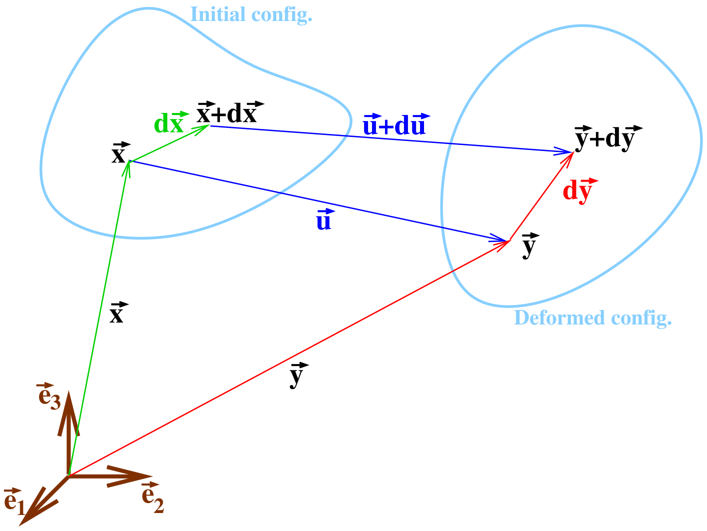
6. Saint-Venant's compatability conditions



Green strain tensor (cartesian coords):

$$
\begin{aligned}
d\mathbf{y}^2 - d\mathbf{x}^2 &= 2\,\varepsilon_{ij}\,dx^i\,dx^j \\
d\mathbf{y}^2 &= \underbrace{(2\,\varepsilon_{ij} + \delta_{ij})}_{C_{ij}}\,dx^i\,dx^j
\end{aligned}
$$

The **right Cauchy-Green deformation tensor** $\underline{\underline{C}}$ measures distances in **euclidean space** of $d\mathbf{y}$'s!

$\underline{\underline{C}}$ is a **metric tensor** for **euclidean space** $\Rightarrow$ it has to verify the **Theorem of Riemann** (basic result from differential geometry).

6.1 Riemann-Christoffel curvature tensor

Let $\underline{\underline{G}}$ be a metric tensor field and let

$$
\Gamma^{\ell}_{ij} = \frac{1}{2}\,g^{k\ell}\left[\frac{\partial g_{ki}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k}\right]
$$

be the Christoffel symbols of the second kind in this metric.

Riemann-Christoffel curvature tensor $\underline{\underline{\underline{\underline{R}}}}$ is defined by:

$$
R^i_{\cdot jk\ell} = \frac{\partial \Gamma^i_{j\ell}}{\partial \xi^k} - \frac{\partial \Gamma^i_{jk}}{\partial \xi^\ell} + \Gamma^m_{j\ell}\cdot\Gamma^i_{mk} - \Gamma^m_{jk}\cdot\Gamma^i_{m\ell}
$$

6.2 Theorem of Riemann

Let $\underline{\underline{G}}$ be a second-order tensor field.

$\underline{\underline{G}}$ **constitutes a metric field** for **euclidean space** if and only if

- $\underline{\underline{C}}$ is symmetric positive definite, and
- it forms a Riemann-Christoffel curvature tensor $\underline{\underline{\underline{\underline{R}}}}$ which is identically zero everywhere.

Compatability conditions of Saint-Venant

Compatability conditions of Saint-Venant = consequence of Riemann theorem for the **right Cauchy-Green "metric" tensor** $\underline{\underline{G}} = \underline{\underline{C}}$, resp. for Green strain tensor $\underline{\underline{\varepsilon}}$

For example, for small deformations (Cauchy strain $e_{r\ell}$), we thus obtain:

$$\epsilon^{irs}\,\epsilon^{j\ell m}\;\nabla_s\nabla_m e_{r\ell} = 0 \qquad \forall i,j = 1,2,3$$

= **6 compatability conditions of Saint-Venant.**

Here, ϵ^{ijk} is the Levi-Civita permutation symbol:

$$\epsilon^{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{otherwise } (i=j \text{ or } i=k \text{ or } j=k). \end{cases}$$