Continuum mechanics
II. Kinematics in curvilinear coordinates

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1. Strain in cartesian coordinates (recapitulation)

- **Green strain tensor**: Lagrange formulation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} + \frac{\partial u_k}{\partial x^i}\frac{\partial u^k}{\partial x^j}\right)$$

- **Cauchy strain tensor**: linearized strain for small deformations

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right)$$

- **Almansi strain tensor**: Euler formulation

$$\mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial y^j} + \frac{\partial u_j}{\partial y^i} - \frac{\partial u_k}{\partial y^i}\frac{\partial u^k}{\partial y^j}\right)$$

1. Strain in curvilinear coordinates

- **Green strain tensor**: Lagrange formulation

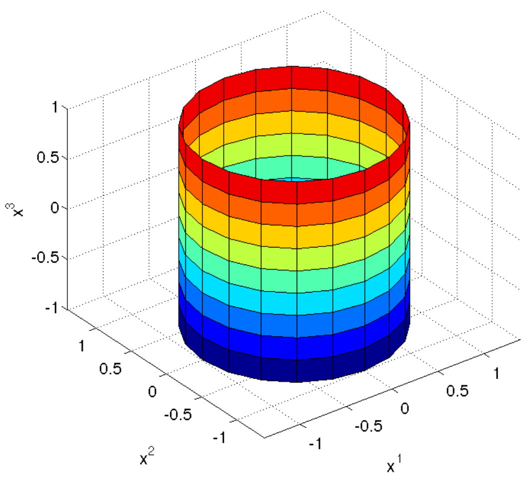
$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla_j u_i + \nabla_i u_j + \nabla_i u_k \cdot \nabla_j u^k\right)$$

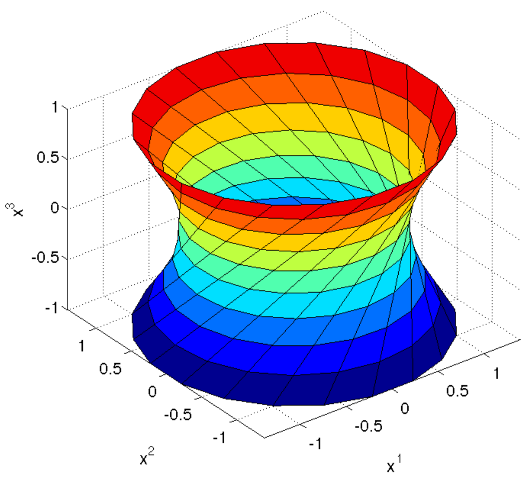
- **Cauchy strain tensor**: linearized strain for small deformations

$$e_{ij} = \frac{1}{2}\left(\nabla_j u_i + \nabla_i u_j\right)$$

2. Example of using curvilinear coordinates

A rotational cylinder is being deformed into a rotational hyperboloid. Calculate the Cauchy strain tensor.





It's advantageous to use the cylindrical coordinates:

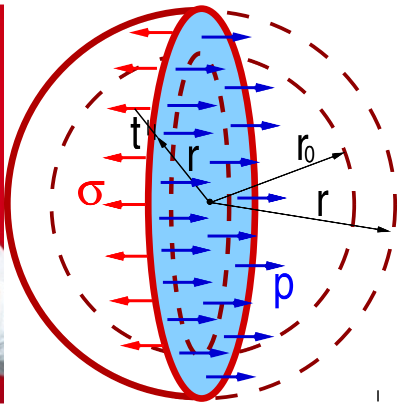
$$\mathbf{x} = \begin{pmatrix} \xi^1 \cos\xi^2 \\ \xi^1 \sin\xi^2 \\ \xi^3 \end{pmatrix} \longrightarrow \mathbf{y}(\mathbf{x}) = \begin{pmatrix} \xi^1 \left(\cos\xi^2 - \frac{\xi^3}{Z}\sin\xi^2\right) \\ \xi^1 \left(\sin\xi^2 + \frac{\xi^3}{Z}\cos\xi^2\right) \\ \xi^3 \end{pmatrix}$$

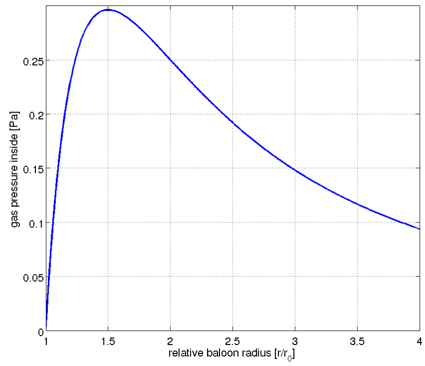
2. Example of using curvilinear coordinates

Advantages of using curvilinear coordinates:

- Simpler analytical formulae for particular deformation modes and particular geometries
- Better intuitive understanding of deformation modes
- Particularly useful for shells and membranes or anisotropic materials

Remember the inflated baloon demonstration?





3. Cauchy strain in cylindrical coordinates

Cauchy strain in curvilinear coordinates:

$$e_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right)$$

Covariant derivative:

$$\nabla_j u_i = \frac{\partial u_i}{\partial \xi^j} - \Gamma^{\ell}_{ij}\, u_\ell$$

Cylindrical coordinates:

$$\mathbf{x}(\xi^1,\xi^2,\xi^3) = \begin{pmatrix} \xi^1 \cos\xi^2 \\ \xi^1 \sin\xi^2 \\ \xi^3 \end{pmatrix} \quad , \quad [g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\xi^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Christoffel symbols of 2nd kind: for cylindrical coordinates

$$\Gamma^1_{22} = -\xi^1 \quad , \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\xi^1} \quad , \quad \Gamma^{\ell}_{ij} = 0 \text{ otherwise.}$$

3. Cauchy strain in cylindrical coordinates

$$
\begin{aligned}
e_{11} &= \nabla_1 u_1 = \frac{\partial u_1}{\partial \xi^1} \\
e_{12} &= \frac{1}{2}(\nabla_1 u_2 + \nabla_2 u_1) = \frac{1}{2}\left[\frac{\partial u_2}{\partial \xi^1} - \Gamma^2_{21} u_2 + \frac{\partial u_1}{\partial \xi^2} - \Gamma^2_{12} u_2\right] \\
&= \frac{1}{2}\left[\frac{\partial u_2}{\partial \xi^1} + \frac{\partial u_1}{\partial \xi^2} - \frac{2}{\xi^1} u_2\right] \\
e_{13} &= \frac{1}{2}(\nabla_1 u_3 + \nabla_3 u_1) = \frac{1}{2}\left[\frac{\partial u_3}{\partial \xi^1} + \frac{\partial u_1}{\partial \xi^3}\right] \\
e_{22} &= \nabla_2 u_2 = \frac{\partial u_2}{\partial \xi^2} - \Gamma^1_{22} u_1 = \frac{\partial u_2}{\partial \xi^2} + \xi^1\, u_1 \\
e_{23} &= \frac{1}{2}(\nabla_2 u_3 + \nabla_3 u_2) = \frac{1}{2}\left[\frac{\partial u_3}{\partial \xi^2} + \frac{\partial u_2}{\partial \xi^3}\right] \\
e_{33} &= \nabla_3 u_3 = \frac{\partial u_3}{\partial \xi^3}
\end{aligned}
$$

Note that physical units of e_{ij} are quite inhomogeneous here!

3. Cauchy strain in cylindrical coordinates

Non-homogeneity of physical units for e_{ij} and u_i

Units of cylindrical coordinates: ξ^1 in $[m]$, ξ^2 in $[rad]$, ξ^3 in $[m]$.

Covariant basis: $\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}$:

$$
\underbrace{\mathbf{g}_1 = \begin{pmatrix} \cos\xi^2 \\ \sin\xi^2 \\ 0 \end{pmatrix}}_{\text{in } [1]}, \quad
\underbrace{\mathbf{g}_2 = \begin{pmatrix} -\xi^1 \sin\xi^2 \\ \xi^1 \cos\xi^2 \\ 0 \end{pmatrix}}_{\text{in } [m]}, \quad
\underbrace{\mathbf{g}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{in } [1]}
$$

Contravariant basis:

$$
\underbrace{\mathbf{g}^1 = \mathbf{g}_1}_{\text{in } [1]}, \quad
\underbrace{\mathbf{g}^2 = \begin{pmatrix} -\frac{1}{\xi^1}\sin\xi^2 \\ \frac{1}{\xi^1}\cos\xi^2 \\ 0 \end{pmatrix}}_{\text{in } [1/m]}, \quad
\underbrace{\mathbf{g}^3 = \mathbf{g}_3}_{\text{in } [1]}
$$

Units for u^i and u_i: displacement $\mathbf{u} = u^i\,\mathbf{g}_i = u_i\,\mathbf{g}^i$ should be in $[m]$:

coordinate	its unit	coordinate	its unit
u^1	$[m]$	u_1	$[m]$
u^2	$[1]$	u_2	$[m^2]$
u^3	$[m]$	u_3	$[m]$

Hence, units for e_{ij}:

coordinate	its unit	coordinate	its unit
e_{11}	$[1]$	e_{22}	$[m^2]$
e_{12}	$[m]$	e_{23}	$[m]$
e_{13}	$[1]$	e_{33}	$[1]$

Correction of unit inhomogeneity:

introduction of **physical components** $e_{(ij)}$ and $u_{(i)}$ by:

$$e_{ij} = \sqrt{g_{ii} \cdot g_{jj}} \cdot e_{(ij)} \qquad \text{and} \qquad u_i = \sqrt{g_{ii}} \cdot u_{(i)}$$

For cylindrical coordinates:

$$\begin{array}{lcr|lcr|lcr}
e_{11} & = & e_{(11)} & e_{21} & = & \xi^1 \cdot e_{(21)} & e_{31} & = & e_{(31)} \\
e_{12} & = & \xi^1 \cdot e_{(12)} & e_{22} & = & (\xi^1)^2 \cdot e_{(22)} & e_{32} & = & \xi^1 \cdot e_{(32)} \\
e_{13} & = & e_{(13)} & e_{23} & = & \xi^1 \cdot e_{(23)} & e_{33} & = & e_{(33)}
\end{array}$$

$$u_1 = u_{(1)} \quad \rightarrow \quad \frac{\partial u_1}{\partial \xi^j} = \frac{\partial u_{(1)}}{\partial \xi^j}$$

$$u_2 = \xi^1 \cdot u_{(2)} \quad \rightarrow \quad \frac{\partial u_2}{\partial \xi^1} = u_{(2)} + \xi^1 \frac{\partial u_{(2)}}{\partial \xi^1} \quad , \quad \frac{\partial u_2}{\partial \xi^2} = \xi^1 \frac{\partial u_{(2)}}{\partial \xi^2}$$

$$\frac{\partial u_2}{\partial \xi^3} = \xi^1 \frac{\partial u_{(2)}}{\partial \xi^3}$$

$$u_3 = u_{(3)} \quad \rightarrow \quad \frac{\partial u_3}{\partial \xi^j} = \frac{\partial u_{(3)}}{\partial \xi^j}$$

Physical components in cylindrical coordinates usually written

$$u_{(1)} = u_r \quad , \quad u_{(2)} = u_\theta \quad , \quad u_{(3)} = u_z$$

$$e_{(11)} = e_{rr} \quad , \quad e_{(12)} = e_{r\theta} \quad , \quad e_{(23)} = e_{\theta z} \quad \ldots$$

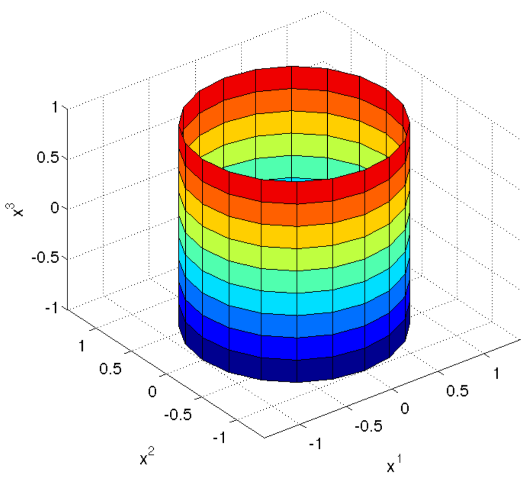
3. Cauchy strain in cylindrical coordinates

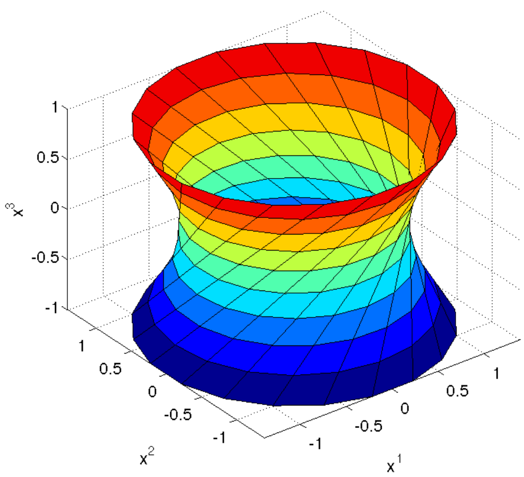
Transforming covariant components to physical components

$$
\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} \\
e_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) \\
e_{rz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
e_{\theta z} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) \\
e_{zz} &= \frac{\partial u_z}{\partial z}
\end{aligned}
$$

4. Back to cylinder → hyperboloid

A rotational cylinder is being deformed into a rotational hyperboloid. Calculate the Cauchy strain tensor.





Use the cylindrical coordinates:

$$
\mathbf{u} = \mathbf{y} - \mathbf{x} = \begin{pmatrix} \xi^1 \left(\cos\xi^2 - \frac{\xi^3}{Z}\sin\xi^2\right) \\ \xi^1 \left(\sin\xi^2 + \frac{\xi^3}{Z}\cos\xi^2\right) \\ \xi^3 \end{pmatrix} - \begin{pmatrix} \xi^1 \cos\xi^2 \\ \xi^1 \sin\xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} -\xi^1 \frac{\xi^3}{Z}\sin\xi^2 \\ \xi^1 \frac{\xi^3}{Z}\cos\xi^2 \\ 0 \end{pmatrix}
$$

4. Back to cylinder → hyperboloid

$$\mathbf{u} = \begin{pmatrix} -\xi^1 \frac{\xi^3}{Z} \sin\xi^2 \\ \xi^1 \frac{\xi^3}{Z} \cos\xi^2 \\ 0 \end{pmatrix} = 0 \underbrace{\begin{pmatrix} \cos\xi^2 \\ \sin\xi^2 \\ 0 \end{pmatrix}}_{\mathbf{g}^1} + (\xi^1)^2 \frac{\xi^3}{Z} \underbrace{\begin{pmatrix} -\frac{1}{\xi^1} \sin\xi^2 \\ \frac{1}{\xi^1} \cos\xi^2 \\ 0 \end{pmatrix}}_{\mathbf{g}^2} + 0 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}^3}$$

Hence

$$u_1 = u_3 = 0 \quad , \quad u_2 = (\xi^1)^2 \frac{\xi^3}{Z} \quad \text{and} \quad u_{(2)} = u_\theta = \xi^1 \frac{\xi^3}{Z}$$

Resulting Cauchy strain:

$$e_{rr} = e_{r\theta} = e_{rz} = e_{\theta\theta} = e_{zz} = 0 \quad \text{and} \quad e_{\theta z} = \frac{r}{2\,Z}$$

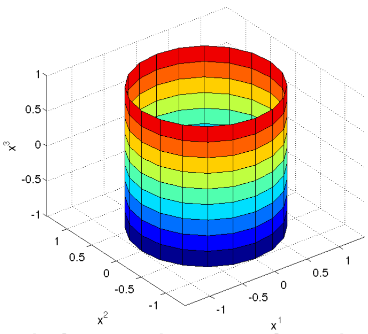
ie. pure shear (ie. distortion of angles) in the (θ, z) tangent-plane.

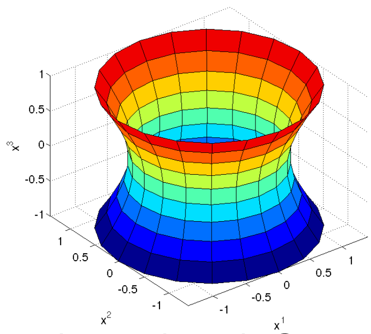
5. Yet another cylinder → hyperboloid example

But different from the previous one!

A rotational cylinder is being deformed into a rotational hyperboloid in the following way (in cylindrical coordinates):

$$\mathbf{x} = \begin{pmatrix} \xi^1 \cos\xi^2 \\ \xi^1 \sin\xi^2 \\ \xi^3 \end{pmatrix} \longrightarrow \mathbf{y}(\mathbf{x}) = \begin{pmatrix} \xi^1 \sqrt{1 + \left(\frac{\xi^3}{Z}\right)^2} \cos\xi^2 \\ \xi^1 \sqrt{1 + \left(\frac{\xi^3}{Z}\right)^2} \sin\xi^2 \\ \xi^3 \end{pmatrix}$$





The resulting shape is the same, but the deformation tensor is different! Why?