Continuum mechanics
III. Conservation laws and mechanical equilibria

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1. Time-derivative of volume integral in Euler formulation

Scalar field $\Phi(\mathbf{y}, t)$ (eg. density, concentration)
Volume integral: over current (deformed) domain Ω_t:

$$\mathcal{I} = \int_{\Omega_t} \Phi(\mathbf{y}, t) dy$$

Change in $\mathcal{I}(t)$: due to two phenomena:

- **Change of $\Phi(., t)$ inside** of the fixed domain Ω_t over time dt:

$$d\mathcal{I}_1 = \int_{\Omega_t} \frac{\partial \Phi(\mathbf{y}, t)}{\partial t} dt\, dy = dt \int_{\Omega_t} \frac{\partial \Phi(\mathbf{y}, t)}{\partial t} dy$$

- **Time flow (in- and out-flow) of continuum** through $\partial\Omega_t$:

$$d\mathcal{I}_2 = \int_{\partial\Omega_t} \Phi\, \mathbf{n} \cdot \mathbf{v}\, dt\, d\Gamma = dt \int_{\partial\Omega_t} \Phi\, n_i\, v^i d\Gamma$$

By the Divergence theorem:

$$d\mathcal{I}_2 = dt \int_{\Omega_t} \frac{\partial}{\partial y^i} \left(\Phi v^i\right) dy$$

1. Time-derivative of volume integral in Euler formulation

Time-derivative of volume integral:

$$\frac{D\mathcal{I}}{Dt} = \frac{d}{dt}(\mathcal{I}_1 + \mathcal{I}_2) = \int_{\Omega_t} \left[\frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial y^i} (\Phi \, v^i) \right] dy$$

$$= \int_{\Omega_t} \left[\frac{\partial \Phi}{\partial t} + v^i \frac{\partial \Phi}{\partial y^i} + \Phi \frac{\partial v^i}{\partial y^i} \right] dy$$

$$= \int_{\Omega_t} \left[\frac{D\Phi}{Dt} + \Phi \frac{\partial v^i}{\partial y^i} \right] dy$$

2. Mass conservation in Euler formulation

Application of the formula for $\frac{D\mathcal{I}}{Dt}$: $\Phi \equiv \varrho$ is the density:

$$\frac{D\mathcal{I}}{Dt} \equiv 0 = \int_{\Omega_t} \left[\frac{D\varrho}{Dt} + \varrho \frac{\partial v^i}{\partial y^i} \right] dy \qquad \text{for all } \Omega_t \subset \Omega$$

We can tend Ω_t to a point to get the point-form of mass-conservation law (called the **continuity equation**):

$$\boxed{0 = \frac{D\varrho}{Dt} + \varrho \frac{\partial v^i}{\partial y^i} = \frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial y^i}(\varrho v^i)}$$

- **Incompressible continuum:** $\frac{D\varrho}{Dt} = 0$:

$$\rho \frac{\partial v^i}{\partial y^i} = 0 \qquad \text{ie.} \qquad \mathrm{div}(\mathbf{v}) = 0$$

- **Continuum at steady state**: $\frac{\partial \varrho}{\partial t} = 0$:

$$\frac{\partial}{\partial y^i}(\rho v^i) = 0 \qquad \text{ie.} \qquad \mathrm{div}(\rho \mathbf{v}) = 0$$

2. Mass conservation in Lagrange formulation

Compare local mass before (Ω_0) and after (Ω_t) deformation:

$$\int_{\Omega_0} \varrho_0(\mathbf{x})dx = \int_{\Omega_t} \varrho(\mathbf{y})dy = \int_{\Omega_0} \varrho(\mathbf{y}(\mathbf{x})) \cdot \det\left[\frac{\partial y^i}{\partial x^j}\right] dx$$

Continuity equation:

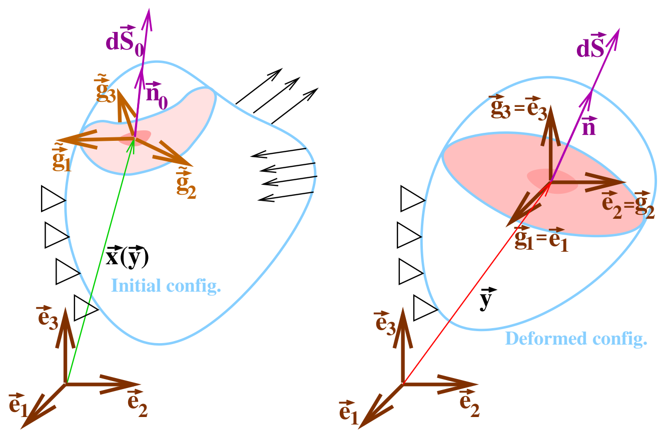
$$\boxed{\varrho_0 = \varrho \cdot \det\left[\frac{\partial y^i}{\partial x^j}\right]}$$

3. Stress in Euler formulation: Euler stress tensor

- cut the **deformed** configuration in two
- cutting-plane normal $\mathbf{n}$, $n_i = \mathbf{n} \cdot \mathbf{g}_i$, area dS.
- delete one of the two parts (eg. the upper one)
- find a force $d\mathbf{F}$ which, applied on a point in $d\mathbf{S}$, simulates the effects of the deleted part
- $d\mathbf{F}$ depends (linearly) on $\mathbf{n}$ and dS

$$d\mathbf{F} = \tau^{ij}\, n_i\, \mathbf{g}_j\, dS$$

τ^{ij} **is the Euler stress tensor**



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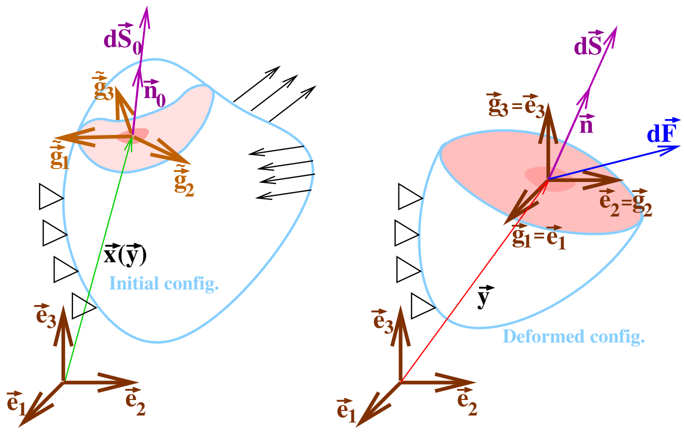
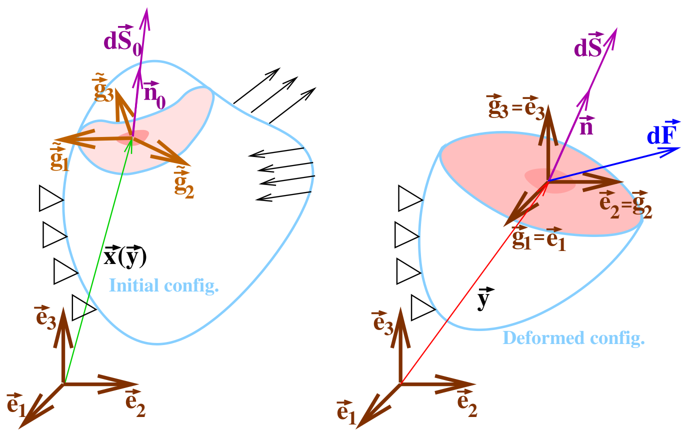


3. Stress in Lagrange formulation: 1st Piola-Kirchhoff

- cut the **deformed** configuration in two
- cutting-plane normal **n**, $n_i = \mathbf{n} \cdot \mathbf{g}_i$, area dS.
- the cutting surface in the initial configuration has a normal $\mathbf{n}_0$ and area dS_0.
- find a force $d\mathbf{F}$ which, applied on a point in $d\mathbf{S}$, simulates the effects of the deleted (eg. upper) part
- recopy $d\mathbf{F}$ to the initial configuration!!
- $d\mathbf{F}$ depends (linearly) on $\mathbf{n}_0$ and dS_0

$$d\mathbf{F} = \sigma^{ij}\, n_{0i}\, \mathbf{g}_j\, dS_0$$

1st Piola-Kirchhoff stress tensor σ^{ij}



↑ Euler formulation ↑

3. Stress in Lagrange formulation: 1st Piola-Kirchhoff

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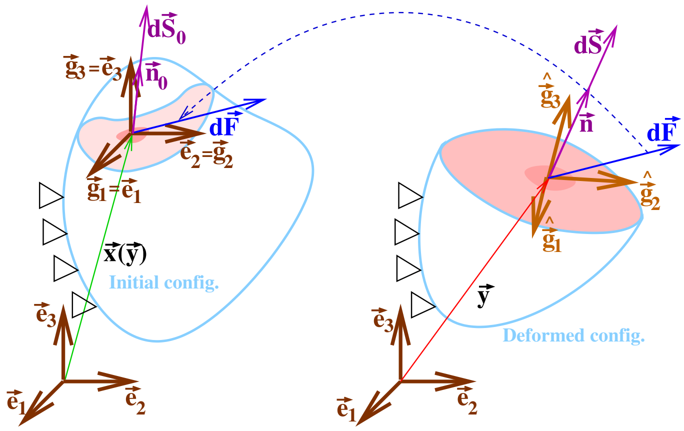
1st Piola-Kirchhoff stress tensor σ^{ij}



↑ Lagrange formulation ↑

3. Stress in Lagrange formulation: 2nd Piola-Kirchhoff

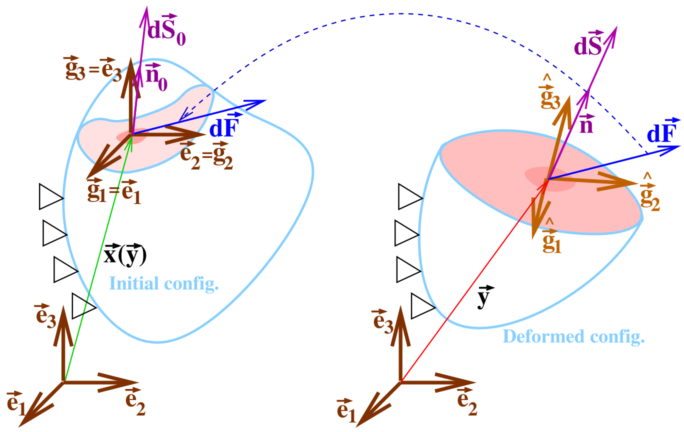
- cut the **deformed** configuration in two
- cutting-plane normal $\mathbf{n}$, $n_i = \mathbf{n} \cdot \mathbf{g}_i$, area dS.
- cutting surface in initial config.: normal $\mathbf{n}_0$ and area dS_0.
- find a force $d\mathbf{F}$ which, applied on a point in $d\mathbf{S}$, simulates the effects of the deleted (eg. upper) part
- transform $d\mathbf{F}$ to $d\mathbf{F}_0$ in initial config.:

$$d\mathbf{F}_0 = dF^j \frac{\partial x^i}{\partial y^j} \mathbf{g}_i$$

- $d\mathbf{F}_0$ depends (linearly) on $\mathbf{n}_0$ and dS_0

$$d\mathbf{F}_0 = T^{ij}\, n_{0i}\, \mathbf{g}_j\, dS_0$$

2nd Piola-Kirchhoff stress tensor T^{ij}



↑ 1st Piola-Kirchhoff ↑

3. Stress in Lagrange formulation: 2nd Piola-Kirchhoff

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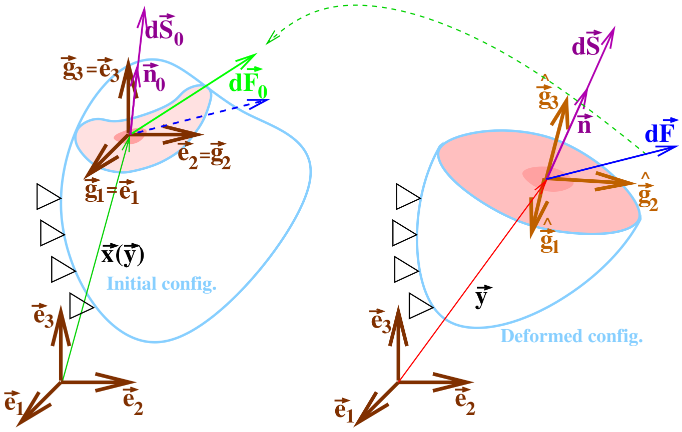
- cutting-plane normal $\mathbf{n}$, $n_i = \mathbf{n} \cdot \mathbf{g}_i$, area dS.
- cutting surface in initial config.: normal $\mathbf{n}_0$ and area dS_0.
- find a force $d\mathbf{F}$ which, applied on a point in $d\mathbf{S}$, simulates the effects of the deleted (eg. upper) part
- transform $d\mathbf{F}$ to $d\mathbf{F}_0$ in initial config.:

$$d\mathbf{F}_0 = dF^j \, \frac{\partial x^i}{\partial y^j} \, \mathbf{g}_i$$

- $d\mathbf{F}_0$ depends (linearly) on $\mathbf{n}_0$ and dS_0

$$d\mathbf{F}_0 = T^{ij} \, n_{0i} \, \mathbf{g}_j \, dS_0$$

2nd Piola-Kirchhoff stress tensor T^{ij}



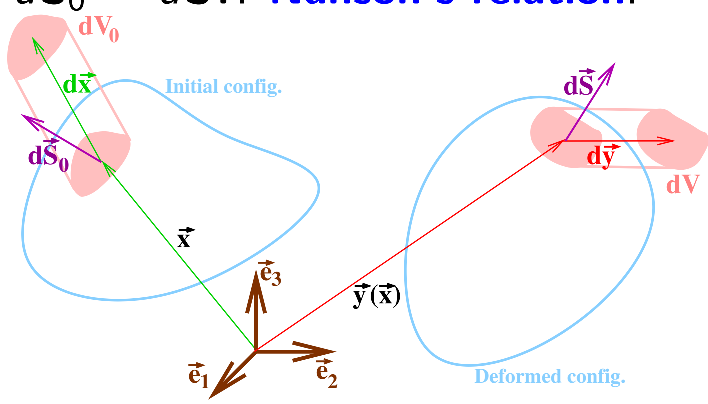
↑ 2nd Piola-Kirchhoff ↑

3. Stress: Cauchy vs. 1st/2nd Piola-Kirchhoff tensors

How to transform areas $dS_0 \to d\mathbf{S}$?: Nanson's relation:

$$dS_j = J \cdot \frac{\partial x^\ell}{\partial y^j} \, dS_{0\ell}$$

with $J = \det \left[\frac{\partial y^i}{\partial x^k} \right]$.



Relation between Euler and 1st Piola-Kirchhoff:

$$dF^i = \tau^{ij} \, n_j \, dS = \tau^{ij} \, dS_j = \tau^{ij} \, J \, \frac{\partial x^\ell}{\partial y^j} \, dS_{0\ell}$$

$$= \sigma^{i\ell} \, n_{0\ell} \, dS_0 = \sigma^{i\ell} \, dS_{0\ell}$$

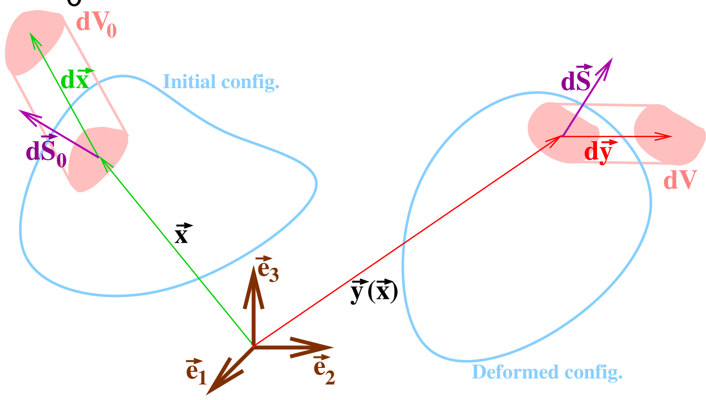
$$\sigma^{i\ell} = \tau^{ij} \, J \, \frac{\partial x^\ell}{\partial y^j}$$

3. Stress: Cauchy vs. 1st/2nd Piola-Kirchhoff tensors

How to transform areas $dS_0 \rightarrow dS$?: Nanson's relation:

$$\boxed{dS_j = J \cdot \frac{\partial x^\ell}{\partial y^j}\, dS_{0\ell}}$$

with $J = \det \left[\frac{\partial y^i}{\partial x^k} \right]$.



Relation between Euler and 2nd Piola-Kirchhoff:

$$\begin{aligned} dF_0^j &= \frac{\partial x^j}{\partial y^k} dF^k = \frac{\partial x^j}{\partial y^k}\, \tau^{km}\, J\, \frac{\partial x^\ell}{\partial y^m}\, dS_{0\ell} \\ &= T^{\ell j}\, dS_{0\ell} \end{aligned}$$

$$\boxed{T^{\ell j} = \tau^{km}\, J\, \frac{\partial x^j}{\partial y^k} \frac{\partial x^\ell}{\partial y^m}}$$

3. Stress: Cauchy vs. 1st/2nd Piola-Kirchhoff tensors

How to transform areas $dS_0 \rightarrow dS$?: Nanson's relation:

$$\boxed{dS_j = J \cdot \frac{\partial x^\ell}{\partial y^j}\, dS_{0\ell}}$$

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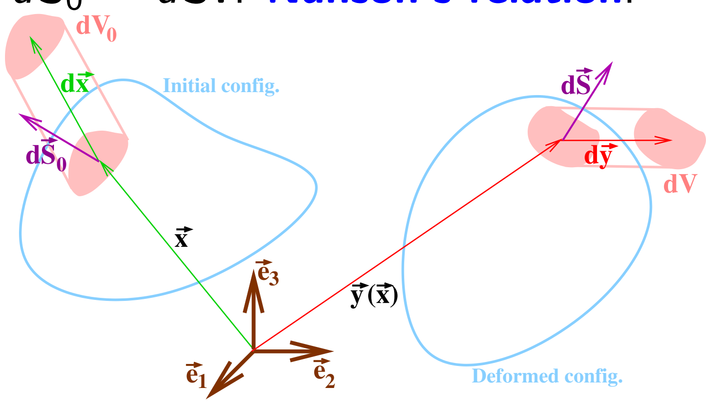

Relation between 1st and 2nd Piola-Kirchhoff:

From relations to Euler stress:

$$T^{\ell j} = \tau^{km}\, J\, \frac{\partial x^j}{\partial y^k} \frac{\partial x^\ell}{\partial y^m} \qquad \text{and} \qquad \sigma^{k\ell} = \tau^{km}\, J\, \frac{\partial x^\ell}{\partial y^m}$$

$$\boxed{T^{\ell j} = \sigma^{k\ell} \frac{\partial x^j}{\partial y^k}}$$

4. Force equilibria: balance of force and of momentum

- **Body loads**: forces on internal points: electro-magnetic field, gravity, acceleration

$$dF^i = f^i \varrho \, dV - \varrho \, a^i \, dV$$

where f^i is the body force per unit mass, a^i is acceleration and ϱ is density
- **Surface (contact) loads**: pressure and drag on the surface: can be expressed by

$$\tau^{ij} n_j$$

- Equilibrium of **forces** and of **momentum of forces** ($\mathbf{F} \times \mathbf{x}$).
- **Euler formulation is more natural** for force equilibria than Lagrange formulation

4. Force equilibria: Euler formulation

$$\int_\omega \varrho \, f^i dV + \int_{\partial\omega} \tau^{ij} \, n_j dS = 0 \qquad \text{for any } \omega \subset \Omega_t$$

By the Divergence theorem:

$$\int_\omega \left(\varrho \, f^i + \frac{\partial \tau^{ij}}{\partial y^j} \right) dV = 0$$

We can tend $\omega \subset \Omega_t$ to one point to get:

$$\boxed{\frac{\partial}{\partial y^j} \tau^{ij} + \varrho \, f^i = 0}$$

in the deformed domain Ω_t.

Using 1st Piola-Kirchhoff stress: and body forces per unit mass of **initial configuration**:

$$dF_0^i = f_0^i\, \varrho_0 dV_0$$

Force equilibrium with 1st Piola-Kirchhoff stress:

$$\boxed{\frac{\partial}{\partial x^j}\sigma^{ij} + \varrho_0 f_0^i = 0}$$

in the initial configuration Ω_0.

Using 2nd Piola-Kirchhoff stress: transform σ^{ij} to $T^{k\ell}$:

$$\boxed{\frac{\partial}{\partial x^\ell}\left(\frac{\partial y^k}{\partial x^j}T^{j\ell}\right) + \varrho_0\, f_0^k = \frac{\partial}{\partial x^\ell}\left[T^{j\ell}\left(\frac{\partial u^k}{\partial x^j} + \delta_j^k\right)\right] + \varrho_0\, f_0^k = 0}$$

Euler formulation: momentum equilibrium about the origin:

$$\int_\omega \mathbf{y} \times \varrho\, f^i\, \mathbf{g}_i dV + \int_{\partial\omega} \mathbf{y} \times \tau^{ij}\, n_j\, \mathbf{g}_i dS = 0$$

Levi-Civita symbol:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if } i=j \text{ or } i=k \text{ or } j=k \end{cases}$$

Properties of the Levi-Civita symbol:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \qquad \Leftrightarrow \qquad c_i = \varepsilon_{ijk}\, a^j\, b^k.$$

Hence for moment equilibrium:

$$\int_\omega \varepsilon_{ijk}\, y^i\, \varrho\, f^j\, dV + \int_{\partial\omega} \varepsilon_{ijk}\, y^i\, \tau^{j\ell}\, n_\ell\, dS = 0$$

4. Force equilibria: momentum conditions

By the Divergence theorem:

$$\int_\omega \varepsilon_{ijk}\, y^i\, \varrho\, f^j dV + \int_\omega \frac{\partial}{\partial y^\ell} \left(\varepsilon_{ijk}\, y^i\, \tau^{j\ell} \right)\, dS = 0$$

Work-out the derivative in the second term to get:

$$\int_\omega \varepsilon_{ijk}\, y^i \left(\frac{\partial \tau^{j\ell}}{\partial y^\ell} + \varrho\, f^j \right) dV + \int_\omega \varepsilon_{ijk}\, \delta^i_\ell\, \tau^{j\ell} dV = 0$$

As the red term is 0 by component-wise conditions, momentum conditions reduce to

$$\varepsilon_{ijk} \tau^{ji} = 0 \qquad \forall k = 1, 2, 3.$$

By definition of the Levi-Civita symbol, this means:

$$\boxed{\tau^{12} = \tau^{21} \quad , \quad \tau^{13} = \tau^{31} \quad , \quad \tau^{23} = \tau^{32}}$$

Euler and 2nd Piola-Kirchhoff stress tensors are symmetric!

5. Transformation of tensors Lagrange ↔ Euler

Deformation gradient:

$$F^i_j = \frac{\partial y^i}{\partial x_j} \qquad {F^{-1}}^i_j = \frac{\partial x^i}{\partial y_j} \qquad J = \det \left[\frac{\partial y^i}{\partial x_j} \right]$$

Strain tensors: Lagrange ↔ Euler

$$\varepsilon_{k\ell} = F^i_k \cdot \mathcal{E}_{ij} \cdot F^j_\ell$$

Stress tensors: Lagrange (2nd Piola-Kirchhoff stress) ↔ Euler

$$T^{ij} = J \cdot {F^{-1}}^j_k \cdot \tau^{km} \cdot {F^{-1}}^i_m$$