ij}(\bar{\mathbf{u}},\mathbf{v}) + D^2\varepsilon_{ij}(\bar{\mathbf{u}},\mathbf{v},\delta\mathbf{u})\right] dx$$

$$= \int_{\Omega_0} F_0^k v_k\, dx + \int_{\Gamma_N} g^k v_k\, d\Gamma + o(\delta\mathbf{u}). \qquad \forall \mathbf{v} \in V_0$$

Collect known terms onto the right-hand side, neglect $o(\delta\mathbf{u})$ terms:

Linearized problem for $\delta\mathbf{u}$ (one iteration of Newton's method):
Given initial guess $\bar{\mathbf{u}}$, find a correction $\delta\mathbf{u} \in V_0$ for which

$$\int_{\Omega_0} \left[E^{ijk\ell}(\bar{\mathbf{u}})\, D\varepsilon_{k\ell}(\bar{\mathbf{u}},\delta\mathbf{u})\, D\varepsilon_{ij}(\bar{\mathbf{u}},\mathbf{v}) + T^{ij}(\bar{\mathbf{u}})\; D^2\varepsilon_{ij}(\bar{\mathbf{u}},\mathbf{v},\delta\mathbf{u})\right] dx$$

$$= \underbrace{\int_{\Omega_0} F_0^k v_k\, dx + \int_{\Gamma_N} g^k v_k\, d\Gamma - \int_{\Omega_0} T^{ij}(\bar{\mathbf{u}}) D\varepsilon_{ij}(\bar{\mathbf{u}},\mathbf{v})}_{R(\bar{\mathbf{u}},\mathbf{v})\text{...nonlinear residual}} \qquad \forall \mathbf{v} \in V_0.$$

3. Incompressible linear elasto-statics

So far, we have treated **compressible** materials,
ie. bulk modulus $K < \infty$, Poisson's ratio $\nu \in [-1, 0.5)$.

Consider now an incompressible linear material, ie. $K = \infty$, $\nu = 0.5$.

The "compressible" formulation of Section 1. does not have a sense, because $\lambda = \infty$ in Hooke's law. We need to reformulate the elasticity problem!

3.1. Incompressible linear elasto-statics

Hooke's law written in deviators: hydrostatic pressure p:

$$\begin{aligned} \tilde{\tau}^{im} &= 2\,\mu\,\tilde{e}_{im} \\ -p = s &= K\,e_{\ell\ell} = (3\,\lambda + 2\,\mu)\,e = \frac{E}{1 - 2\,\nu}\,e \end{aligned}$$

Cauchy stress:

$$\tau^{ij} = \tilde{\tau}^{ij} + s\,\delta^{ij} = \tilde{\tau}^{ij} - p\,\delta^{ij}$$

Mixed formulation: 2 equations for 2 unknowns $(\mathbf{u}, p)$:
force equilibria and volume/pressure relation:

$$\begin{aligned} \frac{\partial}{\partial y^j}\left[\tilde{\tau}^{ij} - p\,\delta_{ij}\right] + \rho f^i &= 0 \\ e_{\ell\ell} + \frac{p}{K} &= 0 \end{aligned}$$

This form is meaningful even if $K \to \infty$ (incompressible limit)!

Mixed formulation: find $(\mathbf{u}, p)$ such that

$$
\begin{aligned}
-\frac{\partial}{\partial y^j}\left[2\,\mu\left(e_{ij}-\frac{1}{3}\,e_{\ell\ell}\delta_{ij}\right)-p\,\delta_{ij}\right] &= \rho\, f^i \\
e_{\ell\ell}+\frac{p}{K} &= 0 \qquad \text{in } \Omega \\
+ \text{ boundary conditions} & \qquad\qquad \text{on } \partial\Omega.
\end{aligned}
$$

In global notation: **Stokes problem:** find $(\mathbf{u}, p)$ such that

$$
\begin{aligned}
-\operatorname{div}\underbrace{\left(2\,\mu\,\underline{\underline{e}}-\frac{2\mu}{3}\operatorname{tr}(\underline{\underline{e}})\underline{\underline{Id}}\right)}_{\underline{\underline{\tilde{\tau}}}}+\nabla p &= \mathbf{F} \\
\operatorname{div}(\mathbf{u})+\frac{p}{K} &= 0 \qquad \text{in } \Omega, \\
\mathbf{u} &= \bar{\mathbf{u}} \qquad \text{on } \Gamma_D\subset\partial\Omega, \\
\left(\underline{\underline{\tilde{\tau}}}-p\,\underline{\underline{Id}}\right)\cdot\mathbf{n} &= \mathbf{g} \qquad \text{on } \Gamma_N=\partial\Omega\setminus\Gamma_D.
\end{aligned}
$$