Continuum mechanics
I. Kinematics in cartesian coordinates

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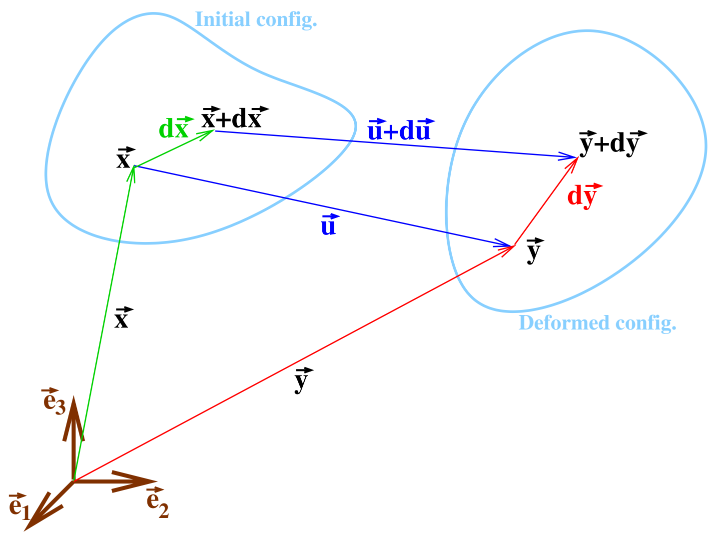
Kinematics: description of position and deformation

- **initial configuration** $\mathbf{x} = x^i \mathbf{e}_i$
  x^i ... material coordinates
- **deformed config.** $\mathbf{y} = y^i \mathbf{e}_i$
  y^i ... spatial coordinates
- displacement $\mathbf{u} = \mathbf{y} - \mathbf{x}$

Two possibilities:

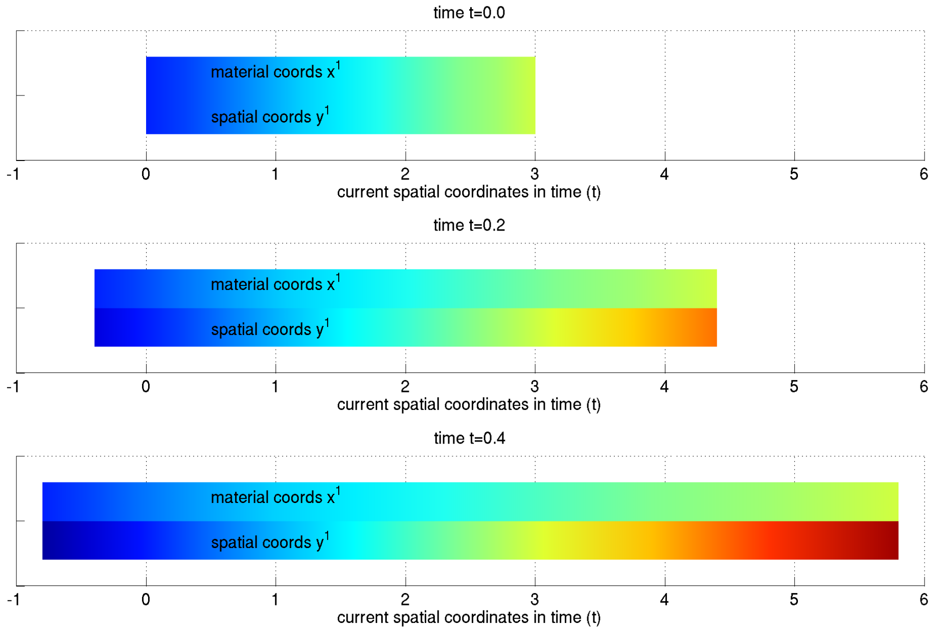
- **Lagrange description:**
  $\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}$
- **Euler description:**
  $\mathbf{u}(\mathbf{y}) = \mathbf{y} - \mathbf{x}(\mathbf{y})$



Aleš Janka | I. Kinematics

Material vs. spatial coordinates: deformation of a 1D rod

- Let $y^1 = y^1(x^1, t) = [(x^1)^2 - 1] \cdot t + x^1$.
- Inversely, $x^1 = x^1(y^1, t) = \frac{1}{2t}\left(\sqrt{1 + 4t(2t + y^1)} - 1\right)$

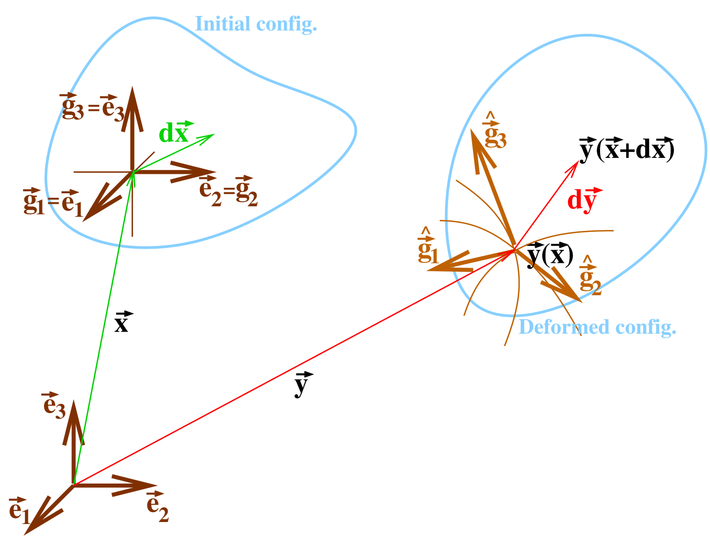


1. Lagrange description

- $\mathbf{x} = x^i \mathbf{e}_i$ and $\mathbf{y} = y^i \mathbf{e}_i$.
- Choice: $\mathbf{y} = \mathbf{y}(\mathbf{x})$
- **Deformation gradient:**

$$F^i_j(\mathbf{x}) = \frac{\partial y^i}{\partial x^j} = y^i_{,j}$$

- $d\mathbf{x} = dx^i \mathbf{e}_i$
- $d\mathbf{y} = dy^i \mathbf{e}_i = \frac{\partial y^i}{\partial x^j} dx^j \mathbf{e}_i$
- $d\mathbf{y} = dx^j\, F^i_j(\mathbf{x})\, \mathbf{g}_i = dx^j\, \hat{\mathbf{g}}_j$



1. Lagrange description: how to measure deformation?

- The "edge" $d\mathbf{x}$ in initial configuration is deformed to $d\mathbf{y}$
- Convenient measure of deformation:

$$d\mathbf{y}^2 - d\mathbf{x}^2 = (dx^i \hat{\mathbf{g}}_i)\cdot(dx^j \hat{\mathbf{g}}_j) - (dx^i \mathbf{g}_i)\cdot(dx^j \mathbf{g}_j) = dx^i\, dx^j\, (\hat{g}_{ij} - g_{ij})$$

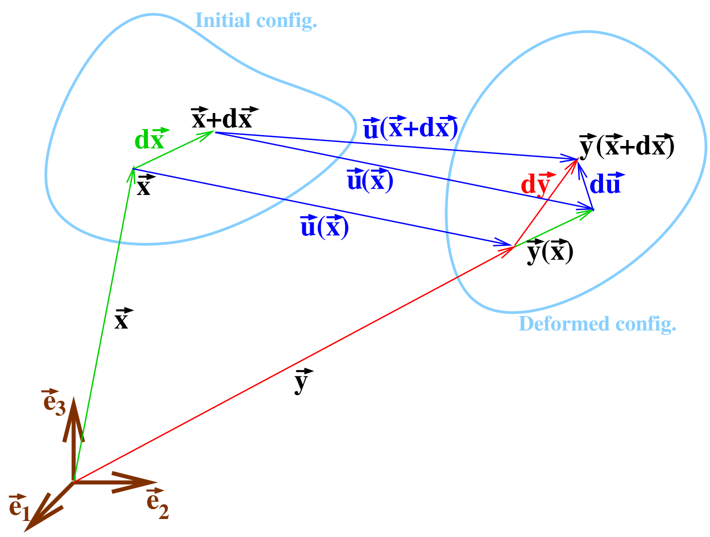
  where $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $\hat{g}_{ij} = \hat{\mathbf{g}}_i \cdot \hat{\mathbf{g}}_j$
  are metric tensors (matrices)
- **Green strain tensor**:

$$\varepsilon_{ij}(\mathbf{x}) = \frac{1}{2}\,(\hat{g}_{ij} - g_{ij})$$

$$\varepsilon_{ij}(\mathbf{x}) = \frac{1}{2}\left(F_i^k(\mathbf{x})\, F_j^\ell(\mathbf{x})\, g_{k\ell} - g_{ij}\right)$$

1. Lagrange descript.: Green strain tensor in displacement

- $\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}$
- $\mathbf{u} = u^i\, \mathbf{g}_i$
- $d\mathbf{y} = d\mathbf{x} + d\mathbf{u}$
- $d\mathbf{u} = \frac{\partial u^i}{\partial x^j}\, dx^j\, \mathbf{g}_i$
- $d\mathbf{y} = \left(dx^i + u^i_{,j}\, dx^j\right) \mathbf{g}_i$



1. Lagrange descript.: Green strain tensor in displacement

$$(d\mathbf{y})^2 = \left(dx^i + u^i_{,j}\, dx^j\right)\left(dx^k + u^k_{,\ell}\, dx^\ell\right) g_{ik}$$

$$(d\mathbf{y})^2 - (d\mathbf{x})^2 = u^i_{,j}\, dx^j\, dx^k\, g_{ik} + u^k_{,\ell}\, dx^\ell\, dx^i\, g_{ik} + u^i_{,j}\, u^k_{,\ell}\, dx^j\, dx^\ell\, g_{ik}$$

$$(d\mathbf{y})^2 - (d\mathbf{x})^2 = \left(u_{i,j} + u_{j,i} + u_{k,i}\, u^k_{,j}\right) dx^i\, dx^j$$

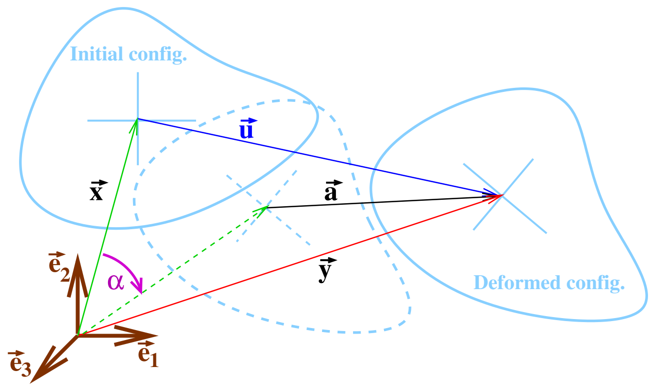
Green strain tensor in displacements:

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}\, u^k_{,j}\right)}$$

Green strain tensor in displacements in cartesian coordinates:

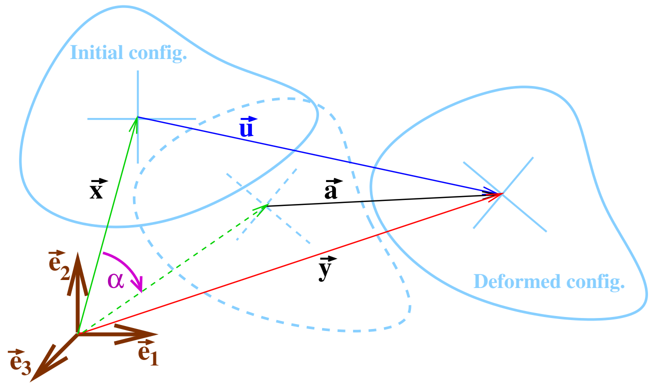
$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)}$$

1. Lagrange description: example 1 – rigid body motion



$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

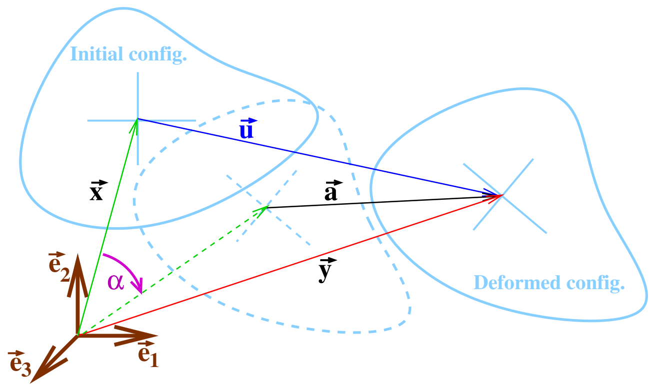
1. Lagrange description: example 1 – rigid body motion



$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha - 1 & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{2}\left(u_{1,1} + u_{1,1} + \sum_{k=1}^{3} u_{k,1} u_{k,1} \right) \\ &= \cos\alpha - 1 + \frac{(\cos\alpha - 1)^2 + \sin^2\alpha}{2} = 0 \end{aligned}$$

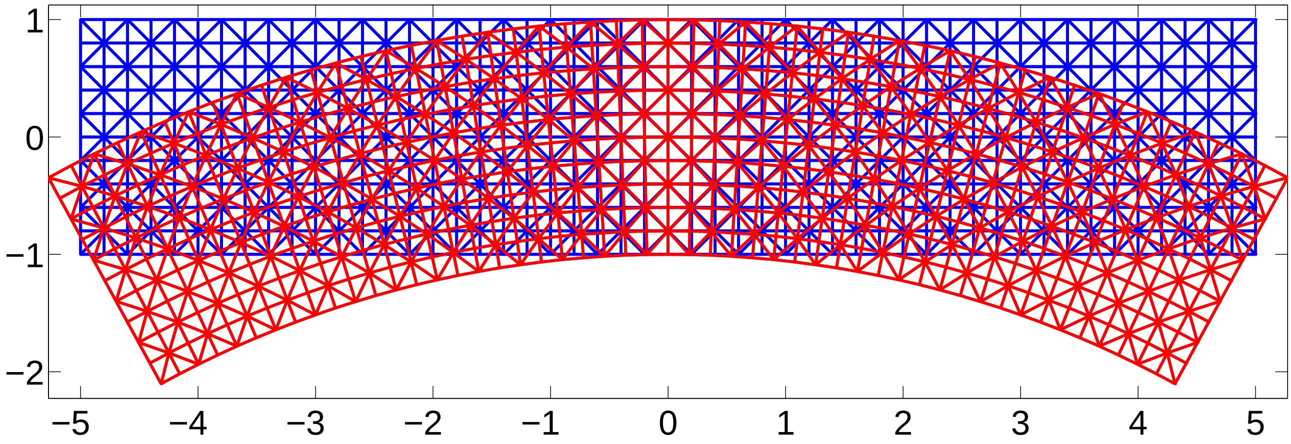
1. Lagrange description: example 1 – rigid body motion



$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha - 1 & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{aligned} \varepsilon_{12} &= \frac{1}{2}\left(u_{1,2} + u_{2,1} + \sum_{k=1}^{3} u_{k,1} u_{k,2} \right) \\ &= \frac{-\sin\alpha + \sin\alpha}{2} + \frac{-\sin\alpha\,(\cos\alpha - 1) + \sin\alpha\,(\cos\alpha - 1)}{2} = 0 \end{aligned}$$

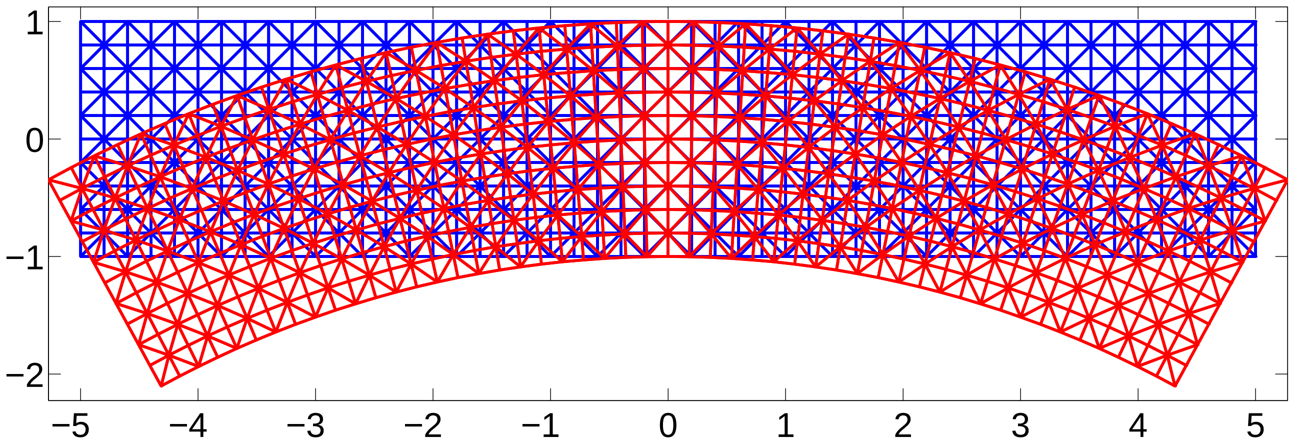
1. Lagrange description: example 2

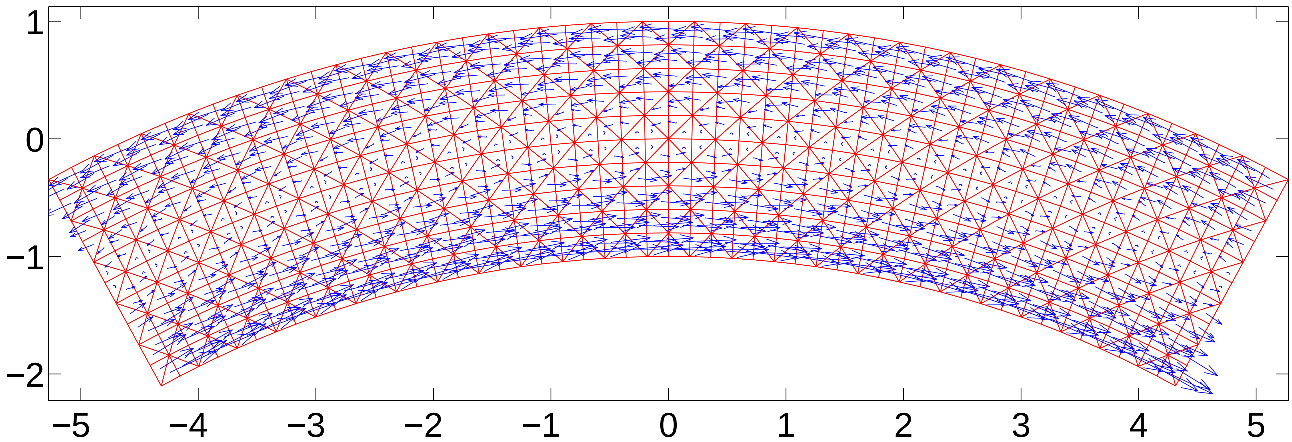


Initial configuration is bent into the deformed configuration

Principal strain of Green strain tensor (Lagrange formulation)?

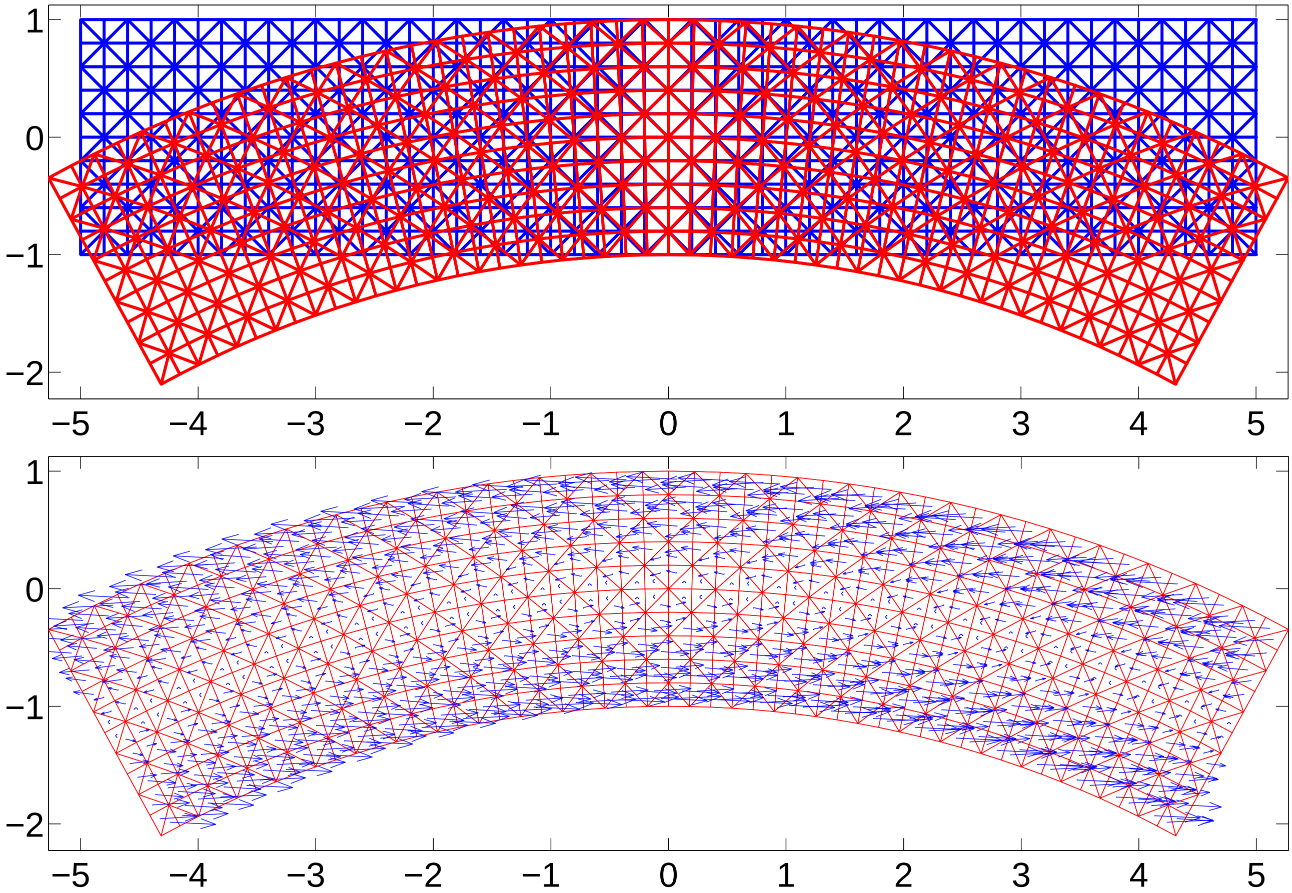
1. Lagrange description: example 2





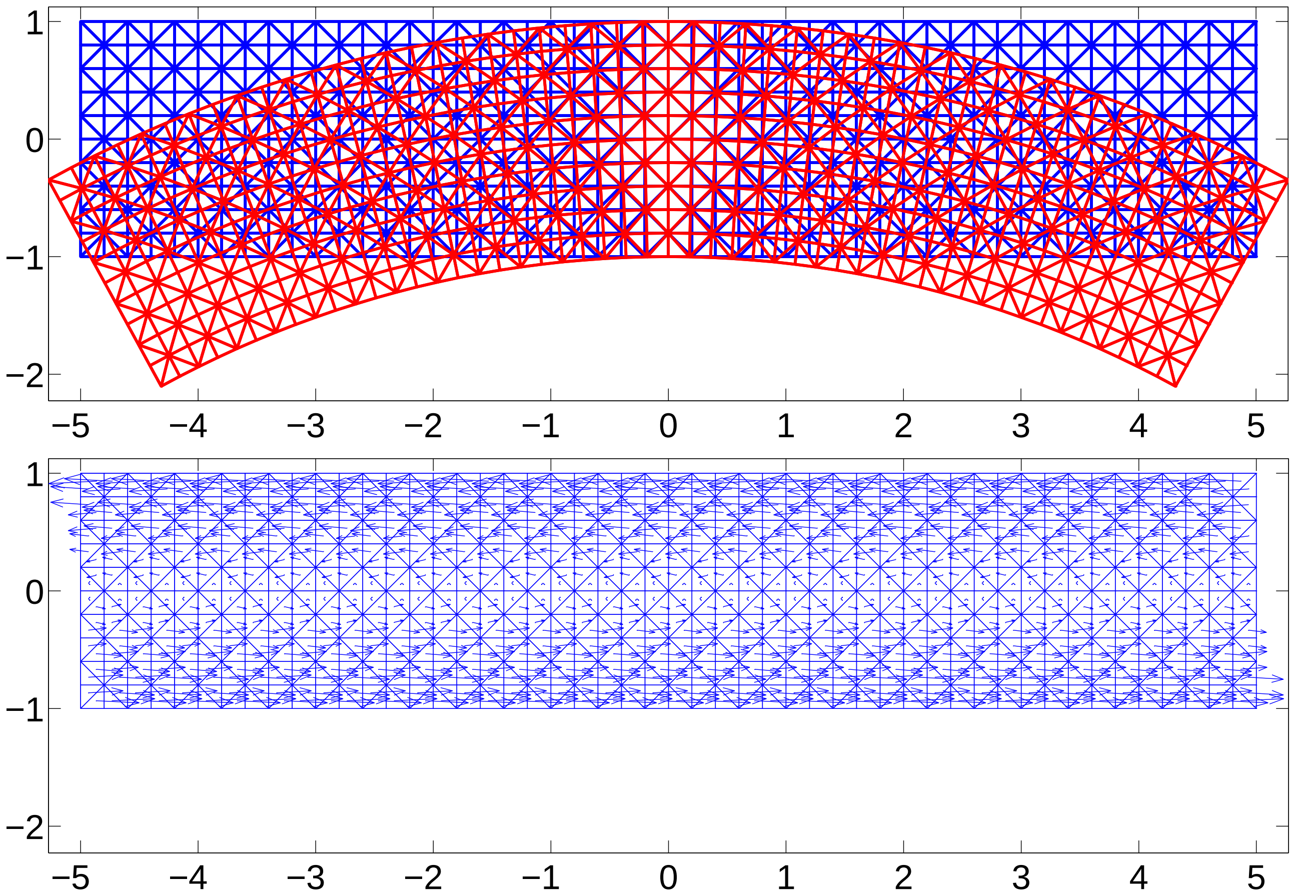
Wrong - this is Almansi strain (Euler formulation)!

1. Lagrange description: example 2



This is the correct Green strain (Lagrange formulation)

1. Lagrange description: example 2



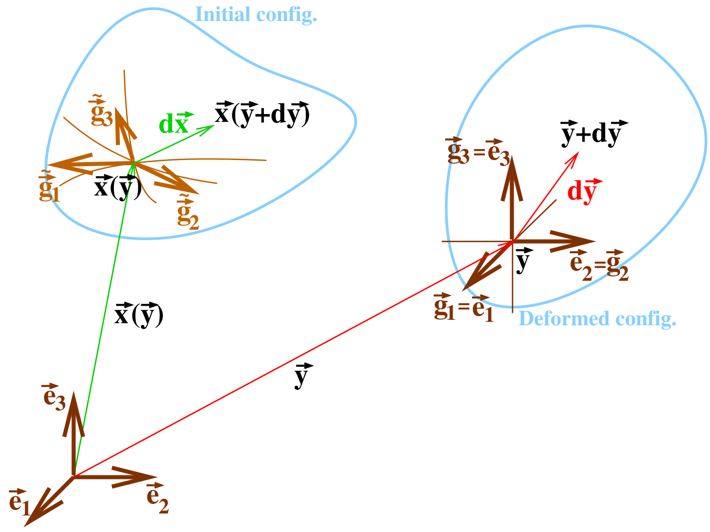
This is the correct Green strain (Lagrange formulation)

2. Euler description

- $\mathbf{x} = x^i \mathbf{e}_i$ and $\mathbf{y} = y^i \mathbf{e}_i$.
- Choice: $\mathbf{x} = \mathbf{x}(\mathbf{y})$
- **Deformation gradient inverse:**

$$F^{-1}{}^i_j = \frac{\partial x^i}{\partial y^j} = x^i_{,j}$$

- $d\mathbf{y} = dy^i \mathbf{e}_i$
- $d\mathbf{x} = dx^i \mathbf{e}_i = \frac{\partial x^i}{\partial y^j} dy^j \, \mathbf{e}_i$
- $d\mathbf{x} = dx^j \, F^{-1}{}^i_j \, \mathbf{g}_i = dx^j \, \tilde{\mathbf{g}}_j$



2. Euler description: Almansi strain tensor

- the deformed "edge" $d\mathbf{y}$ corresponds to the undeformed $d\mathbf{x}$
- Difference of their (lengths)^2:

$$d\mathbf{y}^2 - d\mathbf{x}^2 = (dy^i \mathbf{g}_i)\cdot(dy^j \mathbf{g}_j) - (dy^i \tilde{\mathbf{g}}_i)\cdot(dy^j \tilde{\mathbf{g}}_j) = dy^i \, dy^j \, (g_{ij} - \tilde{g}_{ij})$$

where $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $\tilde{g}_{ij} = \tilde{\mathbf{g}}_i \cdot \tilde{\mathbf{g}}_j$
are metric tensors (matrices)

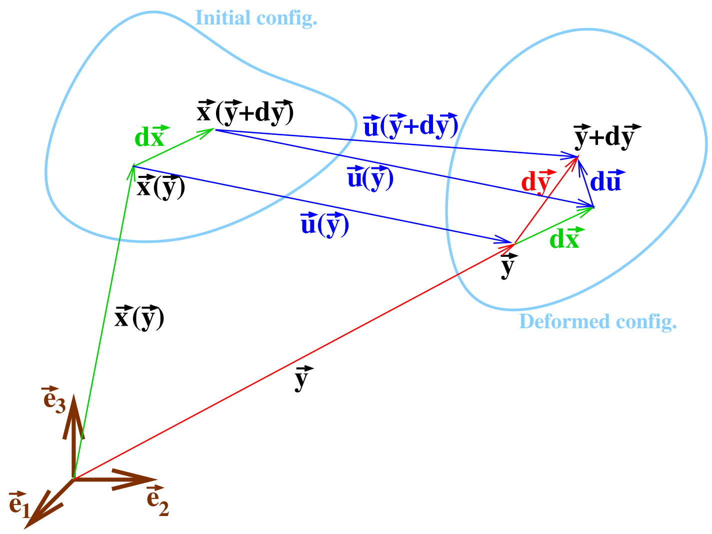
- **Almansi strain tensor**:

$$\mathcal{E}_{ij}(\mathbf{y}) = \frac{1}{2}\left(g_{ij} - \tilde{g}_{ij}\right)$$

$$\mathcal{E}_{ij}(\mathbf{y}) = \frac{1}{2}\left(g_{ij} - F^{-1}{}^k_i \; F^{-1}{}^\ell_j \; g_{k\ell}\right)$$

2. Euler description: Almansi strain tensor in displacement

- $\mathbf{u}(\mathbf{y}) = \mathbf{y} - \mathbf{x}(\mathbf{y})$
- $\mathbf{u} = u^i \, \mathbf{g}_i$
- $d\mathbf{x} = d\mathbf{y} - d\mathbf{u}$
- $d\mathbf{u} = \dfrac{\partial u^i}{\partial y^j} \, dy^j \, \mathbf{g}_i$
- $d\mathbf{x} = \left(dy^i - u^i_{,j} \, dy^j \right) \mathbf{g}_i$



2. Euler description: Almansi strain tensor in displacement

$$(d\mathbf{x})^2 = \left(dy^i - u^i_{,j} \, dy^j \right) \left(dy^k - u^k_{,\ell} \, dy^\ell \right) g_{ik}$$

$$(d\mathbf{y})^2 - (d\mathbf{x})^2 = u^i_{,j} \, dy^j \, dy^k \, g_{ik} + u^k_{,\ell} \, dy^\ell \, dy^i \, g_{ik} - u^i_{,j} \, u^k_{,\ell} \, dy^j \, dy^\ell \, g_{ik}$$

$$(d\mathbf{y})^2 - (d\mathbf{x})^2 = \left(u_{i,j} + u_{j,i} - u_{k,i} \, u^k_{,j} \right) dy^i \, dy^j$$

Almansi strain tensor in displacements:

$$\boxed{\mathcal{E}_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} - u_{k,i} \, u^k_{,j} \right)}$$

Almansi strain tensor in displacements in cartesian coordinates:

$$\boxed{\mathcal{E}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} - \frac{\partial u_k}{\partial y_i} \frac{\partial u_k}{\partial y_j} \right)}$$

3. Green and Almansi strain tensors: mutual relation

- **Green strain tensor**:

$$\varepsilon_{ij}(\mathbf{x}) = \frac{1}{2}\left(\hat{g}_{ij} - g_{ij}\right) = \frac{1}{2}\left(F_i^k\, F_j^\ell\, g_{k\ell} - g_{ij}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

- **Almansi strain tensor**

$$\mathcal{E}_{ij}(\mathbf{y}) = \frac{1}{2}\left(g_{ij} - \tilde{g}_{ij}\right) = \frac{1}{2}\left(g_{ij} - {F^{-1}}_i^k\; {F^{-1}}_j^\ell\, g_{k\ell}\right)$$

$$\mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} - \frac{\partial u_k}{\partial y_i}\frac{\partial u_k}{\partial y_j}\right)$$

3. Green and Almansi strain tensors: mutual relation

Relation between $F_j^i = \frac{\partial y^i}{\partial x^j}$ **and** ${F^{-1}}_k^j = \frac{\partial x^j}{\partial y^k}$:

$$F_j^i \cdot {F^{-1}}_k^j = \sum_{j=1}^{3} \frac{\partial y^i}{\partial x^j}\frac{\partial x^j}{\partial y^k} = \frac{\partial y^i}{\partial y^k} = \delta_k^i$$

by chain rule for the derivatives.

Hence:

$$\varepsilon_{k\ell} = F_k^i \cdot \mathcal{E}_{ij} \cdot F_\ell^j$$

3. Green and Almansi strain tensors: matrix form

Deformation gradient matrix:

$$\mathbf{F} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \frac{\partial y^1}{\partial x^3} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^2}{\partial x^3} \\ \frac{\partial y^3}{\partial x^1} & \frac{\partial y^3}{\partial x^2} & \frac{\partial y^3}{\partial x^3} \end{pmatrix} = \left[F^i_j\right] \qquad \text{and} \qquad \mathbf{F}^{-1} = \left[{F^{-1}}^i_j\right]$$

Green and Almansi matrix for cartesian coords ($\mathbf{g}_i = \mathbf{e}_i$, $g_{ij} = \delta_{ij}$):

$$[\varepsilon_{ij}] = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) \qquad \text{and} \qquad [\mathcal{E}_{ij}] = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right)$$

Mutual relations:

$$[\varepsilon_{ij}] = \mathbf{F}^T \cdot [\mathcal{E}_{ij}] \cdot \mathbf{F} \qquad \text{and} \qquad [\mathcal{E}_{ij}] = \mathbf{F}^{-T} \cdot [\varepsilon_{ij}] \cdot \mathbf{F}^{-1}$$

3. Green and Almansi strain tensors: small deformations

If $u_{i,j} \ll 1$ then $u_{k,i} \cdot u_{k,j}$ is negligible:

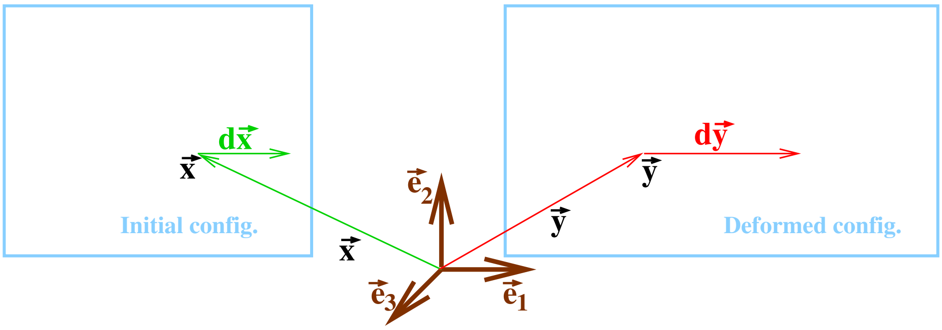
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} + \frac{\partial u^k}{\partial x^i}\frac{\partial u^k}{\partial x^j}\right) \approx \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) = e_{ij}$$

We can replace Green strain ε_{ij} by the Cauchy strain e_{ij}
Advantage: Cauchy strain $e_{ij}(\mathbf{u})$ is **linear** in $\mathbf{u}$

$$\begin{aligned} \mathcal{E}_{ij} &= \frac{1}{2}\left(\frac{\partial u^i}{\partial y^j} + \frac{\partial u^j}{\partial y^i} - \frac{\partial u^k}{\partial y^i}\frac{\partial u^k}{\partial y^j}\right) \approx \frac{1}{2}\left(\frac{\partial u^i}{\partial y^j} + \frac{\partial u^j}{\partial y^i}\right) \\ &= \frac{1}{2}\left(\frac{\partial u^i}{\partial x^k}\frac{\partial x^k}{\partial y^j} + \frac{\partial u^j}{\partial y^k}\frac{\partial x^k}{\partial y^i}\right) \approx \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) = e_{ij} \end{aligned}$$

because $x^k = y^k - u^k$ and $u_{i,k} \cdot u_{k,\ell}$ is negligible.
NB: Green and Almansi simplify to the same Cauchy strain!

4. Physical meaning of Cauchy strain in cartesian coords



Special choices of the deformation mode:
Let $d\mathbf{x} = dx^1\,\mathbf{e}_1$ and $d\mathbf{y} = dy^1\,\mathbf{e}_1$. Then:

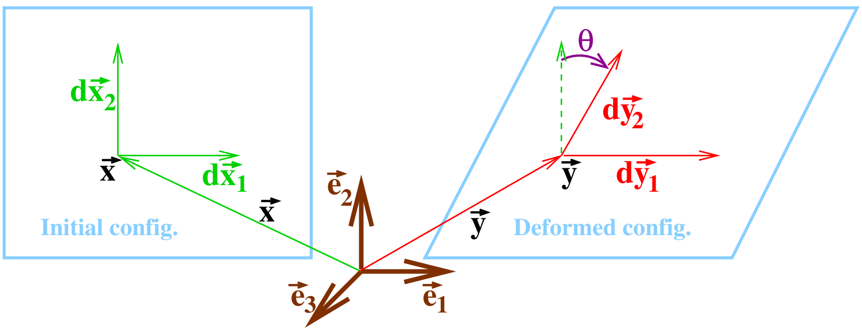
$$d\mathbf{y}^2 - d\mathbf{x}^2 = (dy^1)^2 - (dx^1)^2 = 2\,e_{11}\,(dx^1)^2$$

Hence (for small deformations $dx^1 + dy^1 \approx 2\,dx^1$):

$$e_{11} = \frac{(dy^1)^2 - (dx^1)^2}{2\,(dx^1)^2} = \frac{(dy^1 - dx^1)(dy^1 + dx^1)}{2\,(dx^1)^2} \approx \frac{dy^1}{dx^1} - 1$$

Meaning of e_{kk}: relative elongation along $\mathbf{e}_k$

4. Physical meaning of Cauchy strain in cartesian coords



Special choices of the deformation mode:
Let $d\mathbf{x} = d\mathbf{x}_1 + d\mathbf{x}_2$ and $d\mathbf{y} = d\mathbf{y}_1 + d\mathbf{y}_2$, $d\mathbf{x}_k = dx^k\,\mathbf{e}_k$:

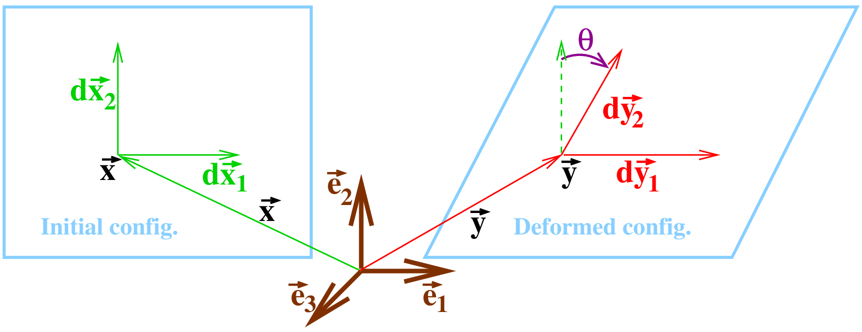
$$\begin{aligned} d\mathbf{y}^2 - d\mathbf{x}^2 &= (d\mathbf{y}_1)^2 + 2\,d\mathbf{y}_1\cdot d\mathbf{y}_2 + (d\mathbf{y}_2)^2 - (d\mathbf{x}_1)^2 - 2\,d\mathbf{x}_1\cdot d\mathbf{x}_2 - (d\mathbf{x}_2)^2 \\ &= 2\left[e_{11}\,(dx^1)^2 + 2\,e_{12}\,dx^1\,dx^2 + e_{22}\,(dx^2)^2\right] \end{aligned}$$

Hence (for small deformations $e_{kk} \ll 1$ and θ is small):

$$\begin{aligned} e_{12} &= \frac{\theta}{2}\frac{dy^1}{dx^1}\frac{dy^2}{dx^2} \approx \frac{\theta}{2}(1 + e_{11})(1 + e_{22}) \\ &\approx \frac{\theta}{2}(1 + e_{11} + e_{22} + e_{11}\,e_{22}) \approx \frac{\theta}{2} \end{aligned}$$

Meaning of e_{12}: half of shear angle θ in the plane $(0, \mathbf{e}_1, \mathbf{e}_2)$

4. Physical meaning of Cauchy strain in cartesian coords



Special choices of the deformation mode:

Let $d\mathbf{x} = d\mathbf{x}_1 + d\mathbf{x}_2$ and $d\mathbf{y} = d\mathbf{y}_1 + d\mathbf{y}_2$, $d\mathbf{x}_k = dx^k\,\mathbf{e}_k$:

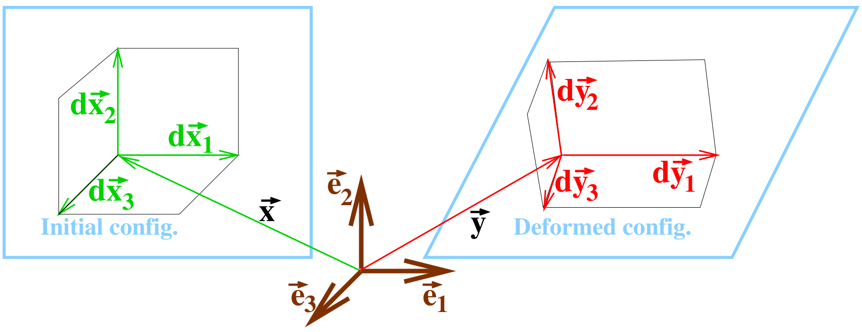
$$\begin{aligned} d\mathbf{y}_1 \cdot d\mathbf{y}_2 &= 2\,e_{12}\,dx^1\,dx^2 \\ &= |d\mathbf{y}_1| \cdot |d\mathbf{y}_2| \cos(\pi/2 - \theta) \approx dy^1\,dy^2\,\sin\theta \approx dy^1\,dy^2 \cdot \theta \end{aligned}$$

Hence (for small deformations $e_{kk} \ll 1$ and θ is small):

$$\begin{aligned} e_{12} &= \frac{\theta}{2}\frac{dy^1}{dx^1}\frac{dy^2}{dx^2} \approx \frac{\theta}{2}(1+e_{11})(1+e_{22}) \\ &\approx \frac{\theta}{2}(1 + e_{11} + e_{22} + e_{11}\,e_{22}) \approx \frac{\theta}{2} \end{aligned}$$

Meaning of e_{12}: half of shear angle θ in the plane $(0, \mathbf{e}_1, \mathbf{e}_2)$

4. Physical meaning of Cauchy strain in cartesian coords



Volume before (dV) and after ($d\hat{V}$) deformation (small deformations):

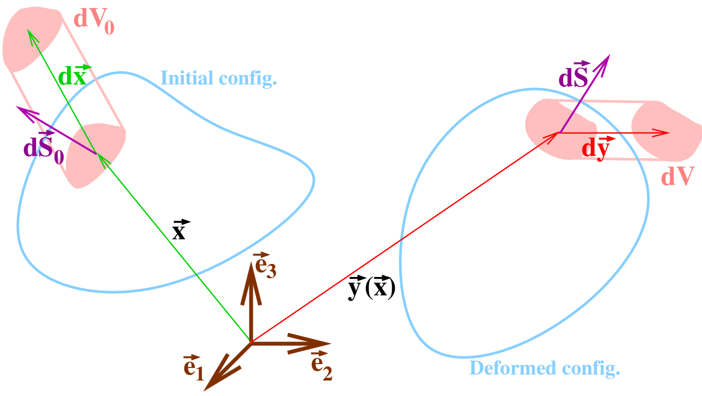
$$dV = dx^1\,dx^2\,dx^3 \qquad d\hat{V} \approx dy^1\,dy^2\,dy^3$$

Relative change of volume (for small deformations $e_{kk} \ll 1$):

$$\begin{aligned} \frac{d\hat{V} - dV}{dV} &= \frac{d\hat{V}}{dV} - 1 \approx \frac{dy^1}{dx^1}\frac{dy^2}{dx^2}\frac{dy^3}{dx^3} - 1 \\ &= (1+e_{11})(1+e_{22})(1+e_{33}) - 1 \approx e_{11} + e_{22} + e_{33} \end{aligned}$$

Meaning of trace of e_{ij}: relative change of volume

Nanson's relation:



- we know how to transform vectors:

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

- we know how to transform volumes:

$$dV = J \cdot dV_0 \qquad \text{with } J = \det \left[\frac{\partial y^i}{\partial x^j}\right]$$

Idea: complete areas to volumes (for any $d\mathbf{x}$, ie. any $d\mathbf{y}$):

$$dS_i\, dy^i = dV = J \cdot dV_0 = J \cdot dS_{0j}\, dx^j = J \cdot dS_{0j} \frac{\partial x^j}{\partial y^i} dy^i$$